ON THE NIELD-KOZNETSOV INTEGRAL FUNCTION AND ITS APPLICATION TO AIRY'S INHOMOGENEOUS BOUNDARY VALUE PROBLEM

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ABSTRACT

In this work, we provide a solution to a two-point boundary value problem that involves an inhomogeneous Airy's differential equation with a variable forcing function. The solution is expressed in terms of the recently introduced Nield-Koznetsov integral function, $Ni(x)$, and another conveniently defined integral function, $Ki(x)$. The resulting expressions involving these integral functions are then evaluated using asymptotic and ascending series.

Indexing terms/Keywords

Airy's Boundary Value Problem, Nield-Kuznetsov Function.

Academic Discipline And Sub-Disciplines

Mathematics, Mathematical Physics.

SUBJECT CLASSIFICATION

Math. Subj. Classification 2010: 34B05, 34B30, 35G15

TYPE (METHOD/APPROACH)

Mathematical Analysis

1. INTRODUCTION

Consider the boundary value problem (BVP) composed of solving the inhomogeneous, second order ordinary differential equation (ODE)

$$y'' - xy = f(x). \qquad \text{...(1)}$$

Subject to the following boundary conditions (BC)

$$y(a) = \alpha \qquad \text{...(2)}$$

$$y(b) = \beta \qquad \text{...(3)}$$

where α and β are real constants and $x \in [a,b]$.

Equation (1) is the well-known inhomogeneous Airy's differential equation. Rooted in Airy's nineteenth century work in optics, Airy's ODE continues to receive interest due to the reduction of many differential equations in mathematical physics to it by an appropriate change of variables (cf. [4,5,8,11,12] and the references therein).

Solution to the homogeneous part of ODE (1) is given by

$$y = c_1 A_i(x) + c_2 B_i(x) \qquad \text{...(4)}$$

where c_1 and c_2 are arbitrary constants, and $Ai(x)$ and $Bi(x)$ are two linearly independent functions, known as Airy's functions and defined by the following integrals (*cf.* [1, 8, 11, 12]):

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right) dt \qquad \text{...(5)}$$

$$Bi(x) = \frac{1}{\pi} \int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt + \frac{1}{\pi} \int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt \qquad \text{...(6)}$$

The wronskian of $Ai(x)$ and $Bi(x)$ is given by, [1]:

$$w(A_i(x), B_i(x)) = A_i(x)B_i'(x) - B_i(x)A_i'(x) = \frac{1}{\pi}. \qquad \text{...(7)}$$

When *f(x)* is a constant function of *x,* the inhomogeneous ODE (1) has a particular solution given by the Scorer functions, [10], as given in what follows:

i) When $f(x) = -\frac{1}{\pi}$, a particular solution is given by

$$Gi(x) = \frac{1}{\pi} \int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt \qquad \text{...(8)}$$

ii) When $f(x) = \frac{1}{\pi}$, a particular solution is given by

$$Hi(x) = \frac{1}{\pi} \int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt. \qquad \text{...(9)}$$

The functions $Gi(x)$ and $Hi(x)$ are known as Scorer's functions. It can be seen from (8), (9) and (6) that

$$Gi(x) + Hi(x) = Bi(x) \qquad \text{...(10)}$$

For real values of x, Airy's and Scorer's functions are real-valued functions, [3]. Extensive analysis has been carried out by Gil *et.al.*, [3], when the argument is complex. Computations of the Airy and Scorer functions continue to receive attention in the literature, and excellent results have been documented (*cf.* [1, 2, 6, 7, 11, 12] and the references therein].

Now, when the forcing function is a constant other than $\mp\frac{1}{\pi}$, or when the forcing function is a variable function of *x*, we need a consistent notation and methodology to find and express the particular solution to ODE (1), and hence the general solution and the solutions to boundary value problems. This is the objective of this work in which we express particular

solutions to equation (1) in terms of the recently introduced Nield-Koznetsov function $N_i(x)$, [9], and in terms of the function $K_i(x)$, [4], revisited in this work.

2. GENERAL SOLUTION OF BVP

General solution to ODE (1) is the sum of the complementary function, given by equation (4) as the solution to the homogeneous Airy's ODE, and the particular solution, y_p, which, using variation of parameters, is assumed to be of the form

$$y_p = u_1 Ai(x) + u_2 Bi(x) \qquad \text{...(11)}$$

where the functions u_1 and u_2 are given by the following forms, respectively, with the help of (7):

$$u_1 = -\pi \int_0^x f(t) Bi(t) dt \qquad \text{...(12)}$$

$$u_2 = \pi \int_0^x f(t) Ai(t) dt . \qquad \text{...(13)}$$

Equation (11) thus takes the form

$$y_p = \pi \left[Bi(x) \int_0^x f(t) Ai(t) dt - Ai(x) \int_0^x f(t) Bi(t) dt \right] \qquad \text{...(14)}$$

and the general solution, y_g, to ODE (1) is thus written as

$$y_g = c_1 A_i(x) + c_2 B_i(x) + \pi \left[Bi(x) \int_0^x f(t) Ai(t) dt - Ai(x) \int_0^x f(t) Bi(t) dt \right] . \qquad \text{...(15)}$$

Equation (15) is valid for both constant forcing function and variable forcing function, as discussed in the following two cases.

2.1. THE CASE OF CONSTANT FORCING FUNCTION

When the forcing function in ODE (1) is constant, say $f(x) = \delta$, equation (15) takes the form

$$y_g = c_1 A_i(x) + c_2 B_i(x) + \pi\delta \left[Bi(x) \int_0^x Ai(t) dt - Ai(x) \int_0^x Bi(t) dt \right] . \qquad \text{...(16)}$$

Equation (16) can be conveniently written in the following form that implements the newly introduced Nield and Koznetsov function, $N_i(x)$, which has been discussed in detail by Hamdan and Kamel [4]:

$$y_g = c_1 A_i(x) + c_2 B_i(x) - \pi\delta N_i(x) \qquad \text{...(17)}$$

where

$$N_i(x) = A_i(x) \int_0^x B_i(t) dt - B_i(x) \int_0^x A_i(t) dt . \qquad \text{...(18)}$$

First and second derivatives of $N_i(x)$ are given by

$$N_i'(x) = A_i'(x) \int_0^x B_i(t) dt - B_i'(x) \int_0^x A_i(t) dt . \qquad \text{...(19)}$$

$$N_i''(x) = A_i''(x)\int_0^x B_i(t)dt - B_i''(x)\int_0^x A_i(t)dt - w(A_i(x), B_i(x)) \qquad \ldots(20)$$

with values at $x = 0$ given by $N_i(0) = N_i'(0) = 0$, $N_i''(0) = -w(A_i(0), B_i(0)) = -\dfrac{1}{\pi}$.

Now, using the BC (2) and (3) in (17), we can determine the following values of the arbitrary constants and render the BVP completely solved:

$$c_1 = \frac{\alpha B_i(b) - \beta B_i(a) + \dfrac{\delta}{N_i''(0)}[N_i(b)B_i(a) - N_i(a)B_i(b)]}{A_i(a)B_i(b) - A_i(b)B_i(a)} \qquad \ldots(21a)$$

$$c_2 = \frac{\alpha A_i(b) - \beta A_i(a) + \dfrac{\delta}{N_i''(0)}[N_i(b)A_i(a) - N_i(a)A_i(b)]}{A_i(b)B_i(a) - A_i(a)B_i(b)}. \qquad \ldots(21b)$$

2.2. THE CASE OF VARIABLE FORCING FUNCTION

When the forcing function in ODE (1) is a variable function *f(x)*, the general solution is given by equation (15). The particular solution given by equation (14) involves the integrals $\int_0^x f(t)A_i(t)dt$ and $\int_0^x f(t)B_i(t)dt$. Using integration by parts, we express these integrals as follows:

$$\int_0^x f(t)A_i(t)dt = f(x)\int_0^x A_i(t)dt - \int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} f'(t)dt \qquad \ldots(22)$$

$$\int_0^x f(t)B_i(t)dt = f(x)\int_0^x B_i(t)dt - \int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} f'(t)dt. \qquad \ldots(23)$$

Substituting (22) and (23) in (14), and using $N_i''(0) = -\dfrac{1}{\pi}$, we obtain:

$$y_p = \frac{A_i(x)}{N_i''(0)}\left\{f(x)\int_0^x B_i(t)dt - \int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} f'(t)dt\right\} - \frac{B_i(x)}{N_i''(0)}\left\{f(x)\int_0^x A_i(t)dt - \int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} f'(t)dt\right\}.$$
$$\ldots(24)$$

Using (18), equation (24) can be written as:

$$y_p = \frac{f(x)N_i(x) - K_i(x)}{N_i''(0)} = \pi\{K_i(x) - f(x)N_i(x)\} \qquad \ldots(25)$$

where we define the integral function $Ki(x)$ as:

$$K_i(x) = A_i(x)\int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} f'(t)dt - B_i(x)\int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} f'(t)dt. \qquad \ldots(26)$$

From (15) and (24), we can express the general solution to (1) as

$$y_g = c_1 A_i(x) + c_2 B_i(x) + \pi\{K_i(x) - f(x)N_i(x)\}. \qquad \ldots(27)$$

Relationship between $K_i(x)$ and $N_i(x)$ can be obtained by multiplying (23) by $A_i(x)$, and (22) by $B_i(x)$, then subtracting, to obtain

$$A_i(x)\int_0^x f(t)B_i(t)dt - B_i(x)\int_0^x f(t)A_i(t)dt = f(x)\left\{A_i(x)\int_0^x B_i(t)dt - B_i(x)\int_0^x A_i(t)dt\right\}$$
$$-\left[A_i(x)\int_0^x\left\{\int_0^t B_i(\tau)d\tau\right\}f'(t)dt - B_i(x)\int_0^x\left\{\int_0^t A_i(\tau)d\tau\right\}f'(t)dt\right] \qquad \text{...(28)}$$

The right-hand-side of (28) is recognized as $f(x)N_i(x) - K_i(x)$. We can thus express $K_i(x)$ as

$$K_i(x) = f(x)N_i(x) - \left\{A_i(x)\int_0^x f(t)B_i(t)dt - B_i(x)\int_0^x f(t)A_i(t)dt\right\}. \qquad \text{...(29)}$$

The first two derivatives of $K_i(x)$ can be obtained from (29) as

$$K_i'(x) = f'(x)N_i(x) + f(x)N_i'(x) - \left\{A_i'(x)\int_0^x f(t)B_i(t)dt - B_i'(x)\int_0^x f(t)A_i(t)dt\right\} \qquad \text{...(30)}$$

$$K_i''(x) = f(x)N_i''(x) + 2f'(x)N_i'(x) + f''(x)N_i(x) + f(x)w(A_i(x),B_i(x))$$
$$- x\left\{A_i(x)\int_0^x f(t)B_i(t)dt - B_i(x)\int_0^x f(t)A_i(t)dt\right\} \qquad \text{...(31)}$$

and the values at $x = 0$ are $K_i(0) = K_i'(0) = K_i''(0) = 0$.

Now, upon using the BC (2) and (3) in (27), we can determine the following values of the arbitrary constants and render the BVP completely solved:

$$c_1 = \frac{[f(b)N_i(b) - K_i(b) - \beta N_i''(0)]B_i(a) - [f(a)N_i(a) - K_i(a) - \alpha N_i''(0)]B_i(b)}{N_i''(0)[A_i(a)B_i(b) - A_i(b)B_i(a)]} \qquad \text{...(32)}$$

$$c_2 = \frac{[f(b)N_i(b) - K_i(b) - \beta N_i''(0)]A_i(a) - [f(a)N_i(a) - K_i(a) - \alpha N_i''(0)]A_i(b)}{N_i''(0)[A_i(b)B_i(a) - A_i(a)B_i(b)]}. \qquad \text{...(33)}$$

3. EVALUATION AND APPROXIMATION OF $N_i(x)$ AND $K_i(x)$

Computing solutions (17) and (27), and evaluation of the arbitrary constants associated with the BVP (1) subject to conditions (2) and (3), necessitates evaluating $N_i(x)$ and $K_i(x)$ on the interval [a,b]. Since these functions are expressed in terms of Airy's functions, we will rely on approximations of Airy's functions to approximate $N_i(x)$ and $K_i(x)$. A number of methods are discussed in the literature to approximate $Ai(x)$ and $Bi(x)$, (cf. [1, 2, 12]). In the current work, we illustrate the calculations using the asymptotic series approximations. The following asymptotic series approximations have been developed for Airy's functions, their derivatives and integrals, as given in [12], wherein $\mu = \frac{2}{3}x^{3/2}$ and $\varphi = \frac{2}{3}t^{2/3}$:

$$A_i(x) = \frac{\exp(-\mu)}{2\sqrt{\pi}x^{1/4}}\left[1 + \frac{3(5)}{1!(-216\mu)} + \frac{5(7)(9)(11)}{2!(-216\mu)^2} + ...\right] \qquad \text{...(34)}$$

$$B_i(x) = \frac{\exp(\mu)}{\sqrt{\pi}x^{1/4}}\left[1 + \frac{3(5)}{1!(216\mu)} + \frac{5(7)(9)(11)}{2!(216\mu)^2} + ...\right] \qquad \text{...(35)}$$

$$A_i'(x) = -\frac{x^{1/4}\exp(-\mu)}{2\sqrt{\pi}}\left[1 - \frac{3(7)}{1!(-216\mu)} - \frac{5(7)(9)(13)}{2!(-216\mu)^2} - ...\right] \qquad \text{...(36)}$$

$$B_i'(x) = \frac{x^{1/4}\exp(\mu)}{\sqrt{\pi}}\left[1 - \frac{3(7)}{1!(216\mu)} - \frac{5(7)(9)(13)}{2!(216\mu)^2} - \cdots\right] \qquad \ldots(37)$$

$$\int_0^x A_i(t)dt = \frac{1}{3} - \frac{\exp(-\mu)}{2\sqrt{\pi}x^{3/4}}\left[1 - \frac{41}{48x^{3/2}} + \frac{9241}{4608x^3} - \cdots\right] \qquad \ldots(38)$$

$$\int_0^x B_i(t)dt = \frac{\exp(\mu)}{\sqrt{\pi}x^{3/4}}\left[1 + \frac{41}{48x^{3/2}} + \frac{9241}{4608x^3} + \cdots\right]. \qquad \ldots(39)$$

For large *x*, we can truncate each of the above series after the first term, and develop the following asymptotic approximations to $N_i(x)$, $N_i'(x)$, $K_i(x)$, and $K_i'(x)$ using equations (18), (19), (29) and (30):

$$N_i(x) \approx -\frac{\exp(\mu)}{3\sqrt{\pi}x^{1/4}} \qquad \ldots(40)$$

$$N_i'(x) \approx -\frac{x^{1/4}\exp(\mu)}{3\sqrt{\pi}} \qquad \ldots(41)$$

$$K_i(x) = -f(x)\frac{\exp(\mu)}{3\sqrt{\pi}x^{1/4}} + \frac{\exp(-\mu)}{2\sqrt{\pi}x^{1/4}}\int_0^x \frac{\exp(\varphi)}{\sqrt{\pi}t^{3/4}}f'(t)dt \qquad \ldots(42)$$

$$K_i'(x) = \frac{f'(x)}{2\pi x} - \frac{x^{1/4}\exp(-\mu)}{2\sqrt{\pi}}\int_0^x \frac{\exp(\varphi)}{\sqrt{\pi}t^{3/4}}f'(t)dt - \frac{\exp(\mu)}{3\sqrt{\pi}}\left\{\frac{f'(x)}{x^{1/4}} + x^{1/4}f(x)\right\}. \qquad \ldots(43)$$

In what follows we will evaluate the solution to ODE (1) subject to BC (2) and (3) when the forcing function is given by $f(x) = \frac{16}{51}x^{17/12}$, over different intervals [a,b], and compare our calculations with the solution obtained using *Maple's dsolve*. General solution is given by equation (27) with $N_i(x)$ and $K_i(x)$ approximated by equations (40) and (42), respectively. Evaluating (42) for forcing function $f(x) = \frac{16}{51}x^{17/12}$, we obtain

$$K_i(x) \approx \frac{\exp(-\frac{2}{3}x^{3/2})}{2\pi x^{1/4}}\left\{\exp(\frac{2}{3}x^{2/3}) - 1\right\} - \frac{\exp(\frac{2}{3}x^{3/2})}{3\sqrt{\pi}}\frac{16}{51}x^{7/6}. \qquad \ldots(44)$$

Now, using the above asymptotic series approximations for $A_i(x), B_i(x), N_i(x)$ and $K_i(x)$ when *x* is large, equation (27) renders the following approximate general solution

$$y_g \approx c_1\frac{\exp(-\mu)}{2\sqrt{\pi}x^{1/4}} + c_2\frac{\exp(\mu)}{\sqrt{\pi}x^{1/4}} + \pi\left[\frac{\exp(-\mu)}{2\pi x^{1/4}}\left\{\exp(\varphi) - 1\right\} - \frac{\exp(\mu)}{3\sqrt{\pi}}\frac{16}{51}x^{7/6}\right] + \pi\frac{16}{51}x^{17/12}\frac{\exp(\mu)}{3\sqrt{\pi}x^{1/4}}$$

$$\ldots(45)$$

where $\mu = \frac{2}{3}x^{3/2}$ and c_1, c_2 are as given by (32) and (33).

Values of c_1 and c_2 are obtained from equations (32) and (33). Values of $A_i(x), B_i(x), N_i(x)$ and $K_i(x)$ at the boundary points are once again approximated using the above asymptotic series of these functions. We carry out these approximations for a number of BC, shown in **Table 1**, where we have chosen $\alpha = 0$ and $\beta = 1$. When the values of **Table 1** are used to construct the solutions to the given BVP, and the solution is plotted on the selected intervals of the *x*-axis, **Figures 1** through **4** are obtained. In these **Figures** we also compare the solutions with those numerical solutions obtained using *Maple's dsolve* numerical built-in function (shown in **Table 2**). As a first approximation in the asymptotic

series, **Figures 1-4** demonstrate better agreement and closeness in the solutions when the end points of the interval [a,b] are close to each other and not too have in values (as can be seen from **Figures 1** and **3**). Even though the asymptotic series approximations are valid for large values of x, and serves Airy's functions well, it may not serve the $K_i(x)$ well, and hence one requires further terms in the asymptotic series to better capture the asymptotic behaviour of the solution for large values of x.

$y(a) = 0$ and $y(b) = 1$	Values of c_1 and c_2 computed using asymptotic approximation
$y(0) = 0$ and $y(1) = 1$	$c_1 = -1.870201738, c_2 = 0.9351008690$
$y(0) = 0$ and $y(10) = 1$	$c_1 = -4.404263988 \times 10^{-9}, c_2 = 2.202131994 \times 10^{-9}$
$y(2) = 0$ and $y(3) = 1$	$c_1 = -9.870966310, c_2 = 0.07524472650$
$y(99) = 0$ and $y(100) = 1$	$c_1 = -8.2247167140 \times 10^{281}, c_2 = 1.6554662320 \times 10^{-289}$

Table 1. Values of c_1 and c_2 computed using asymptotic series approximations

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> ode := diff(y(x), x, x) = x*y(x)+(16/51)*x^(17/12):

Bcs := y(a) = 0, y(b) = 1:

coef := dsolve(ode):

solz:=dsolve({ode,Bcs}, numeric):

inst := dsolve({Bcs,ode}):
```
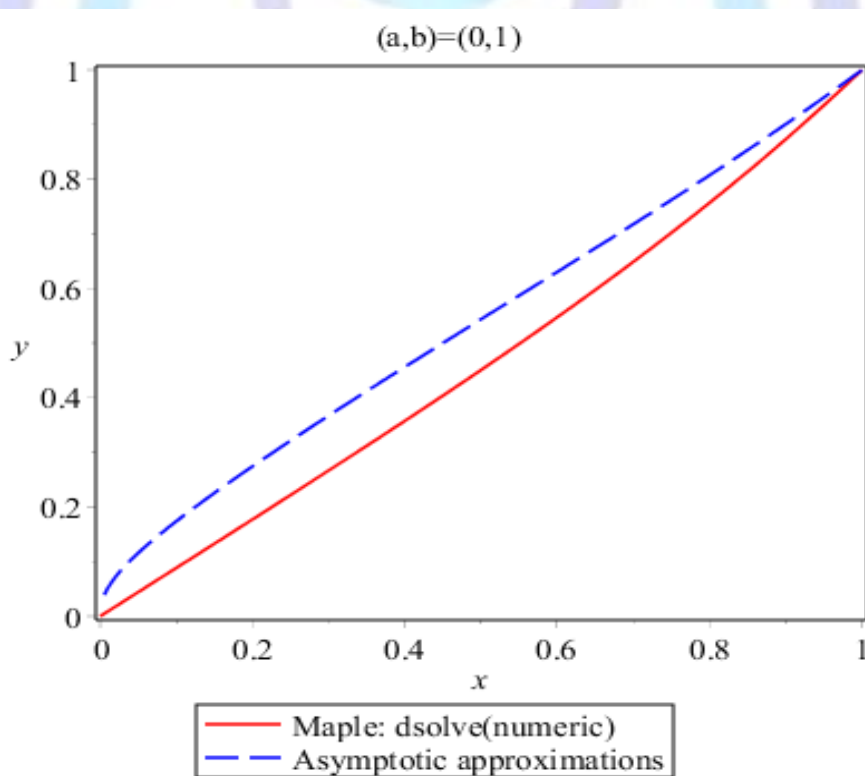
Table 2. Maple's dsolve numerical routine

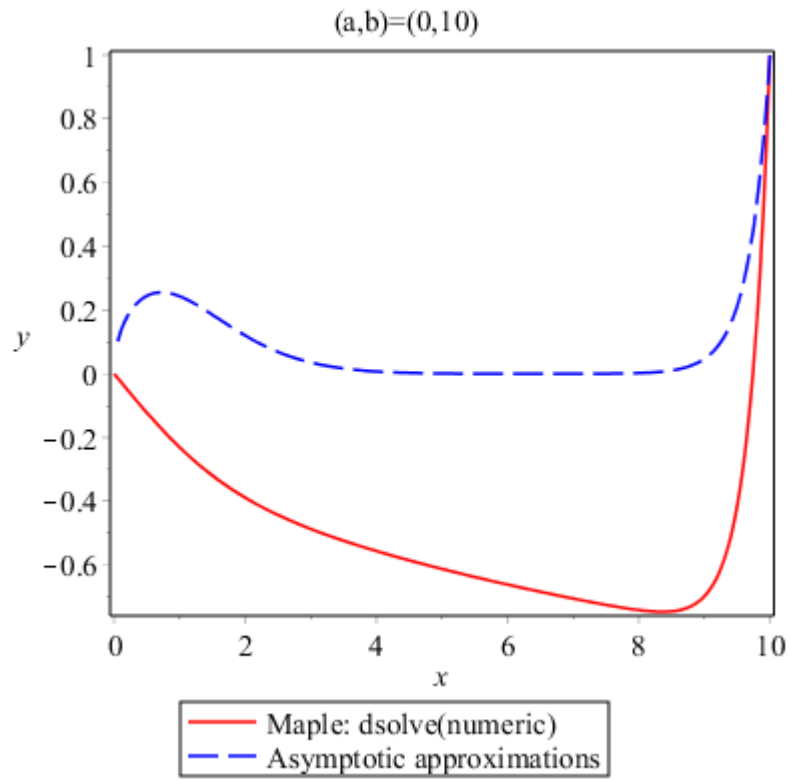**Fig. 1 Solution over the interval [0,1]**

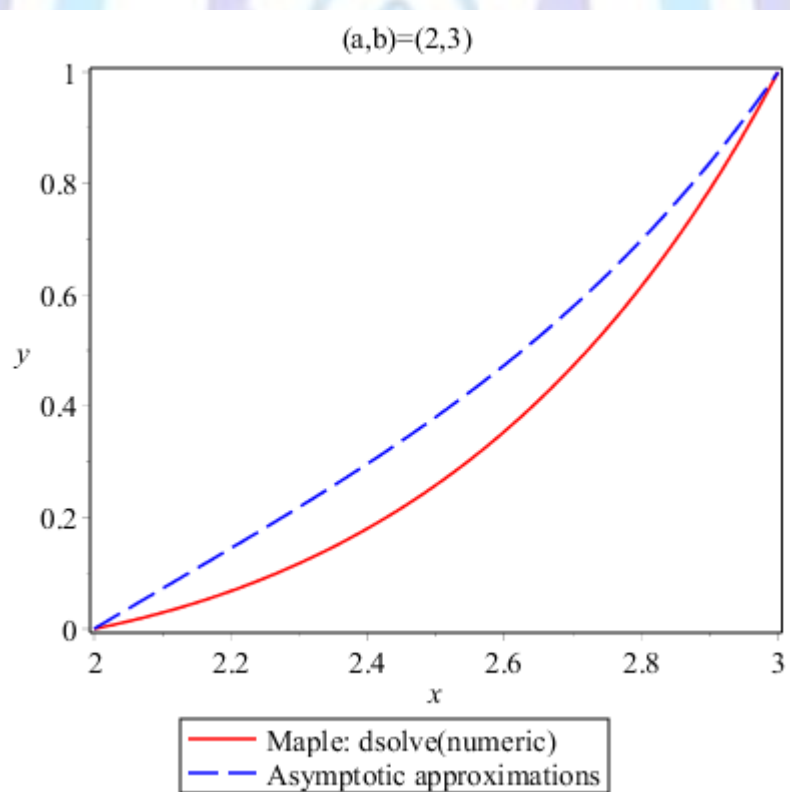**Fig. 2 Solution over the interval [0,10]**

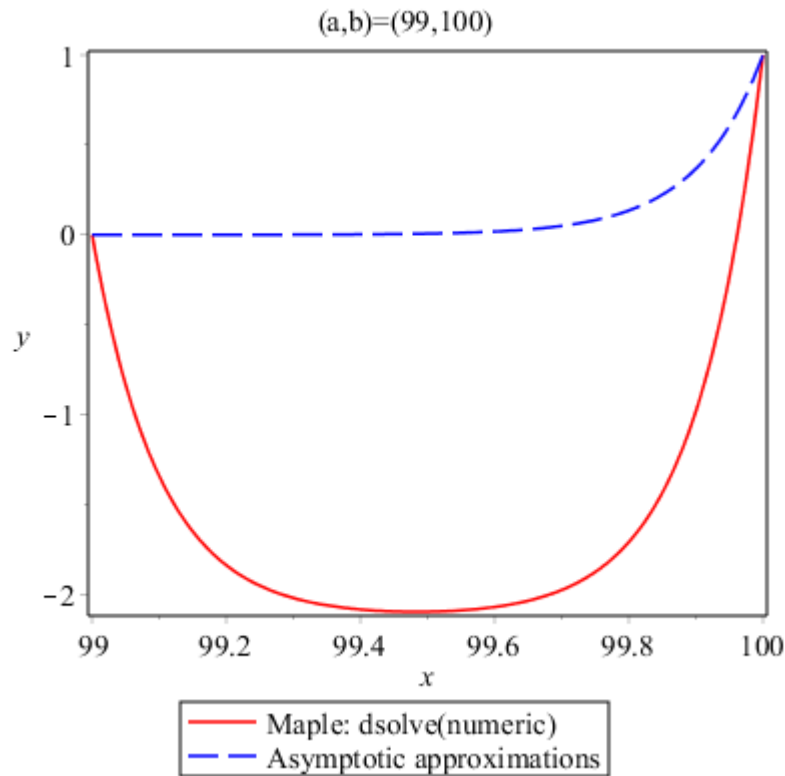**Fig. 3 Solution over the interval [2,3]**

Fig. 4 Solution over the interval [99,100]

4. CONCLUSION

In this work we provided a methodology to solve Airy's inhomogeneous ODE with a two-point boundary value problem. Solutions have been expressed in terms of the newly introduced Nield and Koznetsov function. When the forcing function is a constant function, the general solution to Airy's inhomogeneous ODE is given by equation (17), and when the forcing function is a continuous function of *x*, defined over the real field, the general solution is given by equation (27). The integral functions $A_i(x), B_i(x), N_i(x)$ and $K_i(x)$, used to express the general solution are evaluated, as a first approximation, using asymptotic series expansions.

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