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Existence of a bounded variation solution of a nonlinear integral equation in $L_1(R^+)$

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Abstract

In this paper, we study the existence of a unique solution of a nonlinear integral equation in the space of bounded variation on an unbounded interval by using measure of noncompactness and Darbo fixed point theorem.

Keywords: Nemytskii operator, Volterra integral operator, Hausdorff measure of noncompactness, Functions of bounded variation, Darbo fixed point theorem.

1 Introduction

Integral equations create a very important and significant part of mathematical analysis and their applications to real world problems (cf. [1], [3], [7], [22],[24]). The theory of integral equations has been well developed with the help of various tools from functional analysis, topology and fixed-point theory.
This paper studies the existence of a unique solution of the following nonlinear integral equation

$$x(t) = g(t) + h(t)f(t, x(t)) + \int_0^{\phi(t)} k(t, s)f(s, x(s))ds, \qquad t \in R^+, \tag{1}$$

in the space $L_1(R^+)$ of functions of bounded variation.

2 Preliminaries

This section is devoted to recall some notations and results that will be needed in the sequel. Let R be the field of real numbers and R^+ be the interval $[0, \infty)$. Denote by $L_1 = L_1(R^+)$ the space of Lebesgue integrable functions on the interval $[0, \infty)$, with the standard norm

$$\|x\| = \int_0^{\infty} |x(t)|dt.$$

The most important operator in nonlinear functional analysis is the so-called Nemytskii operator ([2], [11], [12],[21]).

Definition 2.1 *If $f(t, x) = f : R^+ \times R \to R$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in R^+$. Then to every function $x(t)$ being measurable on R^+ we may assign the function*

$$(Fx)(t) = f(t, x(t)), \qquad t \in R^+.$$

The operator F is called the Nemytskii (or superposition) operator generated by f.

Furthermore, we propose a theorem which gives necessary and sufficient condition for the Nemytskii operator to map the space L_1 into itself continuously.

Theorem 2.1 *([2], [15]) If f satisfies Carathéodory conditions, then the Nemytskii operator F generated by the function f maps continuously the space L_1 into itself if and only if*

$$|f(t,x)| \leq a(t) + b|x|,$$

for every $t \in R^+$ and $x \in R$, where $a(t) \in L_1$ and $b \geq 0$ is a constant.

Definition 2.2 *(Volterra integral operator) [25]*
Let $k : \Delta \to R$ be a function that is measurable with respect to both variables, where $\Delta = \{(t,s) : 0 \leq s \leq t < \infty\}$. For an arbitrary function $x \in L_1(R^+)$, we define

$$(Vx)(t) = \int_0^t k(t,s)x(s)ds, \quad t \geq 0.$$

The above operator V is the well-known linear Volterra integral operator. Obviously, if $V : L_1 \to L_1$ then it is continuous [23].

Definition 2.3 *([5], [10], [20])*

The Hausdorff measure of noncompactness $\chi(X)$ (see also [16]–[18]) is defined as

$$\chi(X) = \inf\{r > 0 : \text{there exists a finite subset } Y \text{of } E \text{ such that } x \subset Y + B_r\}.$$

Another regular measure was defined in the space $L_1(I)$ ([4], [9]). For any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X:

$$c(X) = \lim_{\varepsilon \to 0}\{\sup_{x \in X}\{\sup[\int_D |x(\tau)|d\tau : \ D \subset R^+, \ \text{meas}D \leq \varepsilon]\}\} = 0 \tag{2}$$

and

$$d(X) = \lim_{T \to \infty}\{\sup[\int_T^\infty |x(\tau)|d\tau : x \in X]\}, \tag{3}$$

where meas$D$ represents the Lebesgue measure of subset D.
Put

$$\gamma(X) = c(X) + d(X). \tag{4}$$

Then we have the following theorem [19], which connects between the two measures $\chi(X)$ and $\gamma(X)$.

Theorem 2.2 *Let $X \in M_E$ and compact in measure, then*

$$\chi(X) \leq \gamma(X) \leq 2\chi(X).$$

Now, we give Darbo fixed point theorem (cf.[8], [13], [14], [21]).

Theorem 2.3 *If Q is nonempty, bounded, closed and convex subset of E and let $A : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists a constant $k \in [0,1)$ such that*

$$\mu(AX) \leq k\mu(X),$$

for any nonempty subset X of Q. Then A has at least one fixed point in the set Q.

Definition 2.4 *(Functions of bounded variation) ([6], [20])*
Let $x : [a, b] \to R$ be a function. For each partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of the interval $[a, b]$, we define

$$Var(x, [a, b]) = \sup \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|,$$

where the supremum is taken over the interval $[a, b]$. If $Var(x) < \infty$, we say that x has bounded variation and we write $x \in BV$.
We denote by $BV = BV[a, b]$ the space of all functions of bounded variation on $[a, b]$.

Theorem 2.4 *[4] Assume that $X \subset L_1(I)$ is of locally generalized bounded variation, then Conv X (convex hull of X) and $\bar{X}$ are of the same type.*

Corollary 2.1 *[4] Let $X \subset L_1(I)$ is of locally generalized bounded variation, then Conv X is also such.*

Next, we will have the following theorem, which we will further use (cf. [4]).

Theorem 2.5 *Assume that $X \subset L_1$ is a bounded set have the following hypotheses:*

(i) there exists $t_0 \geq 0$ such that the set $x(t_0) : x \in X$ is bounded in R,

(ii) X is of locally generalized bounded variation on R^+,

(iii) for any $a > 0$ the following equality holds

$$\lim_{T \to \infty} \{\sup_{x \in X} \{meas\ \{t > T : |x(t)| \geq a\}\}\} = 0.$$

Then the set X is compact in measure.

Corollary 2.2 *[4] If $X \subset L_1$ is a bounded set satisfies the hypotheses of Theorem 2.5. Then Conv X is compact in measure.*

3 Main result

Equation (1) can be written in operator form as

$$(Gx)(t) = g(t) + h(t)Fx(t) + VFx(t), \tag{5}$$

where $(Fx)(t) = f(t, x)$ and $(Vx)(t) = \int_0^{\phi(t)} k(t, s)x(s)ds.$

We will treat equation (1) under the following assumptions listed below:

(i) $g, h : R^+ \to R$, $g \in L_1(R^+)$ and $h(t)$ is bounded function such that $\sup_{t \in R^+} |h(t)| \leq M$.

(ii) $f : R^+ \times R \to R$ satisfies Carathéodory conditions and there exist a function $a \in L_1(R^+)$ and a constant $b \geq 0$ such that $|f(t, x)| \leq a(t) + b|x|$, for all $t \in R^+$ and $x \in R$.

(iii) there exists $L > 0$ such that

$$|f(t,x) - f(s,y)| \le L(|t-s| + |x-y|).$$

(iv) $k(t,s) : \Delta \to R$ is measurable in both variables and such that the integral operator V generated by k maps L_1 into L_1, $(\Delta = \{(t,s) : 0 \le s \le t < \infty\})$.
Moreover, $\forall\ h > 0$.

$$\lim_{T\to\infty}\{\text{meas }\{t > T : |(Vx)(t)| \ge h\}\} = 0,$$

uniformly with respect to $x \in X$, where X is an arbitrary bounded subset of L_1.

(v) The generalized variation of the function $t \to k(t,s)$ is essentially bounded on $[0,T]$ $\forall\ T > 0$ and uniformly on $s \in [0,T]$.
Also, the function $v(T)$ is defined as

$$v(T) = \operatorname{ess\,sup}\{\text{var}_t k(t,s), [0,T] : s \in [0,T]\},$$

then we get $v(T) < \infty\ \forall\ T \ge 0$.

(vi) $\phi : R^+ \to R^+$ is increasing and continuous function such that $\phi(t) < t$, $\forall\ t \in R^+$ and it is bounded on R^+.

(vii) $b(M + \|V\|) < 1$.

Theorem 3.1 *Under the above assumptions (i)–(vii), equation (1) has at least one solution $x \in L_1(R^+)$ which is a function of locally bounded variation on the interval R^+.*

Proof. First of all observe that by assumption (ii) the operator F maps the space $L_1(R^+)$ into itself and is continuous. By assumption (iv) the Volterra operator V maps $L_1(R^+)$ into itself and is continuous. Finally, for every $x \in L_1(R^+)$ and by assumption (i) we can deduce that $Gx \in L_1(R^+)$.
Moreover, we get

$$\begin{aligned}
\|Gx\| &= \|g + hFx + VFx\| \\
&\le \|g\| + \int_0^\infty |h(t)f(t,x(t)|dt + \int_0^\infty \int_0^{\phi(t)} |k(t,s)f(s,x(s))|dsdt \\
&\le \|g\| + M\int_0^\infty [a(t) + b|x(t)|]dt + \int_0^\infty \int_0^t |k(t,s)||f(s,x(s))|dsdt \\
&\le \|g\| + M\|a\| + bM\int_0^\infty |x(t)|dt + \|V\|\int_0^\infty |f(s,x(s))|dt \\
&\le \|g\| + M\|a\| + bM\|x\| + \|V\|\int_0^\infty [a(t) + b|x(t)|]dt \\
&\le \|g\| + M\|a\| + bM\|x\| + \|V\|\|a\| + b\|V\|\|x\| \\
&\le \|g\| + M\|a\| + \|V\|\|a\| + b(M + \|V\|)r \\
&\le r.
\end{aligned}$$

From the above estimate, the operator G transforms the ball B_r into itself, where

$$r = \frac{\|g\| + M\|a\| + \|V\|\|a\|}{1 - b(M + \|V\|)} > 0.$$

Next, let us choose an arbitrary $x \in B_r$. Observe that

$$\begin{aligned} |(Gx)(0)| &= |g(0) + h(0)f(0,0)| \\ &\leq |g(0)| + |h(0)||f(0,0)| \\ &< \infty. \end{aligned} \tag{6}$$

Hence we infer that all functions belonging to GB_r are bounded at the point $t = 0$ by the same constant.
Further, let us fix $T > 0$ and take a sequence t_i such that $0 = t_0 < t_1 < t_2 \ldots < t_n = T$. Then using our assumptions, we obtain

$$\begin{aligned} \sum_{i=1}^{n} |(Gx)(t_i) - (Gx)(t_{i-1})| &\leq \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| \\ &+ \sum_{i=1}^{n} |h(t_i)f(t_i, x(t_i)) - h(t_{i-1})f(t_{i-1}, x(t_{i-1}))| \\ &+ \sum_{i=1}^{n} |\int_0^{\phi(t_i)} k(t_i, s)f(s, x(s))ds - \int_0^{\phi(t_{i-1})} k(t_{i-1}, s)f(s, x(s))ds| \\ &\leq V(g, T) + \sum_{i=1}^{n} |h(t_i)f(t_i, x(t_i)) - h(t_i)f(t_i, x(t_{i-1}))| \\ &+ \sum_{i=1}^{n} |h(t_i)f(t_i, x(t_{i-1})) - h(t_{i-1})f(t_i, x(t_{i-1}))| \\ &+ \sum_{i=1}^{n} |h(t_{i-1})f(t_i, x(t_{i-1})) - h(t_{i-1})f(t_{i-1}, x(t_{i-1}))| \\ &+ \sum_{i=1}^{n} |\int_0^{\phi(t_i)} k(t_i, s)f(s, x(s))ds - \int_0^{\phi(t_i)} k(t_{i-1}, s)f(s, x(s))ds| \\ &+ \sum_{i=1}^{n} |\int_0^{\phi(t_i)} k(t_{i-1}, s)f(s, x(s))ds - \int_0^{\phi(t_{i-1})} k(t_{i-1}, s)f(s, x(s))ds| \\ &\leq V(g, T) + L\sum_{i=1}^{n} |h(t_i)||x(t_i) - x(t_{i-1})| \\ &+ \sum_{i=1}^{n} |h(t_i) - h(t_{i-1})||f(t_i, x(t_{i-1})| \\ &+ L\sum_{i=1}^{n} |h(t_{i-1})||t_i - t_{i-1}| \\ &+ \int_0^{\phi(t_i)} (\sum_{i=1}^{n} |k(t_i, s) - k(t_{i-1}, s)|)|f(s, x(s))|ds \\ &+ \sum_{i=1}^{n} \int_{\phi(t_i - 1)}^{\phi(t_i)} |k(t_{i-1}, s)||f(s, x(s))|ds \\ V(Gx, T) &\leq V(g, T) + LMV(x, T) + V(h, T) + MN \\ &+ \int_0^{t_i} v(T)[a(s) + b|x(s)|]ds + \sum_{i=1}^{n} \int_{t_i - 1}^{t_i} |k(t_{i-1}, s)|[a(s) + b|x(s)|]ds \\ &+ k_0 \int_0^T a(s)ds + k_0 b \int_0^T |x(s)|ds \\ &\leq V(g, T) + LMV(x, T) + V(h, T) + MN \end{aligned}$$

$$\begin{aligned} &+ \int_0^T v(T)a(s)ds + b\int_0^T v(T)|x(s)|ds + k_0\int_0^T a(s)ds + k_0 b\int_0^T |x(s)|ds \\ &\leq V(g,T) + LMV(x,T) + V(A,T) + MN \\ &+ v(T)\|a\| + bv(T)r + k_0\|a\| + k_0 br < \infty, \end{aligned} \tag{7}$$

where $N = L|t_i - t_{i-1}|$.

In view of the above estimate all functions belonging to GB_r have variation majorized by the same constant on every closed subinterval of the interval R^+.

Next, let us consider the set Q_r=Conv GB_r, obviously $Q_r \subset B_r$. we will prove that Q_r is nonempty, bounded, convex, closed and compact in measure.

Q_r being nonempty follows by considering the nonincreasing function $x(t) = \frac{r}{\pi}(\frac{1}{1+t^2})$ where

$$\|x\| = \int_0^\infty |x(t)|dt = \int_0^\infty |\frac{r}{\pi}(\frac{1}{1+t^2})|dt = \frac{r}{\pi}\arctan|_0^\infty = \frac{r}{\pi}(\frac{\pi}{2}) \leq r.$$

Also Q_r is bounded as a subset of B_r.

To show that Q_r is convex. Let $x_1, x_2 \in Q_r$, then $\|x_i\| \leq r, \quad i = 1, 2.$

Let

$$z(t) = \lambda x_1(t) + (1-\lambda)x_2(t), \quad t \in R^+,\ \lambda \in R^+.$$

Then

$$\begin{aligned} \|z\| &\leq \lambda\|x_1\| + (1-\lambda)\|x_2\| \\ &\leq \lambda r + (1-\lambda)r = r. \end{aligned}$$

So the convexity of Q_r is established.

To show that Q_r is closed. Let $\{x_n\}$ be a sequence of elements in Q_r convergent in $L_1(R^+)$ to x, then the sequence is convergent in measure and as a consequence of the Vitali convergence theorem and the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to x almost uniformly on R^+ which means that $x \in Q_r$ and so the set Q_r is closed.

Further, in virtue of (6),(7) and Theorem 2.5 we conclude that the set GB_r is compact in measure. By Corollary 2.2 this yields that the set Q_r is also compact in measure. Moreover, Corollary 2.1 implies that the set Q_r is of locally generalized bounded variation on R^+. Now, from assumption (i), and since $Q_r \subset B_r$, then G is a self-mapping of the set Q_r into it self and is continuous.

In what follows, we will show that the operator G is a contraction with respect to the measure of noncompactness χ. Assume that X is a nonempty subset of Q_r and let $\varepsilon > 0$ is fixed, then for any $x \in X$ and for a set $D \subset R^+$, meas$D \leq \varepsilon$, we obtain

$$\begin{aligned} \int_D |(Gx)(t)|dt &\leq \int_D g(t|dt + \int_D |h(t)||f(t,x(t))|dt + \int_D\int_0^t |k(t,s)||f(s,x(s))|dsdt \\ &\leq \int_D g(t)dt + M\int_D [a(t) + b|x(t)|]dt + \|V\|\int_D [a(s) + b|x(s)|]dtds \\ &\leq \int_D g(t)dt + M\int_D a(t) + bM\int_D |x(s)|ds + \|V\|\int_D a(s)ds + b\|V\|\int_D |x(s|ds \end{aligned}$$

Now, using the fact that

$$\lim_{\varepsilon\to 0}\sup\{\int_D g(t)dt :\ D \subset R^+,\ \text{meas}D \leq \varepsilon\} = 0,$$

and

$$\lim_{\varepsilon\to 0}\sup\{\int_D a(t)dt :\ D \subset R^+,\ \text{meas}D \leq \varepsilon\} = 0,$$

Then using (2), we get

$$c(GX) \leq b(M + \|V\|)c(X). \tag{8}$$

Furthermore, fixing $T > 0$ we arrive at the following estimate

$$\int_T^\infty |(Gx)(t)|dt \leq \int_T^\infty g(t)dt + M\int_T^\infty a(t)dt + Mb\int_T^\infty |x(t)|dt + \|V\|\int_T^\infty a(s)ds + b\|V\|\int_T^\infty |x(s|ds.$$

As $T \to \infty$, the above inequality yields

$$d(GX) \leq b(M + \|V\|)d(X), \tag{9}$$

where $d(X)$ has been defined before in (3).
Hence combining (8) and (9) we get

$$\gamma(GX) \leq b(M + \|V\|)\gamma(X),$$

Since $X \subset Q_r$ and Q_r is compact in measure, then we have

$$\chi(GX) \leq b(M + \|V\|)\chi(X).$$

Thus in virtue of assumption (vii) we can apply Darbo fixed point theorem which guarantees equation (1) has at least one solution. This completes the proof. ■

4 Example

. Consider the integro-differential equation

$$x(t) = g(t) + \int_0^t p(t,s)f(s,x'(s))ds, \qquad t \in R^+ \tag{10}$$

Differentiate both sides of equation (10) with respect to t, we get

$$x'(t) = g'(t) + p(t,t)f(t,x'(t)) + \int_0^t p'(t,s)f(s,x'(s))ds \tag{11}$$

Put $x'(t) = y(t)$, $g'(t) = h(t)$, $p(t,t) = q(t)$ and $p'(t,s) = k(t,s)$ in (11)
Then we have

$$y(t) = h(t) + q(t)f(t,y(t)) + \int_0^t k(t,s)f(s,y(s))ds, \qquad t \in R^+ \tag{12}$$

Taking into account all assumptions of Theorem 3.1 with $\phi(t) = t$, then equation (10) has at least one solution $x \in L_1(R^+)$ which is a function of locally bounded variation on R^+.

5 Uniqueness of the solution

Now, we can prove the existence of our unique solution.

Theorem 5.1 *Let the assumptions of Theorem 3.1 be satisfied but instead of assumption (vii), let $M + \|V\| < 1$. Then, equation (1) has a unique solution on R^+.*

Proof. To prove the unique solution of equation (1), let $x(t), y(t)$ be any two solutions of equation (1) in B_r, we have

$$\begin{aligned}
\|x-y\| &= \|h(t)[f(t,x(t))-f(t,y(t))] + \int_0^{\phi(t)} k(t,s)[f(s,x(s))-f(s,y(s))]ds\| \\
&\leq \int_0^{\infty} |h(t)||f(t,x(t))-f(t,y(t))|dt + \int_0^{\infty}\int_0^{\phi(t)} |k(t,s)||f(s,x(s))-f(s,y(s))|ds \\
&\leq M\int_0^{\infty} |x(t)-y(t)|dt + \|V\|\int_0^{t} |x(s)-y(s)|ds \\
&\leq (M+\|V\|)\|x-y\|.
\end{aligned}$$

Therefore,

$$[1-(M+\|V\|)]\|x-y\|_{L_1} \leq 0,$$

This yields $\|x-y\| = 0, \Rightarrow x = y$, which completes the proof.

Data Availability (excluding Review articles)

Applicable.

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Supplementary Materials

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

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