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THE FEKETE -SZEGÖ PROBLEM OF ANALYTIC FUNCTIONS BASED ON THE DEFERENtIAL OPERTOR AND CERTAIN SUBCLASSES

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Abstract In this paper I use the operators $D^m_{\beta,\eta}(\eta,\phi)f$, subclass $\mathbb{S}_m\big(d,\beta,\eta\big)$ and $\mathbb{C}_m\big(d,\beta,\eta\big)$.To establish $\Big|a_3-\mu a_2^2\Big|$-functional inequalities for the Fekete-Szegö problem. That's my main result.

Keywords: Fekete-Szegö problem, analytic functions, The linear multiplier differential operator $D^m_{\beta,\eta}(\eta,\phi)f$, subclass $\mathbb{S}_m\big(d,\beta,\eta\big),\mathbb{C}_m\big(d,\beta,\eta\big)$. **Mathematics Subject Classification:** 30c45

Introduction

1 Introducton

Suppose that $\mathbb{M}$ denote the class of all analytic $f\big(z\big)$ in the open unit disk

$$\Omega := \Big\{z\in\mathbb{C} : \big|z\big| < 1\Big\}.$$

With

$$\mathbb{M} := \Big\{g\Big|g\big(z\big) := z+\sum_{n=2}^{\infty}a_nz^n, g:\Omega\to\mathbb{C}\Big\}$$

Also, let $\mathbb{S}\subset\mathbb{M}$ consisting of function which are univalent inside Ω.
A typical problem in geometric function theory is to study a functional made up of com-binations of the coefficients of the original function. Usually, there is a parameter over which the extremal value of the functional is needed. The paper deals with one important functional of this type: the Fekete-Szegö functional. The classical Fekete-Szegö functional is defined by

$$\triangle_\mu\big(f\big) = \Big|a_3-\mu a_2^2\Big|, (0<\mu<1)$$

and it is derived from the Fekete-Szegö inequality. The problem of maximizing the abso-lute value of the functional in subclasses of normalized functions is called the Fekete-Szegö problem. In 1933, Fekete and Szegö [9] found the maximum value of $\Big|a_3-\mu a_2^2\Big|$

as a function of the real parameters μ, for functions belonging to the class $\mathbb{S}$. Since then, several researchers solved the a classial Theorem Fekete-Szegö[9] states that for $f\in\mathbb{S}$ give by

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 3-4\mu & if \quad \mu \leq 0, \\ \\ 1+2exp\Big(\frac{-2\mu}{1-\mu}\Big) & if \quad 0<\mu<1. \\ \\ 4\mu-3 & if \quad \mu \geq 1. \end{cases}$$

Later, Pfluger [3] has considered the complex values of μ and provided, were able to bound the classical functional in the class $\mathbb{S}$ by

$$\left|a_3-\mu a_2^2\right| \leq 1+2exp\{\frac{-2\mu}{1-\mu}\}$$

. Up to this time, several authors have attempted to extend the above inequality to more general classes of analytic functions.

+ A function $f \in \mathbb{M}$ is said to be in the class $\mathbb{S}^*\big(\alpha\big)$ of starlike functions of order α in Ω and is determined

$$\mathbb{S}^*\big(\alpha\big) := \Big\{g \in \mathbb{M}\Big|\mathcal{R}e\Big(\frac{zg'\big(z\big)}{g\big(z\big)}\Big) > \alpha, z \in \Omega, 0 \leq \alpha < 1\Big\}$$

+A function $f \in \mathbb{M}$ is said to be in the class of convex functions of order α in Ω, denoted by $\mathbb{C}(\alpha)$ and is determined

$$\mathbb{C}\big(\alpha\big) := \Big\{g \in \mathbb{M}\Big|\mathcal{R}e\Big(1+\frac{zg''\big(z\big)}{g'\big(z\big)}\Big) > \alpha, z \in \Omega, 0 \leq \alpha < 1\Big\}$$

A notions of α-starlikeness and α-convexity were generalized onto a complex order α see [14],[15],[16].
In particular, the classes $\mathbb{S}^* = \mathbb{S}^*(0)$ and $\mathbb{C} = \mathbb{C}(0)$ are the familiar classes of starlike and convex functions in Ω, respectively.

The paper is organized as follows:
In section preliminaries I remind some basic notations in [3],[4],[7],[8],[9],[10],[14],[15],[16] such as The linear multiplier differential operator $D^m_{\beta,\eta}(\eta,\phi)f$, subclass $\mathbb{S}_m\big(d,\beta,\eta\big)$ and $\mathbb{C}_m\big(d,\beta,\eta\big)$.
Section 3: Stability $\Big|a_3-\mu a_2\Big|$-functional inequalities for d nonzero complex number, $\mu \in \mathbb{C}$ and $f \in \mathbb{S}_m\big(d,\beta,\eta\big)$.
Section 4: Stability $\Big|a_3-\mu a_2\Big|$-functional inequalities when μ, d are real and $f \in \mathbb{S}_m\big(d,\beta,\eta\big)$.

Section 5: Stability $\Big|a_3-\mu a_2\Big|$-functional inequalities when d is a nonzero complex number, $\mu \in \mathbb{R}$ and $f \in \mathbb{S}_m\big(d,\beta,\eta\big)$.

Section 6: Stability $\Big|a_3-\mu a_2\Big|$-functional inequalities when d is a nonzero complex number, $\mu \in \mathbb{C}$ and $f \in \mathbb{C}_m\big(d,\beta,\eta\big)$.

This section should be succinct, with no subheadings.

2 preliminaries

Definition 2.1. Suppose that

$$D^m_{\beta,\eta}f : \mathbb{M} \to \mathbb{M}$$

Then the linear multiplier differential operator $D^m_{\beta,\eta}f$ was defined as follows:
$+D^0_{\beta,\eta}f(z) = f(z)$,

$+D^1_{\beta,\eta}f(z) = D_{\beta,\eta}f(z) = \beta\eta z^2\Big(f(z)\Big)'' + (\beta-\eta)z\Big(f(z)\Big)' + \big(1-\beta+\eta\big)zf(z),$
$+D^2_{\beta,\eta}f(z) = D_{\beta,\eta}\Big(D^1_{\beta,\eta}f(z)\Big),$
.
.
.
$+D^m_{\beta,\eta}f(z) = D_{\beta,\eta}\Big(D^{m-1}_{\beta,\eta}f(z)\Big)$, In there $\beta \geq \eta \geq 0$ and $m \in \mathbb{N}$.

From the definition I lead to consequence

Corollary 2.2. *If $f \in \mathbb{M}$ then the linear multiplier differential operator $D^m(\eta,\phi)f$ identified as*

$$D^m_{\beta,\eta}f(z) = z + \sum_{n=2}^{\infty}[1 + \big(\beta\eta n + \beta - \eta\big)\big(n-1\big)]^m a_n z^n.$$

Definition 2.3. Suppose d be a nonzero complex number, $g \in \mathbb{M}$ and $D^m_{\beta,\eta}g(z) \neq 0$ I define a subclasses as follows:

$$\mathbb{S}_m\big(d,\beta,\eta\big) :=$$
$$\left\{g \in \mathbb{M}\Big|\mathcal{R}e\left(1 + \frac{1}{d}\left(\frac{z\Big(D^m_{\beta,\eta}g(z)\Big)'}{D^m_{\beta,\eta}g(z)} - 1\right)\right) > 0, 0 \leq \eta \leq \beta, m \in \mathbb{N}, z \in \Omega \setminus \{0\}\right\}$$

Definition 2.4. Suppose d be a nonzero complex number, $g \in \mathbb{M}$ and $\Big(D^m_{\beta,\eta}g(z)\Big)' \neq 0$ I define a subclasses as follows:

$$\mathbb{C}_m\big(d,\beta,\eta\big) :=$$
$$\left\{g \in \mathbb{M}\Big|\mathcal{R}e\left(1 + \frac{1}{d}\left(\frac{z\Big(D^m_{\beta,\eta}g(z)\Big)''}{\Big(D^m_{\beta,\eta}g(z)\Big)'} - 1\right)\right) > 0, 0 \leq \eta \leq \beta, m \in \mathbb{N}, z \in \Omega \setminus \{0\}\right\}$$

Let $\mathbb{P}$ be the class of all analytic functions

$$\mathbb{P} := \left\{q(z)|q(z) = 1 + c_1z + c_2z^2 + \cdots, z \in \Omega, Req(z) > 0\right\}$$

Lemma 2.5. *If*

$$q(z) = 1 + c_1z + c_2z^2 + \cdots, \quad \forall q(z) \in \mathbb{P}. \tag{2.1}$$

Then
i)$\Big|c_j\Big| \leq 2, j \geq 1,$
ii)$\Big|c_2 - \frac{c_1^2}{2}\Big| \leq 2 - \frac{\big|c_1\big|^2}{2}.$

In this paper I assume that $k \in \mathbb{N}^*$.

3 Stability $\Big|a_3 - \mu a_2\Big|$-functional inequalities for d nonzero complex number, $\mu \in \mathbb{C}$ and $f \in \mathbb{S}_m\big(d,\beta,\eta\big)$

Theorem 3.1. *Suppose that d be a nonzero complex number and $\mu \in \mathbb{C}$, $0 \leq \eta \leq \beta$. If f of the form:*

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{3.1}$$

is in $\mathbb{S}_m\big(d,\beta,\eta\big)$, *then*

$$\Big|a_2\Big| \leq \frac{2k\big|d\big|}{(2k-1)A^m} \tag{3.2}$$

$$\Big|a_3\Big| \leq \frac{k\big|d\big|}{(3k-1)B^m} max\Big\{1, \Big|1+2kd\Big|\Big\} \tag{3.3}$$

$$\Big|a_3-\mu a_2\Big| \leq \frac{2k\big|d\big|}{(3k-1)B^m} max\Big\{1, \Big|1+2kd-4kd\mu\frac{B^m}{(2k-1)^2A^{2m}}\Big|\Big\} \tag{3.4}$$

In there $A := \Big[1+(2k-1)\big(2k\eta+\beta-\eta\big)\Big]$ *and* $B := \Big[1+(3k-1)\big(3k\beta\eta+\beta-\eta\big)\Big]$. *Consider the functions*

$$\frac{kz\Big(D^m_{\beta,\eta}f\big(z\big)\Big)'}{D^m_{\beta,\eta}} = k+kd\Big[q_1(z)-1\Big] \tag{3.5}$$

$$\frac{kz\Big(D^m_{\beta,\eta}f\big(z\big)\Big)'}{D^m_{\beta,\eta}} = k+kd\Big[q_2(z)-1\Big] \tag{3.6}$$

In there q_1, q_2 *are given in Lemma1. Equality*
in (3.1) *holds if* (3.5)
in (3.2) *holds if* (3.5)*and* (3.6) *for each* μ *in* (3.3) *if* (3.5) *and if* (3.6)

Proof. We put

$$D^m_{\beta,\eta}f\big(z\big) = z+\lambda_2z^2+\lambda_3z^3+\cdots, \tag{3.7}$$

then

$$\lambda_2 = A^m a_2, \lambda_3 = B^m a_3 \tag{3.8}$$

So I have

$$\left(\frac{kz\Big(1+2\lambda_2z+3\lambda_3z^2+\cdots\Big)}{z+\lambda_2z^2+\lambda_3z^3+\cdots}\right) = k-kd+kd\Big(1+c_1z+c_2z^2+\cdots\Big) \tag{3.9}$$

which implies the equality

$$kz+2k\lambda_2z^2+3k\lambda_3z^2+\cdots = kz+\Big(kdc_1+\lambda_2\Big)z^2+\Big(kdc_2+k\lambda_2dc_1+\lambda_3\Big)z^3+\cdots$$

Equating the coefficients of both sides we get

$$\lambda_2 = \frac{kdc_1}{2k-1}, \lambda_3 = \frac{kd^2c_1^2}{3k-1}+\frac{kdc_2}{3k-1} \tag{3.10}$$

Therefore, according to (3.8) and (3.10), I have

$$a_2 = \frac{kdc_1}{(2k-1)A^m}, a_3 = \frac{kd}{(3k-1)B^m}(dc_1^2 + c_2). \tag{3.11}$$

From (3.8) and Lemma 1, I get

$$\left|a_2\right| = \left|\frac{kdc_1}{(2k-1)A^m}\right| \leq \frac{2k\left|d\right|}{(2k-1)A^m} \tag{3.12}$$

and

$$\begin{aligned}
\left|a_3\right| &= \left|\frac{kd}{(3k-1)B^m}\right|\left|\left[c_2 - \frac{c_1^2}{2} + \frac{1+2kd}{2}c_1^2\right]\right| \\
&\leq \left|\frac{kd}{(3k-1)B^m}\right|\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{1+2kd}{2}\left|c_1^2\right|\right] \\
&= \left|\frac{kd}{2(3k-1)B^m}\right|\left[1 + \frac{\left|1+2kd\right|-1}{4}\left|c_1^2\right|\right] \\
&\leq \frac{k\left|d\right|}{(3k-1)B^m} max\left\{1, \left[1 + \left|1+2kd\right| - 1\right]\right\}
\end{aligned} \tag{3.13}$$

Thus, I have

$$\left|a_3\right| \leq \frac{\left|kd\right|}{(3k-1)B^m} max\left\{1, \left|1+2kd\right|\right\}$$

$$\begin{aligned}
\left|a_3 - \mu a_2^2\right| &= \left|\frac{kd}{(3k-1)B^m}\left(dc_1^2 + c_2\right) - \mu\frac{4k^2d^2}{(2k-1)^2A^{2m}}c_1^2\right| \\
&\leq \left|\frac{kd}{(3k-1)B^m}\left(\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left|1+2kd - \mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}}\right|\right)\right. \\
&\leq \left|\frac{kd}{(3k-1)B^m}\right|\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left|1+2kd - \mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}}\right|\right] \\
&= \left|\frac{2kd}{(3k-1)B^m}\right|\left[1 + \frac{\left|c_1^2\right|}{4}\left(\left|1+2kd - \mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}}\right| - 1\right)\right] \\
&\leq \frac{2k\left|d\right|}{(3k-1)B^m} max\left\{1, \left|1+2kd - \mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}}\right|\right\}
\end{aligned} \tag{3.14}$$

I now obtain sharpness of the estimates in (3.1), (3.2) and (3.3). Firstly, in (3.1) the equality holds if $c_1 = 2$. Equivalently, I have $p(z) = p_1(z) = \left(1+z\right)/\left(1-z\right)$. Therefore, the extremal function in $\mathbb{S}_m\left(d, \beta, \eta\right)$ isgiven by

$$\frac{kz\left(D^m_{\beta,\eta}f\left(z\right)\right)'}{D^m_{\beta,\eta}} = \frac{1+\left(2kd-1\right)z}{1-z} \tag{3.15}$$

Next, in (3.2), for first case, the equality holds if $c_1 = c_2 = 2$. Therefore,the extremal functions in $\mathbb{S}_m\left(d, \beta, \eta\right)$ is given by (3.15) and for second case,the equality holds if $c_1 = 0, c_2 = 2$. Equivalently, I have $p(z) = p_2(z) = \left(1+z^2\right)/\left(1-z^2\right)$.Therefore, the extremal function in $\mathbb{S}_m\left(d, \beta, \eta\right)$ is given by

$$\frac{kz\left(D^m_{\beta,\eta}f\left(z\right)\right)'}{D^m_{\beta,\eta}} = \frac{1+\left(2kd-1\right)z^2}{1-z^2} \tag{3.16}$$

Finally, in (3.3), the equality holds. Obtained extremal function for (3.2) is also valid for (3.3). Thus, the proof of Theorem 1 is completed.

$\square$

4 Stability $\left|a_3 - \mu a_2\right|$-functional inequalities when μ, d are real and $f \in \mathbb{S}_m\big(d, \beta, \eta\big)$

Theorem 4.1. *Suppose that $d > 0$ function $f \in \mathbb{S}_m\big(d, \beta, \eta\big)$. Then for $\mu \in \mathbb{R}$ I get*

$$\left|a_3 - \mu a_2\right| \leq \begin{cases} \frac{2kd}{B^m}\Big\{1 + 2kd\Big[1 - \frac{2kd\mu B^m}{(2k-1)^2 A^{2m}}\Big]\Big\} & if \quad \mu \leq \frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \\ \frac{d}{B^m} & if \quad \frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \leq \mu \leq \frac{(1+2kd)A^{2m}}{4kdB^m}, \\ \\ \frac{d}{B^m}\left[\frac{4kd\mu B^m}{A^{2m}} - 2kd - 1\right] & if \quad \mu \geq \frac{(1+2kd)A^{2m}}{4kdB^m} \end{cases} \tag{4.1}$$

In there $A := \Big[1 + (2k-1)\big(2k\eta + \beta - \eta\big)\Big]$ and $B := \Big[1 + (3k-1)\big(3k\beta\eta + \beta - \eta\big)\Big]$. For each μ, the equality holds for functions in (3.5) and (3.6)

Proof. Suppose that $\mu \leq \frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \leq \frac{(1+2kd)A^{2m}}{4kdB^m}$ then from (3.11) and Lemma 1 I get

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{kd}{(3k-1)B^m}\left[\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left(1 + 2kd - \mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}}\right)\right] \\ &\leq \frac{kd}{(3k-1)B^m}\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left(1 + 2kd - \mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}}\right)\right] \\ &= \frac{2kd}{(3k-1)B^m}\left[1 + 2kd\left(1 - \mu\frac{2kdB^m}{(2k-1)^2 A^{2m}}\right)\right] \\ &\leq \frac{2kd}{B^m}\left[1 + 2kd\left(1 - \mu\frac{2kdB^m}{(2k-1)^2 A^{2m}}\right)\right] \end{aligned} \tag{4.2}$$

Next, I consider $\frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \leq \mu \leq \frac{(1+2kd)A^{2m}}{4kdB^m}$. Then, using the above calculations, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2kd}{B^m}$$

Finally, I consider $\mu \geq \frac{(1+2kd)A^{2m}}{4kdB^m}$

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{kd}{(3k-1)B^m}\left[\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left(\mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}} - 1 - 2kd\right)\right] \\ &\leq \frac{kd}{(3k-1)B^m}\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left(\mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}} - 1 - 2kd\right)\right] \\ &= \frac{2kd}{(3k-1)B^m}\left[1 + 2kd\left(1 - \mu\frac{2kdB^m}{(2k-1)^2 A^{2m}}\right)\right] \\ &\leq \frac{2kd}{B^m}\left[\mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}} - 2kd - 1\right)\right] \end{aligned} \tag{4.3}$$

$\square$

Thus, the Theorem has been proved.

5 Stability $\left|a_3 - \mu a_2\right|$-functional inequalities when d is a nonzero complex number, $\mu \in \mathbb{R}$ and $f \in \mathbb{S}_m\big(d, \beta, \eta\big)$

Theorem 5.1. *Suppose that d be a nonzero complex number and for $f \in \mathbb{S}_m\big(d,\beta,\eta\big)$. Then for $\mu \in \mathbb{R}$ I get*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{4k^2\left|d\right|^2}{(2k-1)^2A^{2m}}\Big[H(t_1) - \mu\Big] + \frac{k\left|d\right|\left|sin\theta\right|}{(3k-1)B^m} & if \quad \mu \leq H_1 \\ \frac{2k\left|d\right|}{B^m} & if \quad H_1 \leq \mu \leq L_1 \\ \\ \frac{4k^2\left|d\right|^2}{(2k-1)^2A^{2m}}\Big[H(t_1) - \mu\Big] + \frac{k\left|d\right|\left|sin\theta\right|}{(3k-1)B^m} & if \quad \mu \geq L_1 \end{cases} \tag{5.1}$$

In there $A := \Big[1 + (2k-1)\big(2k\eta + \beta - \eta\big)\Big]$ and $B := \Big[1 + (3k-1)\big(3k\beta\eta + \beta - \eta\big)\Big]$, $k\left|d\right| = kde^{i\theta}$, $t_1 = \frac{(2k-1)^2A^{2m}}{(3k-1)B^m} + \frac{(2k-1)^2A^{2m}e^{i\theta}}{4k\left|d\right|B^m}$, $l_1 = \frac{(2k-1)^2A^{2m}}{4k\left|d\right|B^m}$ $H_1 = H(t_1) - l_1(1 - \left|sin\theta\right|)$ and $L_1 = H(t_1) + l_1(1 - \left|sin\theta\right|)$ For each μ, there is a functin in $f \in \mathbb{S}_m\big(d,\beta,\eta\big)$ such that the equality holds. the equality holds for functions in (3.5) *and* (3.6)

Proof. From (3.14), I have

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &= \left|\frac{kd}{(3k-1)B^m}\Big(kdc_1^2 + c_2\Big) - \mu\frac{k^2d^2}{(2k-1)^2A^{2m}}c_1^2\right| \\ &\leq \frac{k\left|d\right|}{(3k-1)B^m}\left(\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left|1 + 2kd - \mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}}\right|\right) \\ &\leq \frac{k\left|d\right|}{(3k-1)B^m}\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left|1 + 2kd - \mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}}\right|\right] \\ &= \frac{k\left|d\right|}{(3k-1)B^m}\left[\frac{\left|c_1^2\right|}{2}\left(\left|1 + 2kd - \mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}}\right| - 1\right) + 2\right] \\ &= \frac{2k\left|d\right|}{B^m} + \frac{k\left|d\right|}{(3k-1)^2B^m}\left[\left|\mu\frac{4k^2dB^m}{(2k-1)^2A^{2m}} - 2kd - 1\right| - 1\right]\left|c_1^2\right| \\ &= \frac{2k\left|d\right|}{B^m} + \frac{k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left[\left|\mu - \frac{(2k-1)^2A^{2m}}{(3k-1)B^m} - \frac{(2k-1)^2A^{2m}}{4kdB^m}\right| - \frac{(2k-1)^2A^{2m}}{4k\left|d\right|B^m}\right]\left|c_1^2\right| \end{aligned} \tag{5.2}$$

If $k\left|d\right| = kde^{i\theta}$, $t_1 = \frac{(2k-1)^2A^{2m}}{(3k-1)B^m} + \frac{(2k-1)^2A^{2m}e^{i\theta}}{4k\left|d\right|B^m}$, $l_1 = \frac{(2k-1)^2A^{2m}}{4k\left|d\right|B^m}$
in last equation, I have

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{2k\left|d\right|}{B^m} + \frac{k\left|d\right|}{(3k-1)^2B^m}\left[\left|\mu - t_1\right| - l_1\right]\left|c_1^2\right| \\ &\leq \frac{2k\left|d\right|}{B^m} + \frac{k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left[\left|\mu - H(t_1)\right| + l_1\left|sin\theta\right| - l_1\right]\left|c_1^2\right| \\ &= \frac{2k\left|d\right|}{B^m} + \frac{k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left[\left|\mu - H(t_1)\right| + \Big(1 - \left|sin\theta\right|\Big)l_1\right]\left|c_1^2\right| \end{aligned} \tag{5.3}$$

Next, I consider case (5.3) if $\mu \leq H(t_1)$

$$\begin{aligned}
&\left|a_3-\mu a_2^2\right| \\
&\leq \frac{2k\left|d\right|}{B^m}+\frac{k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left[H(t_1)-l_1\left(1-\left|sin\theta\right|\right)-\mu\right]\left|c_1^2\right| \\
&=\frac{2k\left|d\right|}{B^m}+\frac{k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left[H_1-\mu\right]\left|c_1^2\right| \qquad (5.4)
\end{aligned}$$

Next, I consider $\mu \leq H_1 = H(t_1)-l_1\left(1-\left|sin\theta\right|\right)$. According to Lemma 1 and $l_1 = \frac{(2k-1)^2A^{2k}}{2k\left|d\right|B^m}$ From (5.4), we get

$$\begin{aligned}
&\left|a_3-\mu a_2^2\right| \\
&\leq \frac{2k\left|d\right|}{B^m}+\frac{4k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left(H(t_1)-\mu\right)-\frac{4k^2\left|d\right|^2}{(2k-1)^2A^{2m}}\cdot\frac{(2k-1)^2A^{2m}}{2k\left|d\right|B^m}\left(1-\left|sin\theta\right|\right) \\
&=\frac{2k\left|d\right|}{B^m}+\frac{4k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left(H(t_1)-\mu\right)-\frac{2k\left|d\right|}{B^m}\left(1-\left|sin\theta\right|\right) \\
&=\frac{4k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left(H(t_1)-\mu\right)-\frac{2k\left|d\right|}{B^m}\left|sin\theta\right| \qquad (5.5)
\end{aligned}$$

Next, I put $H_1 = H(t_1)-l_1(1-\left|sin\theta\right|) \leq \mu \leq H(t_1)$. So then in (5.4) I have

$$\left|a_3-\mu a_2^2\right| \leq \frac{2k\left|d\right|}{B^m}$$

Next suppose $\mu \geq H(t_1)$. So According to (5.4)

$$\begin{aligned}
&\left|a_3-\mu a_2^2\right| \\
&\leq \frac{2k\left|d\right|}{B^m}+\frac{k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left[\mu-H(t_1)-l_1\left(1-\left|sin\theta\right|\right)\right]\left|c_1^2\right| \\
&=\frac{2k\left|d\right|}{B^m}+\frac{k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left[\mu-H_1\right]\left|c_1^2\right| \qquad (5.6)
\end{aligned}$$

Next, I let $\mu \leq H(t_1)-l_1(1-\left|sin\theta\right|)$. So then in (5.6) I have

$$\left|a_3-\mu a_2^2\right| \leq \frac{2k\left|d\right|}{B^m}$$

Next, I let $\mu \geq H_1 = H(t_1)-l_1(1-\left|sin\theta\right|)$. So According to Lemma1and $l_1 = \frac{(2k-1)^2A^{2m}}{2k\left|d\right|B^m}$ I have

$$\begin{aligned}
&\left|a_3-\mu a_2^2\right| \\
&\leq \frac{2k\left|d\right|}{B^m}+\frac{4k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left(\mu-H(t_1)\right)-\frac{2k\left|d\right|}{B^m}\left(1-\left|sin\theta\right|\right) \\
&=\frac{4k^2\left|d^2\right|}{(2k-1)^2A^{2m}}\left(\mu-H(t_1)-\right)-\frac{2k\left|d\right|}{B^m}\left|sin\theta\right| \qquad (5.7)
\end{aligned}$$

Next suppose $\mu \geq H(t_1)$

Suppose that $\mu \leq \frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \leq \frac{(1+2kd)A^{2m}}{4kdB^m}$ then from (3.11) and Lemma 1 I get

$$\begin{aligned}
\left|a_3 - \mu a_2^2\right| &\leq \frac{kd}{(3k-1)B^m}\left[\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left(1 + 2kd - \mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}}\right)\right] \\
&\leq \frac{kd}{(3k-1)B^m}\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left(1 + 2kd - \mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}}\right)\right] \\
&= \frac{2kd}{(3k-1)B^m}\left[1 + 2kd\left(1 - \mu\frac{2kdB^m}{(2k-1)^2 A^{2m}}\right)\right] \\
&\leq \frac{2kd}{B^m}\left[1 + 2kd\left(1 - \mu\frac{2kdB^m}{(2k-1)^2 A^{2m}}\right)\right]
\end{aligned} \tag{5.8}$$

Next, I consider $\frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \leq \mu \leq \frac{(1+2kd)A^{2m}}{4kdB^m}$. Then, using the above calculations, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2kd}{B^m}$$

Finally, I consider $\mu \geq \frac{(1+2kd)A^{2m}}{4kdB^m}$

$$\begin{aligned}
\left|a_3 - \mu a_2^2\right| &\leq \frac{kd}{(3k-1)B^m}\left[\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left(\mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}} - 1 - 2kd\right)\right] \\
&\leq \frac{kd}{(3k-1)B^m}\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left(\mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}} - 1 - 2kd\right)\right] \\
&= \frac{2kd}{(3k-1)B^m}\left[1 + 2kd\left(1 - \mu\frac{2kdB^m}{(2k-1)^2 A^{2m}}\right)\right] \\
&\leq \frac{2kd}{B^m}\left[\mu\frac{4k^2 dB^m}{(2k-1)^2 A^{2m}} - 2kd - 1\right)\right]
\end{aligned} \tag{5.9}$$

□

Thus, the Theorem has been proved. Suppose I give $\beta = 1$ and $\eta = 0$ in Theorems $1-3$ I have following new results, respectively

Corollary 5.2. (I)*:Suppose that d be a nonzero complex number and $\mu \in \mathbb{C}$, $0 \leq \eta \leq \beta$. If f of the form:*

$$f(z) = z + \sum_{i=2}^{\infty} a_j z^j \tag{5.10}$$

is in $\mathbb{S}_m(d)$, then

$$\left|a_2\right| \leq \frac{|d|}{(2k)^{m-1}} \tag{5.11}$$

$$\left|a_3\right| \leq \frac{k|d|}{(3k)^m} max\left\{1, \left|1 + 2kd\right|\right\} \tag{5.12}$$

$$\left|a_3 - \mu a_2\right| \leq \frac{2k|d|}{(3k-1)(3k)^m} max\left\{1, \left|1 + 2kd - \frac{4kd\mu}{(2k-1)^2}\left(\frac{3}{4k}\right)^m\right|\right\} \tag{5.13}$$

(II) *Suppose that* $d>0$ *function* $f\in\mathbb{S}_m\big(d,\beta,\eta\big)$. *Then for* $\mu\in\mathbb{R}$ *I get*

$$\Big|a_3-\mu a_2\Big|\leq\begin{cases}\frac{2kd}{B^m}\Big\{1+2kd\Big[1-\frac{2kd\mu B^m}{(2k-1)^2A^{2m}}\Big]\Big\} & if\quad \mu\leq\frac{(2k-1)^2A^{2m}}{(3k-1)B^m}\\ \frac{d}{B^m} & if\quad \frac{(2k-1)^2A^{2m}}{(3k-1)B^m}\leq\mu\leq\frac{(1+2kd)A^{2m}}{4kdB^m},\\ \\ \frac{d}{B^m}\Bigg[\frac{4kd\mu B^m}{A^{2m}}-2kd-1\Bigg] & if\quad \mu\geq\frac{(1+2kd)A^{2m}}{4kdB^m}\end{cases}\tag{5.14}$$

(III)*Suppose that* d *be a nonzero complex number and for* $f\in\mathbb{S}_m\big(d,\beta,\eta\big)$. *Then for* $\mu\in\mathbb{R}$ *I get*

$$\Big|a_3-\mu a_2^2\Big|\leq\begin{cases}\frac{4k^2\big|d\big|^2}{(2k-1)^2A^{2m}}\Big[H(t_1)-\mu\Big]+\frac{k\big|d\big|\big|sin\theta\big|}{(3k-1)B^m} & if\quad \mu\leq H_1\\ \frac{2k\big|d\big|}{B^m} & if\quad H_1\leq\mu\leq L_1\\ \\ \frac{4k^2\big|d\big|^2}{(2k-1)^2A^{2m}}\Big[H(t_1)-\mu\Big]+\frac{k\big|d\big|\big|sin\theta\big|}{(3k-1)B^m} & if\quad \mu\geq L_1\end{cases}\tag{5.15}$$

6 Stability $\Big|a_3-\mu a_2\Big|$-functional inequalities when d is a nonzero complex number, $\mu\in\mathbb{C}$ and $f\in\mathbb{C}_m\big(d,\beta,\eta\big)$

Definition 6.1. Suppose d be a nonzero complex number, $g\in\mathbb{M}$ and $\Big(D^m_{\beta,\eta}g\big(z\big)\Big)'\neq0$ I define a subclasses as follows:

$$\mathbb{C}_m\big(d,\beta,\eta\big):=$$
$$\left\{g\in\mathbb{M}\Big|\mathcal{R}e\left(1+\frac{1}{d}\left(\frac{z\Big(D^m_{\beta,\eta}g\big(z\big)\Big)''}{\Big(D^m_{\beta,\eta}g\big(z\big)\Big)'}-1\right)\right)>0,0\leq\eta\leq\beta,m\in\mathbb{N},z\in\Omega\setminus\{0\}\right\}$$

Theorem 6.2. *Suppose that* d *be a nonzero complex number and* $\mu\in\mathbb{C}$, $0\leq\eta\leq\beta$. *If* f *of the form:*

$$f(z)=z+\sum_{i=2}^{k}ka_jz^j\tag{6.1}$$

is in $\mathbb{C}_m\big(d,\beta,\eta\big)$, *then*

$$\Big|a_2\Big|\leq\frac{2k\big|d\big|}{(2k-1)A^m}\tag{6.2}$$

$$\Big|a_3\Big|\leq\frac{k\big|d\big|}{(3k-1)B^m}max\Big\{1,\Big|1+2kd\Big|\Big\}\tag{6.3}$$

$$\Big|a_3-\mu a_2\Big|\leq\frac{2k\big|d\big|}{(3k-1)B^m}max\Big\{1,\Big|1+2kd-4kd\mu\frac{B^m}{(2k-1)^2A^{2m}}\Big|\Big\}\tag{6.4}$$

In there $A:=\Big[1+(2k-1)\big(2k\eta+\beta-\eta\big)\Big]$ *and* $B:=\Big[1+(3k-1)\big(3k\beta\eta+\beta-\eta\big)\Big]$. *Consider the functions*

Theorem 6.3. *Suppose that* $d>0$ *function* $f\in\mathbb{C}_m\big(d,\beta,\eta\big)$. *Then for* $\mu\in\mathbb{R}$ *I get*

$$\Big|a_3-\mu a_2\Big|\leq\begin{cases}\frac{2kd}{B^m}\Big\{1+2kd\Big[1-\frac{2kd\mu B^m}{(2k-1)^2A^{2m}}\Big]\Big\} & if\quad \mu\leq\frac{(2k-1)^2A^{2m}}{(3k-1)B^m}\\ \frac{d}{B^m} & if\quad \frac{(2k-1)^2A^{2m}}{(3k-1)B^m}\leq\mu\leq\frac{(1+2kd)A^{2m}}{4kdB^m},\\ \\ \frac{d}{B^m}\Bigg[\frac{4kd\mu B^m}{A^{2m}}-2kd-1\Bigg] & if\quad \mu\geq\frac{(1+2kd)A^{2m}}{4kdB^m}\end{cases}\tag{6.5}$$

In there $A := \Big[1 + (2k-1)\big(2k\eta + \beta - \eta\big)\Big]$ *and* $B := \Big[1 + (3k-1)\big(3k\beta\eta + \beta - \eta\big)\Big]$. *For each* μ, *the equality holds for functions in* (3.5) *and* (3.6)

Theorem 6.4. *Suppose that* d *be a nonzero complex number and for* $f \in \mathbb{C}_m\big(d, \beta, \eta\big)$. *Then for* $\mu \in \mathbb{R}$ *I get*

$$\Big|a_3 - \mu a_2^2\Big| \leq \begin{cases} \frac{4k^2\left|d\right|^2}{(2k-1)^2 A^{2m}}\Big[H(t_1) - \mu\Big] + \frac{k\left|d\right|\left|sin\theta\right|}{(3k-1)B^m} & if \quad \mu \leq H_1 \\ \frac{2k\left|d\right|}{B^m} & if \quad H_1 \leq \mu \leq L_1 \\ \\ \frac{4k^2\left|d\right|^2}{(2k-1)^2 A^{2m}}\Big[H(t_1) - \mu\Big] + \frac{k\left|d\right|\left|sin\theta\right|}{(3k-1)B^m} & if \quad \mu \geq L_1 \end{cases} \tag{6.6}$$

In there $A := \Big[1 + (2k-1)\big(2k\eta + \beta - \eta\big)\Big]$ *and* $B := \Big[1 + (3k-1)\big(3k\beta\eta + \beta - \eta\big)\Big]$, $k\big|d\big| = kde^{i\theta}$, $t_1 = \frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} + \frac{(2k-1)^2 A^{2m} e^{i\theta}}{4k\left|d\right|B^m}$, $l_1 = \frac{(2k-1)^2 A^{2m}}{4k\left|d\right|B^m}$ $H_1 = H(t_1) - l_1(1 - \big|sin\theta\big|)$ *and* $L_1 = H(t_1) + l_1(1 - \big|sin\theta\big|)$ *For each* μ, *there is a functin in* $f \in \mathbb{S}_m\big(d, \beta, \eta\big)$ *such that the equality holds.*

the equality holds for functions in (3.5) *and* (3.6)

Corollary 6.5. (I)*:Suppose that* d *be a nonzero complex number and* $\beta = 1, \eta = 0$ *and* $\mu \in \mathbb{C}$.
If f *of the form:*

$$f(z) = z + \sum_{i=2}^{\infty} a_j z^j \tag{6.7}$$

is in $\mathbb{C}_m\big(d\big)$, *then*

$$\Big|a_2\Big| \leq \frac{\big|d\big|}{(2k)^{m-1}} \tag{6.8}$$

$$\Big|a_3\Big| \leq \frac{k\big|d\big|}{(3k)^m} max\Big\{1, \Big|1 + 2kd\Big|\Big\} \tag{6.9}$$

$$\Big|a_3 - \mu a_2\Big| \leq \frac{2k\big|d\big|}{(3k-1)(3k)^m} max\Big\{1, \Big|1 + 2kd - \frac{4kd\mu}{(2k-1)^2}\left(\frac{3}{4k}\right)^m \Big|\Big\} \tag{6.10}$$

(II) *Suppose that* $d > 0$, *function* $f \in \mathbb{C}_m\big(d\big)$. *Then for* $\mu \in \mathbb{R}$ *I get*

$$\Big|a_3 - \mu a_2\Big| \leq \begin{cases} \frac{2kd}{B^m}\Big\{1 + 2kd\Big[1 - \frac{2kd\mu B^m}{(2k-1)^2 A^{2m}}\Big]\Big\} & if \quad \mu \leq \frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \\ \frac{d}{B^m} & if \quad \frac{(2k-1)^2 A^{2m}}{(3k-1)B^m} \leq \mu \leq \frac{(1+2kd)A^{2m}}{4kdB^m}, \\ \\ \frac{d}{B^m}\left[\frac{4kd\mu B^m}{A^{2m}} - 2kd - 1\right] & if \quad \mu \geq \frac{(1+2kd)A^{2m}}{4kdB^m} \end{cases} \tag{6.11}$$

(III)*Suppose that* d *be a nonzero complex number and for* $f \in \mathbb{C}_m\big(d\big)$. *Then for* $\mu \in \mathbb{R}$ *I get*

$$\Big|a_3 - \mu a_2^2\Big| \leq \begin{cases} \frac{4k^2\left|d\right|^2}{(2k-1)^2 A^{2m}}\Big[H(t_1) - \mu\Big] + \frac{k\left|d\right|\left|sin\theta\right|}{(3k-1)B^m} & if \quad \mu \leq H_1 \\ \frac{2k\left|d\right|}{B^m} & if \quad H_1 \leq \mu \leq L_1 \\ \\ \frac{4k^2\left|d\right|^2}{(2k-1)^2 A^{2m}}\Big[H(t_1) - \mu\Big] + \frac{k\left|d\right|\left|sin\theta\right|}{(3k-1)B^m} & if \quad \mu \geq L_1 \end{cases} \tag{6.12}$$

)

7 CONCLUSION

In this paper, I construct the $\left|a_3-\mu a_2^2\right|$-inequality function I based on The linear multiplier differential operator $D^m_{\beta,\eta}(\eta,\phi)f$ and subclass $\mathbb{S}_m\left(d,\beta,\eta\right),\mathbb{C}_m\left(d,\beta,\eta\right)$.

8 Conflicts of Interest

The author declares no conflicts of interest.

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References

[1] Al-Abbadi, M. H and Darus, M, (2011), The Fekete-Szegö theorem for a certain class of analytic functions. Sains Malaysiana. 40(4): 385-389.

[2] Al-Shaqsi, K and Darus, M, (2008), On Fekete-Szegö problems for certain subclass of analytic functions. Appl. Math. Sci. 2(9): 431-441

[3] A. Pfluger, The Fekete-Szegö inequality by a variational method, Ann. Acad. Sci.Fenn. Ser. A I Math. 10 (1984).

[4] C. Pommerenke, Univalent Functions, in: Studia Mathematica Mathematische Lehr- bucher, Vandenhoeck and Ruprecht, 1975..

[5] D. Răducanu, H. Orhan, Subclasses of analytic functions defined by a generalized differential operator, Int. J. Math. Anal. 4(1) (2010), 1-15.

[6] Darus, M, (2002), The Fekete-Szegö problems for close-to-convex functions of the class $K_{sh}(\alpha,\beta)$. Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis18(1): 13-18.

[7] E. Deniz, H. Orhan, The Fekete-Szegö problem for a generalized subclass of analytic functions, Kyungpook Math. J. 50 (2010), 37-47.

[8] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Math. Sci. 27 (2004), 1429-1436.

[9] Fekete, M, Szegö, G: Eine Bemerkung über ungerade schlichte Funktionen. J. Lond. Math. Soc. 8, 85-89 (1933).

[10] G. S. Sălăgean, Subclasses of univalent functions, Complex analysis, Proc. 5th Rom.- Finn. Semin., Bucharest 1981, Part 1, Lect. Notes Math. 1013 (1983), 362-372.

[11] Ma, W and Minda, D, (1994), A unified treatment of some speial classes of univalent functions, in: Proceeding of the conference on complex analysis, Z. Li. F.Ren, L.Yang, and S. Zhang (Eds.), Int. Press , 157-169.

[12] Mohammed A. and Darus M, (2010), On Fekete-Szegö problem for certain subclass of analytic functions. Int. J. Open Problems Compt. Math. 3(4): 510-520.

[13] . Ma, W. and Minda, D.A. 1994. Unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis. Eds Li, Z., Ren, F., Yang, L. and Zhang, S. Int.Press, pp.157-169.

[14] M. A. Nasr, M. K. Aouf, Starlike function of complex order, J. Natur. Sci. Math. 25(1985), 1-12.

[15] M. A. Nasr, M. K. Aouf, On convex functions of complex order, Mansoura Sci. Bull.(1982), 565-582.

[16] P. Wiatrowski, The coefficients of a certain family of holomorphic functions, Zeszyty Nauk. Uniw. Lódz., Nauki. Mat. Przyrod. Ser. II (1971), 75-85.

[17] Oshah, A and Darus, M, (2014), Differential sandwich theorems with a new generalized derivative operator. Advances in Mathematics: Scientific Journal, 3(2),117-124.

[18] RØnning, F, (1993), Uniformly convex function and a corresponding class of starlike functions, Proc. Amer. Math. Soc., 118, 189-196.

[19] Ravichandran, V, Gangadharan, A. and Darus, M, (2004), On Fekete-Szegö inequality for certain class of Bazilevic functions. Far East Journal of Mathematical Sciences15(2): 171-180.

[20] Ramadan, S.F and Darus, M, (2011), On the Fekete- Szegö inequality for a class of analytic functions defined by using generalized differential operator, Acta Universitatis Apulensis, 26, 167-178.

[21] . Ravichandran, V., Gangadharan, A. and Darus, M. 2004. Fekete-Szego inequality for certain class of Bazilevic functions. Far East J. Math. Sci. 15:171-180.

[22] . Ravichandran, V. Darus, M., Khan, M.H. and Subramanian, K.G. 2004. Fekete-Szego inequal-ity for Certain Class of Analytic Functions. Austral. J. Math. Anal. Appl. 1, Art. 4:1-7.

[23] . Ravichandran, V., Bolcal, M., Polotoglu, Y. and Sen, A. 2005. Certain subclasses of starlike and convex functions of complex order. Hacet. J. Math. Stat. 34:9-15.

[24] Srivastava, H. M. and Mishra, A. K, (2000), Applications of fractional calculus to parabolic starlike and uniformly convex functions, Comput. Math. Appl., 39,57-69.

[25] Srivastava H. M., Mishra A. K. and Das M. K. (2001), The Fekete-Szegö problem for a subclass of close-to-convex functions. Complex Variables, Theory and Application 44: 145-163.

[26] Srivastava, H. M. and Owa, S, (1989), Univalent functions, fractional calculus and their applications., John Wiley and Sons, New Jersey.

[27] . Shanmugam, T.N. and Sivasubramanian, S. 2005. On the Fekete-Szego problem for some subclasses of analytic functions. J. Inequal. Pure Appl. Math.6, Art. 71:1-6.