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Approximation of New Sequence of Integral Type Operators with two Parameters

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Abstract

In our paper, we provide and study a new sequence of positive and linear operators of integral type $M_{n,r,s}(f;y)$. This sequence depends on two parameters, positive integers r and s. We mention some of the properties of this sequence and describe a Voronovskaja type asymptotic formula. Besides, we find the error estimates of this approximation in terms of the modulus of continuity. lastly, we introduce a numerical example and compare the results obtained.

Keywords: positive and linear operators, Voronovskaja-type asymptotic formula, Modulus of continuity.

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1. Introduction

Szasz in 1950, [10] introduced a sequence of positive and linear operators to approximate the unbounded continuous functions in the interval $[0,\infty)$ as:

$$Z_n(f;y) = \sum_{l=0}^{\infty} q_{n,k}(y) f\left(\frac{l}{n}\right), \tag{1.1}$$

where $q_{n,k}(y) = \frac{(ny)^l}{l! e^{ny}}, y \in [0,\infty)$.

After that, several researchers are modified for many sequences of operators [2], [3], and [4].

Rempulska and et.al. in 2009,[9] studied the following sequence of improvement Szasz -Mirakyan operators $Z_{n,r}(f;y)$ as:

$$Z_{n,r}(f(\tau);y) = \frac{1}{A_r(ny)} \sum_{l=0}^{\infty} \frac{(ny)^{rl}}{(rl)!} f\left(\frac{rl}{n}\right), \tag{1.2}$$

$y \in [0,\infty), n \in N = \{1,2,\dots\}$, and for every fixed $r \in N$,

where, $A_r(y) = \sum_{l=0}^{\infty} \frac{y^{rl}}{(rl)!}$

clearly $A_1(y) = e^y and\ A_2(y) = \cosh(y)$.

After that, many researchers presented various studies in this aspect as [1], [7], [8], and [11]

Mohammad and Hassan in 2019 [6] introduce a new sequence of integral type operators on the space

$C_\alpha[0,\infty) = \{f \in C[0,\infty): |f(\tau)| = O(e^{\alpha\tau}), \text{for some } \alpha > 0\}$

and the norm $\|f\|_{C_\alpha[0,\infty)} = sup_{\tau\in[0,\infty)} |f(\tau)| e^{-\alpha\tau}$ as:

$$M_{n,r}(f(\tau);y) = \frac{1}{A_r(ny)} \int_0^y A_r'(n\tau) f(\tau) d\tau \tag{1.3}$$

In the percent paper, we generalized the sequence (1.3) on the space $C_\alpha[0,\infty)$as:

$$M_{n,r,s}(f(\tau);y)=\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)f(y+(\tau-y)^s)d\tau \qquad (1.4)$$

where $r,s\in N$ are parameters. We denote, $A_{r,m}(y)=\sum_{l=0}^{\infty}\frac{y^{rl+m}}{(rl+m)!}, m\in N^0$.

Such that, $M_{n,r,s}(f(\tau);y)=M_{n,r}(f(\tau);y)$ at $s=1$.

We study the properties of the sequence $M_{n,r,s}(f;y)$, also, we discuss a formula of Voronovskaja and estimate the error in view of the modulus of continuity. A numerical example for the sequence $M_{n,r,s}(f;y)$ is given for the test function $f(\tau)=\sin(10\tau)$ where $\tau\in[0,2]$, Finally, we discuss the results.

2. Auxiliary Results

Here, we introduce some properties for the sequence $M_{n,r,s}(f;y)$

Lemma 2.1

Let $n,r\in N$ and $y\in[0,\infty)$ we get:

1. $\lim\limits_{n\to\infty}\frac{A_{r,i}(ny)}{A_r(ny)}=1, for\ i\in N$
2. $\lim\limits_{n\to\infty}\frac{n^i}{A_r(ny)}=0, i\in N^0=\{0,1,2,\dots\}$

Lemma 2.2

Let $y\in[0,\infty)$ we have:

(i) $M_{n,r,s}(1;y)=1-\frac{1}{A_r(ny)}\to 1\ as\ n\to\infty$;

(ii) $M_{n,r,s}(\tau;y)=y\left(1-\frac{1}{A_r(ny)}\right)-\frac{(-y)^s}{A_r(ny)}+\frac{(-1)^s s!}{n^s}\frac{A_{r,s}(ny)}{A_r(ny)}\to y\ as\ n\to\infty$;

(iii) $M_{n,r,s}(\tau^2;y)=y^2\left(1-\frac{1}{A_r(ny)}\right)-\frac{(-1)^s 2y}{A_r(ny)}+\left\{y^s-\frac{s!}{n^s}A_{r,s}(ny)\right\}-\frac{1}{A_r(ny)}\left\{y^{2s}-\frac{(2s)!}{n^{2s}}A_{r,2s}(ny)\right\}\to y^2\ as\ n\to\infty.$

Proof

We can easily prove this lemma by direct computation.

Definition 2.1

For $k\in N^0$ the $k-th$ order moment $T^s_{n,k,r}(y)$ for the operator $M_{n,r,s}(f(\tau);y)$ is define as:

$$T^s_{n,k,r}(y)=M_{n,r,s}((\tau-y)^k;y)=\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)\,(\tau-y)^{ks}d\tau$$

Lemma 2.3

For the moment function $T^s_{n,k,r}(y)$, we obtain:

(1) $T^s_{n,0,r}(y)=1-\frac{1}{A_r(ny)}$;

(2) $T^s_{n,1,r}(y)=\frac{(-1)^s}{A_r(ny)}\left\{\frac{s!}{n^s}A_{r,s}(ny)-y^s\right\}$;

(3) $T^s_{n,2,r}(y)=\frac{1}{A_r(ny)}\left\{\frac{(2s)!}{n^{2s}}A_{r,2s}(ny)-y^{2s}\right\}$;

(4) $T^s_{n,k,r}(y)=\frac{(-1)^{ks}}{A_r(ny)}\left\{\frac{(ks)!}{n^{ks}}A_{r,ks}(ny)-y^{ks}\right\}, k\geq 1.$

Further, we have:

(i) $T^s_{n,k,r}(y)$ is a polynomial in y not exceed of ks, whenever n is sufficiently large.

(ii) for every $y\in[0,\infty), T^s_{n,k,r}(y)=O(n^{-ks})$.

Proof

By using Lemma 2.2 we can prove this lemma immediately.

Lemma 2.4

For each real number $\alpha, \delta > 0$ and $[a,b] \subset (0,\infty)$

$$\sup_{y\in[a,b]} \left| \int_{|y-t\geq\delta} \frac{A_r^{'}(n\tau)}{A_r(ny)} e^{\alpha\tau} d\tau \right| = O\left(n^{-\lambda}\right), \lambda > 0$$

Proof

We can prove this lemma by using Taylor's expansion and (2) with $k = i$ in lemma 2.3,

3. Main Results

Theorem 3.1

For $f \in C_\alpha[0,\infty)$, the sequence of positive and linear operators $M_{n,r}(f(\tau);y)$ is converge uniformly to f as $n \to \infty$.

Proof

From lemmas 2.1 and 2.2 we have:

$M_{n,r,s}(1;y) \to 1$ uniformly as $n \to \infty$ (3.1)

$M_{n,r,s}(\tau;y) \to y$ uniformly as $n \to \infty$ (3.2)

$M_{n,r,s}\left(\tau^2;y\right) \to y^2$ uniformly as $n \to \infty$ (3.3)

Hence, from Korovkin theorem [5], we obtaine:

$$\lim_{n\to\infty} M_{n,r,s}\left(f(\tau);y\right) = f(y)\ uniformly\ as\ n \to \infty \quad (3.4)$$

Theorem 3.2

For $f \in C_\alpha[0,\infty), \alpha > 0$ and f has two derivatives at a point $y \in (0,\infty)$, we have,

$$\lim_{n\to\infty} n^s \left(M_{n,r,s}(f(\tau);y) - f(y)\right) = (-1)^s s!\, f^{'}(y)$$

Proof

Applying Taylor's formula for the function f, we obtain:

$f(\tau) = f(y) + (\tau - y)f'(y) + \frac{(\tau-y)^2}{2} f^{''}(y) + \xi(\tau,y)(\tau - y)^2$, where $\xi(\tau,y) \to 0\ as\ \tau \to y$.

Operating by $M_{n,r,s}$ we have:

$$M_{n,r,s}(f(\tau);y) = M_{n,r,s}(1;y)f(y) + M_{n,r,s}\left((\tau - y);y\right)f^{'}(y)$$

$$+\frac{1}{2}M_{n,r,s}((\tau - y)^2;y)f^{''}(y) + M_{n,r,s}(\xi(\tau,y)(\tau - y)^2;y)$$

$$M_{n,r,s}(f(\tau);y) = T_{n,0,r}^{s}(y)f(y) + T_{n,1,r}^{s}(y)f^{'}(y) + \frac{1}{2}T_{n,2,r}^{s}(y)f^{''}(y) + R(n,y)$$

Where, $R(n,y) = M_{n,r,s}(\xi(t,y)\left(\tau - y)^2;y\right)$,

from Lemmas 2.1 and 2.3 we get

$$\lim_{n\to\infty} n^s \left(M_{n,r,s}(f(\tau);y) - f(y)\right) = (-1)^s s!\, f^{'}(y) + \lim_{n\to\infty} n^s R(n,y).$$

Now, we prove that $\lim_{n\to\infty} n^s R(n,y) \to 0\ as\ n \to \infty$.

since $\xi(y + (\tau - y)^s, y) \to 0$ as $\tau \to y$, for a given $\varepsilon > 0$, $\exists\delta > 0$ such that $|\xi(y + (\tau - y)^s, y)| < \varepsilon$, whenever

$0 < |(\tau - y)^s| < \delta$, for $|(\tau - y)^s| \geq \delta$, $\exists C > 0$ such that $|\xi(y + (\tau - y)^s, y)|(\tau - y)^{2s} \leq Ce^{\alpha(y+(\tau-y)^S)}$. Hence,

$$n^s|R(n,y)| \leq \frac{n^s}{A_r(ny)} \int_{y-t<\delta} A_r^{'}(n\tau)|\xi(y + (\tau - y)^s, y)|(\tau - y)^{2s}\, d\tau + \frac{n^s}{A_r(ny)} \int_{y-\tau\geq\delta} A_r^{'}(n\tau)|\xi(y + (\tau - y)^s, y)|(\tau - y)^{2s}\, d\tau$$

$:= I_1 + I_2$.

$I_1 \leq n^s\, \varepsilon\, T_{n,2,r}^{s}(y)$

$= n^s \varepsilon O(n^{-2s})$.

$= \varepsilon O(n^{-s})$.

$$I_2 \leq \frac{n^s}{A_r(ny)} \int_{y-t\geq\delta} A_r^{'}(n\tau)C\, e^{\alpha(y+(\tau-y)^S)} d\tau$$

Then, from Lemma 2.4, we have:

$I_2 = Cn^s O(n^{-\lambda})$

$= CO(n^{s-\lambda})$

$= o(1), \lambda > s$

Since $\varepsilon > 0$ is arbitrary, so we get $I_1 \to 0$ as $n \to \infty$. As well $I_2 \to 0\ as\ n \to \infty$.

Then, $\lim_{n\to\infty} n^s\, R(n,y) \to 0\ as\ n \to \infty$.

Now, we find an error for $f \in C_\alpha[0,\infty)$, $f \in C_\alpha^1[0,\infty)$ and $f \in C_\alpha^2[0,\infty)$.

Theorem 3.3

If $f \in C_\alpha[0,\infty)$, $\alpha > 0$ and $x \in [0,\infty)$, then:

$$|M_{n,r,s}(f(\tau); y) - f(y)| \leq \left(2 + \frac{1}{A_r(ny)}\right)\omega_f(\mu_s) + \frac{|f(y)|}{A_r(ny)}$$

Where, $\mu_s = M_{n,r,s}(y - \tau; y) = \frac{(-1)^s}{A_r(ny)}\left\{y^s - \frac{s!}{n^s}A_{r,s}(ny)\right\}$

Proof

For any continuous function f, the modulus of continuity is defined as follows:

$\omega_f(\delta) = \omega(f,\delta) = \max_{|\tau-y|\leq\delta} |f(\tau) - f(y)|$

since $\omega_f(\delta) \geq |f(\tau) - f(y)|$ whenever $|\tau - y| \leq \delta$,

and $f(\tau) - f(y) \leq |f(\tau) - f(y)|$

then, $f(\tau) - f(y) \leq \omega_f(|\tau - y|)$

hence, $f(\tau) - f(y) \leq \left(1 + \frac{|\tau-y|}{\delta}\right)\omega_f(\delta)$

$$M_{n,r,s}(f(\tau); y) \leq \left(M_{n,r,s}(1; y) + \frac{1}{\delta}M_{n,r,s}(y - \tau; y)\right)\omega_f(\delta) + f(x)M_{n,r,s}(1; y)$$

From Lemma 2.2 we obtain:

$M_{n,r,s}(f(\tau); y) - f(y) \leq \left(1 - \frac{1}{A_r(ny)} + \frac{\mu_s}{\delta}\right)\omega_f(\delta) - \frac{f(y)}{A_r(ny)}$,

$$|M_{n,r,s}(f(\tau); y) - f(y)| \leq \left(1 + \frac{1}{A_r(ny)} + \frac{\mu_s}{\delta}\right)\omega_f(\delta) + \frac{|f(y)|}{A_r(ny)}$$

Let $\delta = \mu_s$, hence

$$\left|M_{n,r,s}(f(\tau);y) - f(y)\right| \leq \left(2 + \frac{1}{A_r(ny)}\right)\omega_f(\mu_s) + \frac{|f(y)|}{A_r(ny)}$$

Theorem 3.4

Suppose that $f \in C_\alpha^1[0,\infty), \alpha > 0$ and $y \in [0,\infty)$, then:

$$\left|M_{n,r,s}(f(\tau);y) - f(y)\right| \leq \left(\frac{1}{A_r(ny)}\right)|f(y)| + \mu_s\left|f^{'}(y)\right| + (\mu_s + 1)\omega_{f^{'}}(\delta)$$

Where, $\mu_s = M_{n,r,s}(y-\tau;y) = \frac{(-1)^s}{A_r(ny)}\left\{y^s - \frac{s!}{n^s}A_{r,s}(ny)\right\}$.

Proof

By Taylors expansion for the function $f(\tau)$ about $\tau = y$, we obtain:

$$f(\tau) - f(y) = (\tau - y)f^{'}(y) + (\tau - y)\left(f^{'}(\xi) - f^{'}(y)\right), \text{ where } \xi \in (\tau, y)$$

Hence,

$$f(\tau) - f(y) \leq |\tau - y|f^{'}(y) + |\tau - y|\omega_{f^{'}}(|\tau - y|)$$

$$\leq |\tau - y|f^{'}(y) + |\tau - y|\left(1 + \frac{|\tau - y|}{\delta}\right)\omega_{f^{'}}(\delta)$$

$$\begin{aligned} M_{n,r,s}(f(\tau);y) - M_{n,r,s}(1;y)f(y) \\ &\leq M_{n,r,s}((y-\tau);y)f^{'}(y) \\ &+ \left\{M_{n,r,s}((y-\tau);y) + \frac{1}{\delta}M_{n,r,s}((\tau-y)^2;y)\right\}\omega_{f^{'}}(\delta) \end{aligned}$$

Consequently,

$$M_{n,r,s}(f(\tau);y) - f(y) \leq \frac{1}{A_r(ny)}f(y) + \mu_s f^{'}(y) + \left(\mu_s + \frac{\gamma_s}{\delta}\right)\omega_{f^{'}}(\delta)$$

Where $\gamma_s = M_{n,r,s}((\tau-y)^2;y) = \frac{1}{A_r(ny)}\left\{\frac{(2s)!}{n^{2s}}A_{r,2s}(ny) - y^{2s}\right\}$,

if we take $\delta = \gamma_s$ then,

$$M_{n,r,s}(f(\tau);y) - f(y) \leq \frac{1}{A_r(ny)}f(y) + \mu_s f^{'}(y) + (\mu_s + 1)\omega_{f^{'}}(\gamma_s)$$

then,

$$\left|M_{n,r,s}(f(\tau);y) - f(y)\right| \leq \frac{1}{A_r(ny)}|f(y)| + \mu_s\left|f^{'}(y)\right| + (\mu_s + 1)\omega_{f^{'}}(\gamma_s)$$

Theorem 3.5

Assume that $f \in C_\alpha^2[0,\infty), \alpha > 0$ then for any $n \in N$ it follows that:

$$\left|M_{n,r,s}(f(\tau);y) - f(y)\right| \leq \frac{1}{A_r(ny)}\|f(y)\| + \mu_s\left\|f^{'}(y)\right\| + \frac{\gamma_s}{2}\left\|f^{''}(y)\right\|,$$

Where, $\mu_s = M_{n,r,s}(y-\tau;y) = \frac{(-1)^s}{A_r(ny)}\left\{y^s - \frac{s!}{n^s}A_{r,s}(ny)\right\}$ and $\gamma_s = M_{n,r,s}((\tau-y)^2;y)$

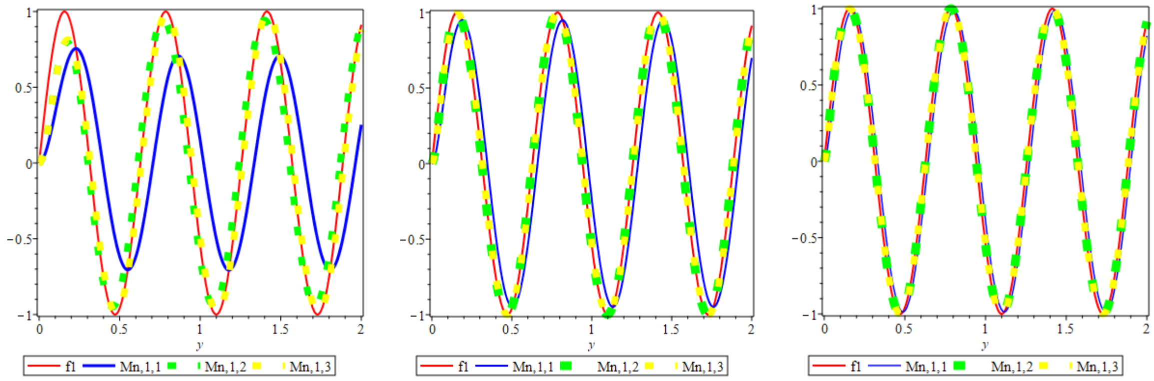
$$= \frac{1}{A_r(ny)}\left\{\frac{(2s)!}{n^{2s}}A_{r,2s}(ny) - y^{2s}\right\}$$

Proof

By using the same manner in theorem 3.4 and by applying Lemma 2.3 we get the proof.

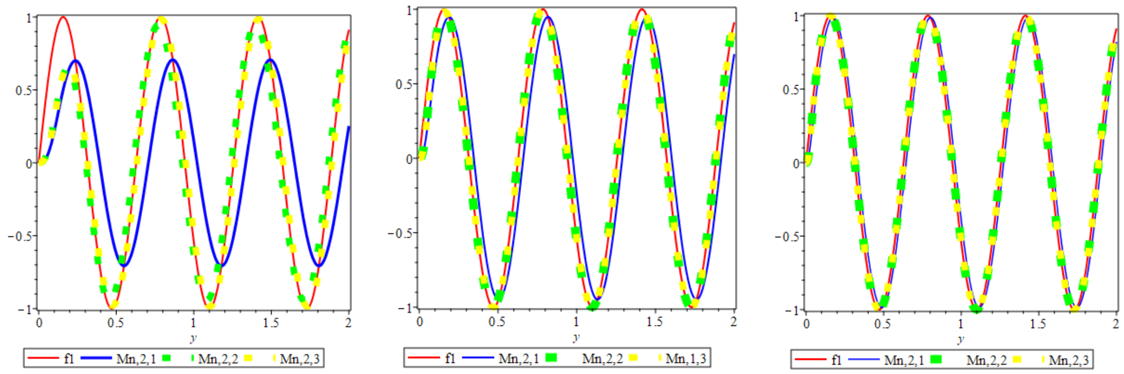
Example

For $n = 10, 30, 60$ the sequences $M_{n,r,s}(f_1; y)$ converge to the function $f_1(\tau) = \sin(10\tau), \tau \in [0,2]$ at s=1, 2, and 3, and $r = 1,2$ in figures 1, and 2. The comparison between these sequences by the error functions $E(y) = |M_{n,r,s}(f_1; y) - f_1(\tau)|$ in figures 3 and 4.



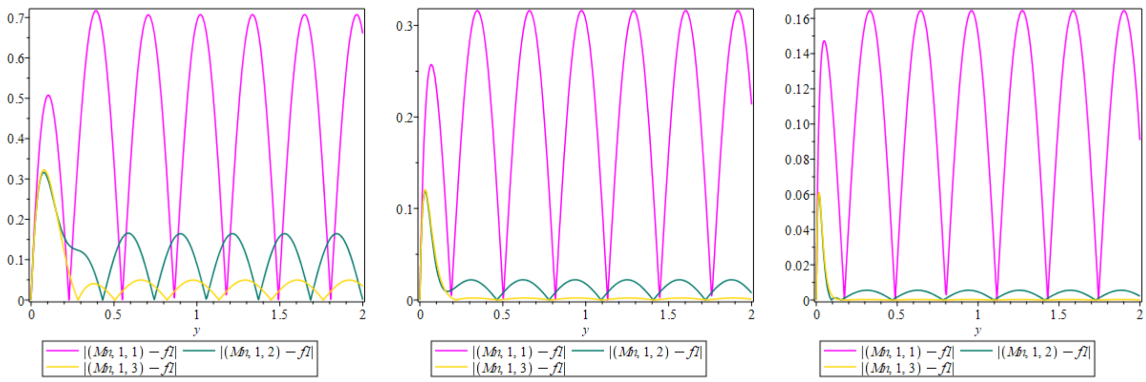
(a1) (b1) (c1)

Fig. 1: The convergence of $M_{n,r,s}(f_1(\tau); y)$ at r=1 with s=1,2 and 3 at n=10 in (a1), at n=30 in (b1), and n=60 in (c1) respectively.



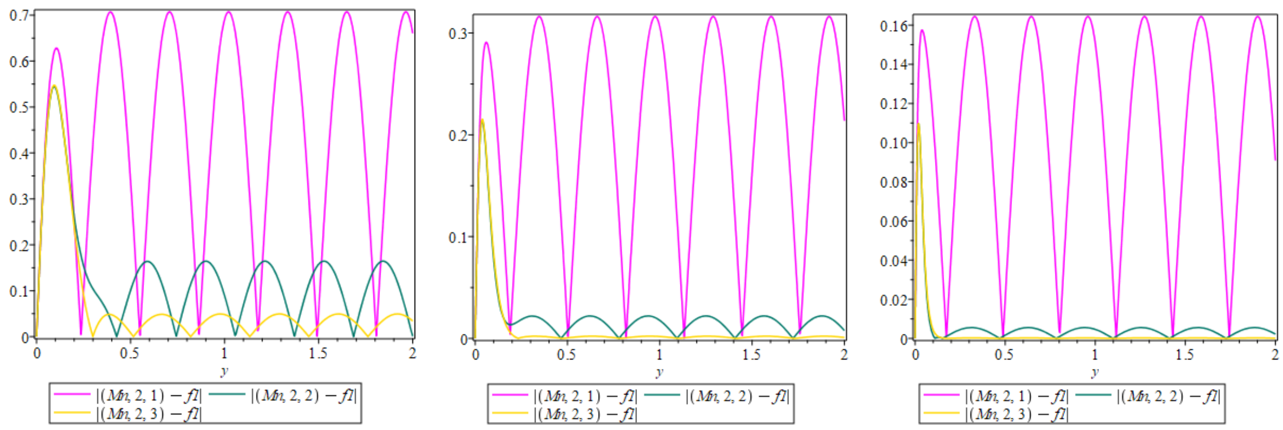
(a2) (b2) (c2)

Fig. 2: The convergence of $M_{n,r,s}(f_1(\tau); y)$ at r=2 with s=1,2 and 3 at n=10 in ($a2$), at n=30 in ($b2$), and n=60 in ($c2$) respectively.



($a3$) ($b3$) ($c3$)

Fig. 3: The error function $E(y)$ for the sequence $M_{n,r,s}(f_1(\tau); y)$ at r=1 with s=1,2 and 3 at n=10 in ($a3$), *at* n=30 in ($b3$), and n=60 in ($c3$) respectively.



($a4$) ($b4$) ($c4$)

Fig. 4: The error function $E(y)$ for the sequence $M_{n,r,s}(f_1(\tau);y)$ at r=2 with s=1,2 and 3 at n=10 in ($a4$), at n=30 in ($b4$), and n=60 in ($c4$) respectively.

4. Conclusions

We gave a numerical example for the sequence $M_{n,r,s}(f;y)$ to approximate the test function $f(\tau) = \sin(10\tau)$ in cases $n = 10, 30$ *and* 60 with arbitrary r= *1,2* respectively for some values of $s = 1,2,3$ and compared the results, it turns out the sequence $M_{n,r,s}(f(\tau);y)$ gave better results when s is bigger.

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