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Triads Of Metallic Ratios, The Mathematical Relations between different Metallic Means, And Geometric Substantiation of Metallic Numbers with the Right Angled Triangles

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Abstract

This paper synergizes the newly discovered geometry of all Metallic Means and the recently published mathematical formulae those provide the precise correlations between different Metallic Ratios. The paper also illustrates the concept of the "TRIADS of Metallic Means". The Metallic Means and their TRIADS can be geometrically substantiated, in an intriguing manner, as described herein.

Keywords: Metallic Mean, Pythagoras Theorem, Fibonacci, Pell, Lucas, Pi, Phi, Silver Ratio, Right Triangle, Metallic Numbers, Metallic Ratio Triads, 3 6 9, Pythagorean Triples, Bronze Ratio, Golden Ratio, Pascal's Triangle, Metallic Ratio

Introduction

This paper brings together following, recently discovered, new aspects of Metallic Ratios.

- 1) The Generalised Geometric Construction of all Metallic Ratios: cited by the Wikipedia page: "Metallic Mean" [6]. This generalised geometric substantiation of all Metallic Means was published in January 2021 [7]
- 2) The Mathematical Formula that provides the precise correlation between different Metallic Means. This explicit formula has been recently published in the month of May 2021 [1].

These couple of important aspects of Metallic Means: viz. the generalised geometric constructions of all Metallic Means and the concerned mathematical formulae, were brought together in the work mentioned in Reference [2].

The prime objective of current paper is to further synergize the these two newly discovered aspects of Metallic Means.

The *synergism* between above two features of Metallic Means unveils an intriguing pattern of Metallic Ratios, which asserts that the mathematical implications of these Means have not been fully appreciated so far. The abovementioned Geometry and Mathematics synergically enable us to recognize the full worth of these Metallic Means, as described in this paper.



As a brief introduction, each Metallic Mean δ_n is the root of the simple Quadratic Equation $X^2 - nX - 1 = 0$, where n is any positive natural number.

Thus, the fractional expression of the n^{th} Metallic Ratio is $\delta_n = \frac{n + \sqrt{n^2 + 4}}{2}$

Moreover, each Metallic Ratio can be expressed as the continued fraction:

$$\delta_n = n + \frac{1}{n + \frac{1}{n + \frac{1}{n + \dots}}} ; \text{ And hence, } \delta_n = n + \frac{1}{\delta_n} \quad \dots\text{References: [3], [4], [5]}$$

GEOMETRIC CONSTRUCTION OF ALL METALLIC MEANS :

Each Metallic Ratio can be constructed geometrically with a special Right Angled Triangle. Any n^{th} Metallic Mean can be represented by the Right Triangle having its catheti 1 and $\frac{n}{2}$. Hence, the right triangle with one of its catheti = 1 may substantiate any Metallic Mean, having its second cathetus = $\frac{n}{2}$, where $n = 1$ for Golden Ratio, $n = 2$ for Silver Ratio, $n = 3$ for Bronze Ratio, and so on. Such Right Triangle provides the precise value of n^{th} Metallic Mean by the generalised formula:

$$\text{The } n^{\text{th}} \text{ Metallic Mean } (\delta_n) = \text{Hypotenuse} + \text{Cathetus } \frac{n}{2}$$

Such Right Triangle not only provides for the accurate geometric construction and precise fractional expression of any n^{th} Metallic Mean (δ_n), but its every geometric feature is the prototypical form of that Metallic Mean [6], [7], [8], [9].

The characteristic geometry of such Right Triangle having its catheti 1 and $\frac{n}{2}$, is resplendent with the corresponding n^{th} Metallic Mean (δ_n) embedded in its every geometric aspect.

For example, the remarkable expression of Golden Ratio in every geometric feature of $1:2:\sqrt{5}$ triangle, including all its angles and side lengths, its 'Incenter-Excenters Orthocentric system', its Gergonne and Nagel triangles, and also the Nobbs points and the Gergonne line, various triangle centers as well as the Incircle of $1:2:\sqrt{5}$ triangle, make this triangle the quintessential form of the Golden Ratio (ϕ) and also of the fourth Metallic Mean (ϕ^3). [7]

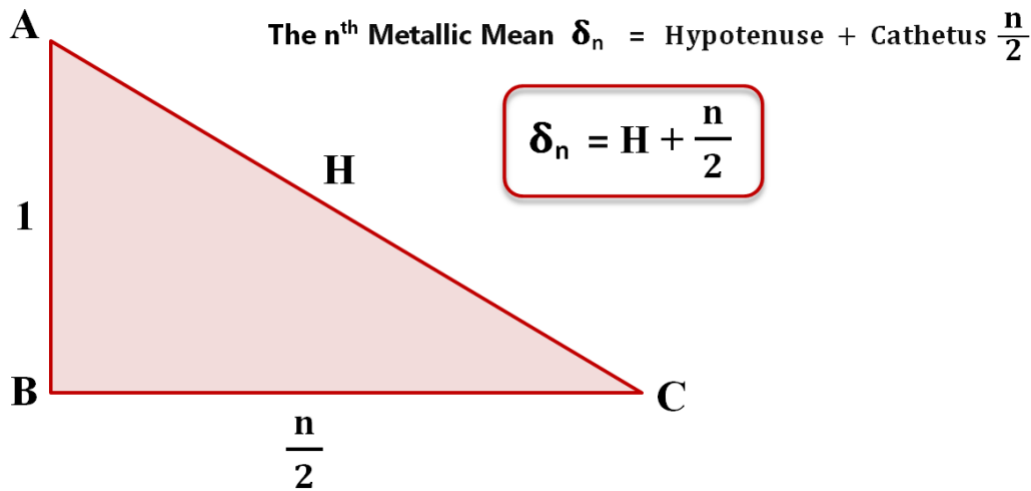


Figure 1: Generalised Right Triangle to represent any n^{th} Metallic Mean δ_n

MATHEMATICAL CORRELATIONS AMONG DIFFERENT METALLIC RATIOS :

If k , m and n are three positive integers such that n is the smallest of the three integers and $\frac{mn + 4}{m - n} = k$

then, it is observed that

$$\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k \text{ where } \delta_k, \delta_m \text{ and } \delta_n \text{ are the } k^{\text{th}}, m^{\text{th}} \text{ and } n^{\text{th}} \text{ Metallic Means respectively.}$$

This explicit formula, among several other formulae those give the precise mathematical relations between different Metallic Means, has been recently published in the work mentioned in References [1] and [2].

The “TRIADS” Of Metallic Means :

The abovementioned explicit formula gives the “Triads” of Metallic Means as $[\delta_n, \delta_m, \delta_k]$

Where $\frac{mn + 4}{m - n} = k$ and $\frac{kn + 4}{k - n} = m$

hence, $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$ and also $\frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m$

Moreover,

$$\frac{km - 4}{k + m} = n \quad \text{and} \quad \frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n$$



For example, if $n=6$, the three integers 6, 11 and 14 satisfy the prerequisite $\frac{mn + 4}{m - n} = k$;

Hence, the three Merallic means δ_6, δ_{11} and δ_{14} form a **Triad** [$\delta_6, \delta_{11}, \delta_{14}$] such that :

$$\frac{\delta_{11} \times \delta_6 + 1}{\delta_{11} - \delta_6} = \delta_{14} \quad \text{and also} \quad \frac{\delta_{14} \times \delta_6 + 1}{\delta_{14} - \delta_6} = \delta_{11} \quad \text{Also,} \quad \frac{\delta_{14} \times \delta_{11} - 1}{\delta_{14} + \delta_{11}} = \delta_6$$

Noticeably, $n=6$ forms such multiple triads:

n	6	6	6	6	6	6	6	6
m	7	8	10	11	14	16	26	46
k	46	26	16	14	11	10	8	7

: Shaded Triads have been exemplified above.

And, just like $n=6$ exemplified above, every integer forms such multiple triads:

For example, $n=10$

n	10	10	10	10	10	10	10	10
m	11	12	14	18	23	36	62	114
k	114	62	36	23	18	14	12	11

Noticeably, Odd n forms Fewer Triads

n	5	5	5	5	5	5	5	5
m	6	34						
k	34	6						

It may be noticed from above Tables that every n^{th} Metallic Mean can give precise values of various Metallic Means by the formula $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$, maximum upto $(n^2 + n + 4)^{\text{th}}$ Metallic Mean:

$$m_{\text{max}} = k_{\text{max}} = (n^2 + n + 4)$$



Also noticeably, the Even Integers (Even n_s) form comparatively more Triads than the Odd n_s . Several such patterns about these Triads of Metallic Means have been discussed in detail in Reference [2]. Here, let us consider the classical correspondence of the abovementioned Formula and the TRIADS with the Geometry of Metallic Ratios.

Remarkably, the abovementioned Triads of Metallic Means can be represented geometrically, as shown below.

For instance, the **Triad** [$\delta_n, \delta_m, \delta_k$] is illustrated geometrically in following **Figure 2**.

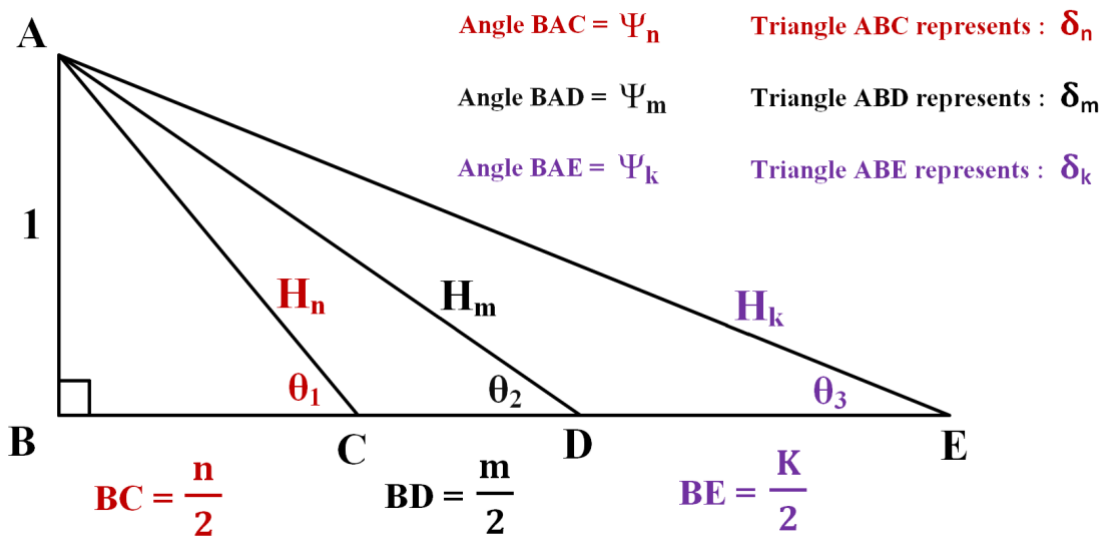


Figure 2: Three Right Triangles representing the “Triad of Metallic Means”

Remarkably, in above Figure 2: if the three Metallic Means δ_n, δ_m and δ_k constitute a **Triad** as $\frac{mn + 4}{m - n} = k$, then

$$\theta_1 = \theta_2 + \theta_3 \quad (\text{and also } \Psi_n + 90^\circ = \Psi_m + \Psi_k)$$

In other words,

$$\arctan \frac{2}{n} = \arctan \frac{2}{m} + \arctan \frac{2}{k}$$

Solving it gives the correlations: $\frac{mn + 4}{m - n} = k$ and $\frac{kn + 4}{k - n} = m$

And also $\frac{km - 4}{k + m} = n$

And hence satisfy the prerequisite for $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$ and $\frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m$

And also $\frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n$

Moreover, **the Hypotenuses** of these Triad-Triangles : H_n , H_m and H_k in **Figure 2** exhibit following relations.

$$\frac{H_k}{H_m} = \frac{2}{m-n} \times H_n$$

Simplifying this, we get $\frac{k^2 + 4}{m^2 + 4} = \frac{n^2 + 4}{(m-n)^2}$

And solving it gives $k = \frac{mn + 4}{m-n}$; which is the prerequisite for formation of a Triad $[\delta_n, \delta_m, \delta_k]$

Moreover, entire geometry of such Triad-Triangles is resplendent with the precise correlation among the three Metallic Means δ_n , δ_m and δ_k . Following intriguing relations are observed in above **Figure 2**. Consider the **larger acute angles** of the three triangles;

$$\Psi_m + \Psi_k = 2 \arctan \delta_n$$

$$\Psi_m - \Psi_n = 2 \arctan \frac{1}{\delta_k}$$

$$\Psi_k - \Psi_n = 2 \arctan \frac{1}{\delta_m}$$

Similarly, $\arctan \frac{1}{\delta_n} = \arctan \frac{1}{\delta_m} + \arctan \frac{1}{\delta_k}$

simplifying which we get : $\frac{km - 4}{k + m} = n$ and hence $\frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n$

And,

$$\arctan \frac{1}{\delta_n} + \arctan \frac{1}{\delta_m} + \arctan \frac{1}{\delta_k} = 2 \arctan \frac{1}{\delta_n} = (\theta_1) = [\theta_2 + \theta_3]$$

$$= \arctan \frac{2}{n} = [180^\circ - (\Psi_m + \Psi_k)]$$

= Half of the Smaller Acute Angle of the Pythagorean Triple associated with the Right Triangle for δ_n (Described in following Section)



Similarly,

$$\arctan \delta_n + 90^\circ = \arctan \delta_m + \arctan \delta_k$$

And,

$$\arctan \delta_n + \arctan \delta_m + \arctan \delta_k = \Psi_n + 180^\circ = (\Psi_m + \Psi_k) + 90^\circ$$

Another interesting aspect of this geometry of Metallic Means is its classical correspondence with Primitive Pythagorean Triples. Each of the Right Triangles representing various Metallic Means, and hence the abovementioned Triads of Metallic Means are associated with a particular primitive Pythagorean triple, depending upon the value of n . For example, Right Triangle for 6th Metallic Mean (δ_6), and hence all Triads with $n=6$, are associated with 3-4-5 Pythagorean Triple. For instance, consider the Triad [$\delta_6, \delta_{11}, \delta_{14}$]. If this Triad is constructed geometrically, as in above **Figure 2**, then it is observed that:

$$2\theta_1 = (\theta_1 + \theta_2 + \theta_3) = \text{The Smaller Acute Angle of 3-4-5 Pythagorean Triangle i.e. } \arctan \frac{3}{4}$$

$$2\Psi_n - 90^\circ = (\Psi_n + \Psi_m + \Psi_k) - 180^\circ = \text{The Larger Acute Angle of 3-4-5 Pythagorean Triangle i.e. } \arctan \frac{4}{3}$$

Likewise, Right Triangle for 3rd Metallic Mean (and all Triads with $n=3$), or Right Triangle for 10th Metallic Mean (and Triads with $n=10$), are all related to 5-12-13 Pythagorean Triple, and so on.

Several such intriguing properties of Metallic Means and their abovementioned TRIADS are described in details in the works mentioned below in the References. For instance, the presence of these TRIADS of Metallic Ratios in Pascal's Triangle [10]; these TRIADS are found to be closely associated with Pythagorean Triples and Pythagorean Primes [2][11]; the geometric substantiation of Metallic Ratios and their TRIADS [2] [6] [7] [8] [9]; and special positions of Integers 3, 6 and 9 in the realm of Metallic Means [2][12]. Further, all imperical formulae those provide the precise mathematical relations between different Metallic Means are described in the work mentioned in Reference [1].

Conclusion:

This paper synergized the geometry of all Metallic Means and the mathematical formula that provides the precise correlations between different Metallic Ratios. The synergy between these two aspects of Metallic Ratios provided the geometric substantiation of the "TRIADS of Metallic Means" in an intriguing manner, as communicated in this paper.

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