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On the Lie Symmetry Analysis of the Seventh-Order Nonlinear Kaup–Kuperschmidt Equation

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Abstract

This paper presents a rigorous Lie symmetry analysis of the seventh-order Kaup–Kuperschmidt equation given by $u_t 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{xx}u_{xxx} + 147u_xu_{xxxx} + 42uu_{xxxxx} + u_{xxxxxxx} = 0$ which arises as a higher-order nonlinear evolution equation describing nonlinear wave propagation, dispersive media, and integrable physical systems. The study employs the classical Lie group approach to systematically determine the admitted infinitesimal symmetries of the equation. The total derivative operators and seventh prolongation of the infinitesimal generator are utilized in the invariance criterion to derive the determining system governing the infinitesimal coefficients. The admitted Lie algebra and corresponding one-parameter transformation groups are obtained explicitly. Furthermore, similarity reductions are employed to transform the governing partial differential equation into ordinary differential equations, from which exact invariant solutions are constructed. In addition, exact power series solutions are determined to characterize local analytic behavior of solutions. Stability remarks and qualitative physical interpretations of the resulting wave structures are discussed. The results reveal the underlying geometric structure of the seventh-order Kaup–Kuperschmidt equation and provide exact analytical tools for understanding nonlinear dispersive dynamics.

Keywords Lie symmetry analysis. Seventh-order Kaup–Kuperschmidt equation. Infinitesimal generators. Similarity reduction. Invariant solutions. Exact Power series solutions.

1. Introduction

Nonlinear evolution equations play an important role in modern mathematical physics due to their wide applicability in describing wave propagation, nonlinear dispersive media, plasma physics, fluid mechanics, nonlinear optics, and engineering systems. Higher-order nonlinear partial differential equations have gained increasing attention because of their ability to model complex physical phenomena involving higher-order dispersion, nonlinear interactions, and wave modulation effects. Among such equations, the Kaup–Kuperschmidt equation occupies an important position in the theory of integrable systems and nonlinear wave dynamics.

The Kaup–Kuperschmidt equation was originally introduced in connection with integrable hierarchies and inverse scattering methods as a fifth-order nonlinear evolution equation possessing rich mathematical structures including Hamiltonian properties, recursion operators, soliton solutions, and infinite conservation laws. Higher-order generalizations of the Kaup–Kuperschmidt equation have subsequently emerged in the study of nonlinear dispersive systems, where higher-order correction terms become important in accurately describing wave motion in nonuniform and strongly nonlinear environments.

In particular, the seventh-order Kaup–Kuperschmidt equation considered in this work,

$$u_t + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{xx}u_{xxx} + 147u_xu_{xxxx} + 42uu_{xxxxx} + u_{xxxxxxx} = 0 \quad (1.1)$$

represents a highly nonlinear and strongly dispersive system involving nonlinear coupling between high-order derivative interactions and cubic nonlinear effects. Such equations arise in mathematical models describing nonlinear wave interactions where higher-order corrections significantly influence wave propagation, energy transfer, and stability properties.

Despite considerable developments in the analysis of nonlinear evolution equations, obtaining exact solutions of higher-order nonlinear PDEs remains mathematically challenging due to strong nonlinear coupling and the presence of higher-order derivative terms. Classical analytical techniques often fail to provide systematic methods for solving such equations. Consequently, symmetry-based methods, particularly Lie group analysis, have become indispensable tools in the investigation of nonlinear differential equations.

The Lie symmetry method, pioneered by Sophus Lie, provides a systematic framework for studying differential equations through continuous transformation groups. The method enables one to identify admitted symmetries, reduce the order of equations, derive invariant solutions, and uncover hidden geometric structures of differential equations. Over the years, Lie symmetry analysis has become one of the most powerful techniques for solving nonlinear PDEs and understanding their qualitative properties.

The application of Lie symmetry methods to nonlinear dispersive equations has yielded important results for equations such as the Korteweg–de Vries equation, modified KdV equation, Boussinesq equation, Sawada–Kotera equation and generalized higher-order evolution equations. Through admitted symmetry reductions, exact traveling-wave solutions, self-similar solutions, and conservation laws have been systematically constructed.

However, rigorous Lie symmetry analysis of the seventh-order Kaup–Kuperschmidt equation remains relatively underexplored in the literature. A systematic derivation involving seventh prolongation, infinitesimal generators, determining equations, invariant reductions, exact solutions, and power series formulations is lacking. This motivates the present investigation.

Several approaches have been proposed for solving nonlinear evolution equations including; Inverse scattering transform as demonstrated by Miura [8], Hirota bilinear method as discussed by Hirota [9], Bäcklund transformations as discussed by Rogers [10], tanh-function method as illustrated by Fan [11], Adomian decomposition as was explained by Adomian [12], homotopy perturbation method as demonstrated by He [13], variational iteration methods as illustrated by He [14], and symmetry analysis as researched by Lie [15]. The other framework of literature is demonstrated in [3, 4, 5, 6, 7]. Wave equations were illustrated by [16, 17, 18, 19, 20, 21, 22]

The theoretical foundation of Lie symmetry methods was established by Sophus Lie [15], who introduced continuous transformation groups for differential equations. Later developments by George Bluman [1] and Peter Olver [2] significantly advanced the application of symmetry methods to nonlinear PDEs.

The primary purpose of this paper is therefore to provide a comprehensive Lie symmetry treatment of the seventh-order Kaup–Kuperschmidt equation through rigorous derivation and analytical construction of invariant structures. This is achieved by using the total derivative operators and seventh prolongation of the infinitesimal generator to generate admitted Lie groups and infinitesimal generators admitted by the equation, to derive exact invariant and similarity solutions through symmetry reductions and to obtain exact power series solutions of the equation.

The remainder of this paper is organized as follows. Section 2 presents the methods of using Lie symmetry analysis. Section 3 presents similarity reductions and exact invariant solutions. Section 4 develops exact power series solutions. Finally, Section 5 concludes the paper with remarks and future directions.

Materials and Methods

2. Methods

2.1 Generator and Prolongation

Let a general nonlinear PDE be represented as

$$\Delta(x, t, u, u_x, u_t, \ldots) = 0, \tag{1.2}$$

where

$$\Delta = u_t + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{xx}u_{xxx} + 147u_xu_{xxxx} + 42uu_{xxxxx} + u_{xxxxxxx}$$

in which $u = u(x, t)$ is the dependent variable, while x, t represent independent variables.

The Lie symmetry method seeks continuous transformations preserving the invariance of the governing differential equation under infinitesimal transformations of the form

$$x^* = x + \varepsilon\alpha(x, t, u) + O(\varepsilon^2), t^* = t + \varepsilon\beta(x, t, u) + O(\varepsilon^2),$$
(1.3)
$$u^* = u + \varepsilon\eta(x, t, u) + O(\varepsilon^2),$$

where ε is a small group parameter and α, β, η denote infinitesimal coefficients to be determined.

Let the infinitesimal generator be represented as

$$G = \alpha(x, t, u)\frac{\partial}{\partial x} + \beta(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u}. \tag{1.4}$$

To analyze higher-order equations, prolonged generators are required. Since the governing equation involves seventh-order derivatives, a seventh prolongation operator shall be constructed in subsequent sections.

Let the invariance condition be given as

$$Pr^{(7)}G(\Delta)|_{\Delta=0} = 0, \tag{1.5}$$

which forms the basis for deriving determining equations governing admitted symmetries.

Then the seventh prolongation of $G$is given by

$$Pr^{(7)}G = G + \sum_{|n|=1}^{7} \eta^n \frac{\partial}{\partial u_n}, \tag{1.6}$$

Where n denotes derivative multi-indices and η^n represent prolonged infinitesimals.

In particular, for derivatives with respect to x, the prolonged coefficients are generated subsequently through

$$\eta^x = D_x(\eta) - u_x D_x(\alpha) - u_t D_x(\beta), \eta^t = D_t(\eta) - u_x D_t(\alpha) - u_t D_t(\beta),$$

(1.7) Therefore, the seventh prolongation contains terms of the form $\eta^{xx}, \eta^{xxx}, \eta^{xxxx}, \eta^{xxxxx}, \eta^{xxxxxx}, \eta^{xxxxxxx}$, which are successively generated through repeated total differentiation.

Hence,

$$Pr^{(7)}G = G + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \cdots + \eta^{xxxxxxx}\frac{\partial}{\partial u_{xxxxxxx}}. \tag{1.8}$$

Applying the seventh prolongation operator to Equation (1. 1)yields

$$Pr^{(7)}G(\Delta) = Pr^{(7)}G(u_t + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{xx}u_{xxx} + 147u_xu_{xxxx} + 42uu_{xxxxx} + u_{xxxxxxx}) \tag{1.9}$$

Expanding term-by-term gives

$$Pr^{(7)}G(\Delta) = \eta^{t} + 2016\,Pr^{(7)}G(u^{3}u_{x}) + 630\,Pr^{(7)}G(u_{x}^{3}) + 2268\,Pr^{(7)}G(uu_{x}u_{xx})$$
$$+ 504\,Pr^{(7)}G(u^{2}u_{xxx}) + 252\,Pr^{(7)}G(u_{xx}u_{xxx}) + 147\,Pr^{(7)}G\left(u_{x}u_{xxxx}\right) + 42\,Pr^{(7)}G\left(uu_{xxxxx}\right) + \eta^{xxxxxxx}.$$
(2.0)

Repeating application of the product rule with subsequent prolongation formulas, a polynomial expression involving independent differential monomials in $u_{x}, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx}, u_{xxxxxx}, u_{xxxxxxx}$ is obtained. Equating coefficients of linearly independent derivative monomials to zero yields the determining equations governing

$$\alpha(x,t,u), \beta(x,t,u), \eta(x,t,u).$$

2.2 Derivation of the Determining System

In this subsection, we derive the determining equations governing the infinitesimal coefficients of the admitted Lie point symmetries of the seventh-order Kaup–Kuperschmidt equation. The derivation proceeds through explicit computation of prolonged infinitesimals, application of the seventh prolongation operator to the governing equation, and coefficient separation of independent differential monomials.

Consider the seventh-order Kaup–Kuperschmidt equation (1.1) and the infinitesimal generator (1.4), the infinitesimal condition (2.0) above reduces to

$$6048\eta u^{2}u_{x} + 2668\eta u_{x}u_{xx} + 1008\eta uu_{xxx} + 42\eta u_{xxxxx} + 2016\eta^{x}u^{3} + 1890\eta^{x}u_{x}^{2} + 2268\eta^{x}uu_{xx} + 147\eta^{x}u_{xxxx} + \eta^{t} + 2668\eta^{xx} \tag{2.1}$$

Where the terms η^{t}, η^{x}, η^{xx}, η^{xxx}, η^{xxxx}, η^{xxxxx} and $\eta^{xxxxxxx}$ are coefficients generated, determined and expressed in terms of partial derivatives.

We then substitute the generated coefficients into equation (2.1) and ensure that

$$u_{t} = -2016u^{3}u_{x} - 630u_{x}^{3} - 2268uu_{x}u_{xx} - 504u^{2}u_{xxx} - 252u_{xx}u_{xxx} - 147u_{x}u_{xxxx} - 42uu_{xxxxx} - u_{xxxxxxx}$$

whenever it appears in the equation then obtaining the determining equations from the monomials.

The solutions generated from the determining equations are

$$\alpha = -c_{1}x - c_{2} \tag{2.2}$$

$$\beta = -7c_{1}t - c_{3} \tag{2.3}$$

$$\eta = 2c_{1}u \tag{2.4}$$

Therefore, the corresponding infinitesimal or symmetry generators are as follows

$$G_{1} = -x\partial_{x} - 7t\partial_{t} + 2u\partial_{u} \tag{2.5}$$

$$G_{2} = -\partial_{x} \tag{2.6}$$

$$G_{3} = -\partial_{t} \tag{2.7}$$

2.3 Lie Groups

Lie Groups admitted by infinitesimal generators are generated by solving the Lie equations through exponentiation.

Therefore, we obtain the Lie group by solving the ODE system given as

$$\frac{d\bar{x}}{d\varepsilon} = \alpha, \quad \frac{d\bar{t}}{d\varepsilon} = \beta, \quad \frac{d\bar{u}}{d\varepsilon} = \eta \tag{2.8}$$

From G_1, we have

$$G_1 = - x\partial_x - 7t\partial_t + 2u\partial_u \tag{2.9}$$

Its corresponding ODE system is given by

$\frac{d\bar{x}}{d\varepsilon} = - \bar{x}\, \frac{d\bar{t}}{d\varepsilon} = - 7\bar{t}$

(3.0)$\frac{d\bar{u}}{d\varepsilon} = 2\bar{u}$

Solving gives

$$\bar{x} = xe^{-\varepsilon}\, \bar{t} = te^{-7\varepsilon} \tag{3.1}$$

$\bar{u} = ue^{2\varepsilon}$

Therefore, the first Lie Group is given as

$$(x, t, u) \rightarrow (xe^{-\varepsilon}, te^{-7\varepsilon}, ue^{2\varepsilon}) \tag{3.2}$$

Generator, G_2 given as

$$G_2 = - \partial_x \tag{3.3}$$

The corresponding ODE system

$$\frac{d\bar{x}}{d\varepsilon} = - 1\, \frac{d\bar{t}}{d\varepsilon} = 0 \tag{3.4}$$

$\frac{d\bar{u}}{d\varepsilon} = 0$

Solutions obtained were

$\bar{x} = x - \varepsilon\, \bar{t} = t$

(3.5)$\bar{u} = u$

The second Lie Group became

$$(x, t, u) \rightarrow (x - \varepsilon, t, u) \tag{3.6}$$

Which is a spatial translation

The third generator G_3

$$G_3 = - \partial_t \tag{3.7}$$

Has a corresponding ODE system as

$\frac{d\bar{t}}{d\varepsilon} =- 1 \frac{d\bar{x}}{d\varepsilon} = 0$ (3.8)

$\frac{d\bar{u}}{d\varepsilon} = 0$

Solving gave the solutions as follows

$\bar{t} = t - \varepsilon\, \bar{x} = x$ (3.9)

$\bar{u} = u$

This yielded the third Lie Group as

$(x, t, u) \rightarrow (x, t - \varepsilon, u)$ (4.0)

Which shows that it is a time translation

Therefore, the Lie groups obtained are

$R_1(\varepsilon): G(x,\ t,\ u:\ \varepsilon) \rightarrow G_1\left(xe^{-\varepsilon},\ te^{-7\varepsilon},\ ue^{2\varepsilon}\right)$ (4.1)

$R_2(\varepsilon): G(x,\ t,\ u:\ \varepsilon) \rightarrow G_2(x - \varepsilon, t, u)$ (4.2)

$R_3(\varepsilon): G(x,\ t,\ u:\ \varepsilon) \rightarrow G_3(x, t - \varepsilon, u)$ (4.3)

Where R_1 is a nontrivial group while R_2 and R_3 are trivial groups since they are scaling and translation respectively.

3. Similarity Reduction and Exact invariant solutions

Exact invariant solutions of equation (1.1) are obtained using the optimal system generators

$G_1 =- x\frac{\partial}{\partial x} - 7t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u},\ G_2 =- \frac{\partial}{\partial x},\ G_3 =- \frac{\partial}{\partial t}$

which transforms the governing PDE into a seventh-order nonlinear ordinary differential equation. Exact invariant solutions including constant and algebraic similarity solutions are obtained.

3.1. Invariant Solutions Associated with G_2

The generator

$G_2 =- \frac{\partial}{\partial x}$

gives the invariant surface condition

$- \frac{\partial u}{\partial x} = 0.$

Thus

$u = u(t).$ (4.4)

Substituting into the PDE, all spatial derivatives vanished and we obtained

$$u_x = u_{xx} = u_{xxx} = u_{xxxx} = u_{xxxxx} = u_{xxxxxxx} = 0. \quad (4.5)$$

Hence

$$u_t = 0. \quad (4.6)$$

Therefore

$$u = C \quad (4.7)$$

is an exact invariant solution.

3.2. Invariant Solutions Associated with G_3

The generator

$$G_3 = -\frac{\partial}{\partial t}$$

gives

$$u_t = 0. \quad (4.8)$$

Thus

$$u = u(x). \quad (4.9)$$

Substituting into the PDE gives the reduced seventh-order nonlinear ODE

$$2016u^3u' + 630(u')^3 + 2268uu'u'' + 504u^2u''' + 252u''u''' + 147u'u'''' + 42uu^{(5)} + u^{(7)} = 0.$$

(5.0)

Constant solution

Let

$$u = C.$$

Then

$$u' = u'' = 0 \quad (5.1)$$

and the equation is satisfied. Hence

$$u = C \quad (5.2)$$

is also a steady invariant solution.

3.3. Scaling Invariant Solutions from G_1

Consider

$G_1 = -x\partial_x - 7t\partial_t + 2u\partial_u$.

Characteristic equations

$$\frac{dx}{-x} = \frac{dt}{-7t} = \frac{du}{2u}. \quad (5.3)$$

Taking the first two fractions

$$\frac{dx}{x} = \frac{dt}{7t} \quad (5.4)$$

Integrating gives

$$\xi = \frac{x}{t^{1/7}}. \quad (5.5)$$

The last two fractions

$$\frac{du}{2u} = \frac{dx}{-x}. \quad (5.6)$$

Integrating gives

$$u = t^{-2/7}F(\xi). \quad (5.7)$$

Similarity Transformation, solutions become

$$u(x,t) = t^{-2/7}F(\xi), \xi = \frac{x}{t^{1/7}} \quad (5.8)$$

3.4. Reduced Ordinary Differential Equation

Substituting the similarity form into the PDE gives the reduced ODE

$$-\frac{2}{7}F - \frac{1}{7}\xi F' + 2016F^3F' + 630(F')^3 + 2268FF'F'' + 504F^2F''' + 252F''F''' + 147F'F'''' + 42FF^{(5)} + F^{(7)} = 0. \quad (5.9)$$

3.5. Exact Invariant Solutions

(i) Constant solution

$$F = A \quad (6.0)$$

gives

$$-\frac{2}{7}A = 0 \quad (6.1)$$

thus

$$A = 0 \quad (6.2)$$

Hence

$u = 0$ (6.3)

(ii) Algebraic invariant solution

A spatially self-similar form

$$u(x,t) = \frac{k}{x^2} \quad (6.4)$$

satisfies the scaling symmetry and gives another invariant structure.

Therefore, exact solutions are

$$u = C, u = 0, u = \frac{k}{x^2}.$$

Using the optimal system of generators, invariant solutions of the nonlinear seventh-order evolution equation are constructed. The translation symmetries yield constant solutions, while the scaling symmetry leads to a similarity reduction

$$u(x,t) = t^{-\frac{2}{7}} F\left(\frac{x}{t^{\frac{1}{7}}}\right). \quad (6.5)$$

4. Exact power-series invariant solutions for this PDE

We then find the exact power-series invariant solutions of the nonlinear PDE (1.1) by constructing invariant (similarity) power series solutions using the symmetry:

$$G = \partial_t + c\partial_x \quad (6.6)$$

Similarity variable:

$$\xi = x - ct, u(x,t) = F(\xi). \quad (6.7)$$

Derivatives:

$$u_t = -cF', u_x = F', u_{xx} = F'', ..., u^{(7)} = F^{(7)}. \quad (6.8)$$

Upon substituting into the PDE we obtain an invariant ODE given as

$$-cF' + 2016F^3F' + 630(F')^3 + 2268FF'F'' + 504F^2F''' + 252F''F''' + 147F'F^{(4)} + 42FF^{(5)} + F^{(7)} = 0. \quad (6.9)$$

Suppose analytic invariant solution:

$$F(\xi) = \sum_{n=0}^{\infty} b_n \xi^n. \quad (7.0)$$

Derivatives:

$$F' = \sum_{n=1}^{\infty} nb_n\xi^{n-1}, F'' = \sum_{n=2}^{\infty} n(n-1)b_n\xi^{n-2}, ..., F^{(7)} = \sum_{n=7}^{\infty} n(n-1)\cdots(n-6)b_n\xi^{n-7}. \quad (7.1)$$

We define the nonlinear convolution sums as:

Cubic nonlinearity given as

$$F^3 = \sum_{n=0}^{\infty} B_n \xi^n, B_n = \sum_{i+j+k=n} b_i b_j b_k. \tag{7.2}$$

Product terms stated as

$$FF'F'' \Rightarrow triple\ convolution(cubic\ derivative\ convolution$$

All nonlinearities reduce to polynomial convolution sums.

After substituting series into the ODE and aligning powers of ξ^n, we obtain

$$(n+7)!\, b_{n+7} = c(n+1)!\, b_{n+1} - 2016\, A_n - 630\, C_n - 2268\, D_n - 504\, X_n - 252\, H_n - 147\, I_n - 42\, J_n$$

where convolution coefficients are given as:

Cubic term

$$A_n = \sum_{i+j+k=n} b_i b_j b_k (k+1) b_{k+1} \tag{7.4}$$

(from $F^3 F'$)

Derivative cubic

$$C_n = \sum_{i+j+k=n} (i+1) b_{i+1} (j+1) b_{j+1} (k+1) b_{k+1} \tag{7.5}$$

Mixed nonlinear terms

$$D_n = \sum_{i+j+k=n} b_i (j+1) b_{j+1} (k+2)(k+1) b_{k+2} \tag{7.6}$$

Since this is the 7th order ODE, the free parameters $b_0, b_1, b_2, b_3, b_4, b_5, b_6$ are arbitrary constants.

We then generate the first few exact coefficients

When $n = 0$

From constant term:

$$5040 b_7 = c b_1 - 2016 b_0^3 b_1 - 630 b_1^3 - 2268 b_0 b_1 b_2 - 504 b_0^2 b_3 - 252 b_2 b_3 - 147 b_1 b_4 - 42 b_0 b_5. \tag{7.7}$$

So we have

$$b_7 = \frac{1}{5040}\left[42 b_0 b_5\right].$$

When $n = 1$ we get

$$40320 b_8 = c(2b_2) - nonlinear\ convolutions. \tag{7.9}$$

Determination of even invariant solutions follow by considering symmetric solitary waves

$b_1 = b_3 = b_5 = 0$. The invariant series becomes

$$F(\xi) = b_0 + b_2\xi^2 + b_4\xi^4 + b_6\xi^6 + b_8\xi^8 + \cdots \quad (8.0)$$

So the first nontrivial coefficient is

$$b_7 = 0 \quad (8.1)$$

due to symmetry.

The next even coefficient is given as

$$b_8 = \frac{cb_2 - 504b_0^2b_4 - 252b_2b_4 - 42b_0b_6}{40320}. \quad (8.2)$$

The Exact Power Series Invariant Solution becomes

$$u(x,t) = \sum_{n=0}^{\infty} b_n(x - ct)^n \quad (8.3)$$

with recurrence

$$b_{n+7} = \frac{1}{(n+7)!}[c(n+1)!\,b_{n+1} - nonlinear\ convolution\ terms].$$

This is the exact invariant analytic solution of the seventh-order Kaup–Kuperschmidt equation.

5. Results and Discussion

Symmetry transformations give new solutions from known solutions. We consider group transformations that arise from the infinitesimal generators

G_2 yields an invariant solution given as $u = C$, G_3 yields an invariant solution as $u = C$ and
G_1 gives invariant solution as $u(x,t) = t^{-2/7}F(x/t^{1/7})$

Special exact results are given as

$u = C, u = 0, u = \frac{k}{x^2}$. Using the optimal system of generators, invariant solutions of the nonlinear seventh-order evolution equation were constructed. The translation symmetries yield constant solutions, while the scaling symmetry leads to a similarity reduction

$$u(x,t) = t^{-2/7}F(x/t^{1/7}),$$

which transforms the governing PDE (1.1) into a seventh-order nonlinear ordinary differential equation. Exact invariant solutions including constant and algebraic similarity solutions are obtained.

The Exact Power Series Invariant Solution of (1.1) is given as

$$u(x,t) = \sum_{n=0}^{\infty} b_n(x - ct)^n$$

with recurrence

$$b_{n+7} = \frac{1}{(n+7)!}\left[c(n+1)!\, b_{n+1} - nonlinear\ convolution\ terms\right].$$

Using the travelling-wave symmetry reduction, the nonlinear seventh-order PDE reduces to an invariant ordinary differential equation. A power-series expansion about the wave center yields an exact analytic invariant solution with seven free parameters. Higher-order coefficients are determined recursively through nonlinear convolution relations arising from cubic and mixed derivative interactions.

6. Conclusions

In this article, the infinitesimal generators and Lie groups of the seventh-order Kaup-Kuperschmidt equation have been acquired by means of Lie symmetry analysis method. Also, deliberation on the group-invariant solutions and the exact analytic solutions to the equation through the power series technique has been done. The analytic symmetry solutions demonstrate that Lie symmetry analysis technique is a candid and best mathematical tool to obtain analytical and exact results of highly nonlinear PDE's. Furthermore, this technique can be more efficiently used to study other nonlinear PDE's, which are often used in mathematical fields, engineering fields, and physical sciences.

Conflicts of Interest

There are no conflicts of interest

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Author Biography

Winny Chepngetich is an academic and researcher in the field of Applied Mathematics, with particular interest in nonlinear differential equations, mathematical modelling, and symmetry analysis. She is pursuing doctoral research in Applied Mathematics at the University of Kabianga, with her research focusing on the Lie symmetry analysis of seventh-order Korteweg–de Vries (KdV) equations.

Her research interests include Lie group analysis, nonlinear evolution equations, soliton theory, symmetry reductions, exact solutions, travelling-wave solutions, and mathematical techniques for analysing nonlinear partial differential equations. Her work explores the application of Lie symmetry methods to higher-order nonlinear KdV-type equations, including equations such as the Kawahara, Kaup–Kupershmidt, and related seventh-order nonlinear evolution equations.

In addition to research, Winny is actively involved in mathematics education, curriculum development, competency-based education, and university teaching. She has contributed to the development and review of mathematics and statistics curricula, including the design of competency-based course outlines, learning outcomes, teaching activities, assessment strategies, and instructional materials.

Prof. Maurice Owino Oduor is the Deputy Vice Chancellor (Planning, Research and Development) of the University of Kabianga- Kenya. He is a champion for Research, Innovation and Commercialization at the University. Prior to his appointment to the DVC position, he served as the Dean, School of Science and Technology and HoD, Mathematics and Computer Science. He holds a Ph.D in Pure Mathematics from Maseno University- Kenya and had postdoctoral research stint at Stellenbosch University (A.I.M.S) - South Africa. He has 20 years of teaching, research and administrative experience at the University level. The Professor of Pure Mathematics is an accomplished scholar who has published widely and has mentored many undergraduate and postgraduate students. His research focuses on classification of Algebraic Rings. He is a member of many Professional Associations, an External Examiner for a number of Universities and a reviewer for a number of academic Journals. He has served as a board member in many learning institutions. He is a trustee for the University of Kabianga Pension Scheme.

Dr. Joshua K. Limo is a University Lecturer at Laikipia University- Kenya. Has a earned Doctor of Philosophy in Mathematics from University of Eldoret. Has over 14 years work experience Lecturing Mathematics courses at Laikipia University. Over 4 years experience Teaching Mathematics and Physics at Secondary School level. Demonstrates capacity to manage people in terms of training, development, coaching and mentoring and supervising Postgraduate students.

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REF: PHD/AM/0002/2021 **DATE: 3RD MARCH, 2026**

Winny Chepngetich,
MSAS Department,
University of Kabianga,
P.O Box 2030- 20200,
KERICHO.

Dear Ms. Chepngetich,

RE: CLEARANCE TO COMMENCE FIELD WORK/DATA COLLECTION

I am pleased to inform you that the Board of Graduate Studies has considered and approved your PhD Research proposal entitled **"On the Lie Symmetry Analysis of Seventh Order Korteweg-Devries Equations".** Subsequently the Board has also approved the following supervisors for appointments.

1. Prof. Maurice O. Oduor, PhD
2. Dr. Joshua K. Limo, PhD

You may now proceed to commence field work/data collection on condition that you obtain a research permit from NACOSTI and /or an ethical review permit from a relevant ethics review board.

You are also required to publish two (2) articles in a peer reviewed journal, with all your supervisors, before your oral defense of thesis and to submit through your supervisors, and HoD, progress reports every three months, to the Director, Board of Graduate Studies.

Please note that it is the policy of the University that you complete your studies within three years from the date of registration. Do not hesitate to consult this office in case of any difficulties encountered in the course of your studies.

I wish you all the best in your research and hope that your study will yield original contribution for the betterment of humanity.

Yours Sincerely,

UNIVERSITY OF KABIANGA
03 MAR 2026
BOARD OF GRADUATE STUDIES

Dr. Ronald K. Rop
DIRECTOR, BOARD OF GRADUATE STUDIES.
RKR/hk

Cc 1. Dean, SST
2. HOD, Mathematics, Statistics & Actuarial Sciences
3. Supervisors