New Conditions of The Existence of Fixed Point in $\Delta -$ Ordered Banach Algebra

Boushra Youssif Hussein

Department of Mathematics, College of Education, University of AL-Qadisiyah

Boushra.alshebani@qu.edu.iq

Abstract

The main idea is to construct a new algebra and find new necessary and sufficient conditions equivalent to the existence of fixed point. In this work, an algebra is constructed, called Δ- ordered Banach algebra, we define convergent in this new space, Topological structure on $\Delta -$ ordered Banach Algebra and prove this as Housdorff space. Also, we define new conditions as $\Delta - lipshtiz$, , $\Delta - contraction\ \ conditions$ in this algebra construct, we prove this condition is the existence and uniqueness results of the fixed point. In this paper , we prove a common fixed point if the self-functions satisfy the new condition which is called $\varphi - contraction$.

Keywords: Fixed point, Ordered Banach algebra**,** lipschitz mapping, and contraction mappimg

Introduction

It is known Banach contraction principle and a number of generality in background of metric spaces play a fundamental role for several complications of functional analysis, differential and integral equations.

Gahler (1963) [6] presented the notion of 2-metric spaces as a generalization of an usual metric space. Gahler proved that geometrically $d(a, b, c)$ represents the region of a triangle formed by the point $a, b, c\ \epsilon X$ equally its vertices.

An usual metric space is a continuous function, but Ha, Cho and While (1988) [15] examined that a 2- metric space is not a continuous mapping. Dhage (1984) [5] introduced the notion of a D- metric space as a generality of a 2- metric space; and studied the topological properties of D- metric space

Mustafa and Sim (2006) [17] introduced a newfangled metric called G-metric space . They show the topological constructions of Dhages [4] work unacceptable, after Sedghi, Shobe and Zhom (2007) [20] presented concept, which is named D^*- metric space, but Fernardcz ,Sle, Saxena, Malviya and Kuman (2017)[13] generalized an S-metric space to A-metric space.

Many researchers have their consideration to generalizing mertic (see Yan and Shao Yuan on (2011) [25] , Sastry, Srinivas, Chandra and Balaiah (2011) [14], Kim and Soo (2012)[20], Dey and Saha (2013) [4],

Liu and Xu (2013) [8] introduced some concepts of a cone metric space over Banach algebra. Some researchers then developed many concepts as, Nashine and Altun, (2012)[9], Tiwari and Dubey (2013) [22], Arun and Zaheer (2014) [3]. But Nashine and Altun (2012) [10] defined cone metric spaces and proved some fixed point theorems of contractive maps in such a space using the normality condition. Also, Rahimi & Soleimani (2014) [12] used the notion ordered cone metric space.

But some scholars have attention about fixed point theorem such as Badshah, Bhagatand and Shukla(2016)[23] how introduced some fixed point theorem for α-ϕ- metric mapping in 2- metric spaces and Ma, Jiang and Hongkaisun (2014) [24] state fixed point theorem on C*-algebra valued metric spaces.

The point x- that satisfies the equation x = T (x) is called a fixed point of the function T which is considered the root of the equation above. To find this root, we first find an initial holding value of x_0. Then, we calculate the value of the function T in x_0 to get another root called x_1 that is x_1 = T (x_0) ; and then repeat the process can get a new approximate value x_2 = T (x_1). Thus, a sequence of root values can be generated by applying the formula x_{n+1} = T (x_n) for n = 0.1,2, …

The fixed point a in the equation above represents the distance of the intersection point of the curves of y = x, y = T (x) for each axis x, y . If x_0 is the initial fixed point, then T(x_0) is the length of the column from x_0 on the x axis until it intersects the curve of the T- function and since the points on the rectangle y = x are equal to the distance from both axes y and x, so the line passing at the point (x_0, T (x_0)) rectangle the x-axis will intersect the line y=x in the x- axis, represent x_1 where

$$x_1=T(x_0)$$

In a similar way, we find the remaining points where x_{n+1} = T (x_n). Here, we ask the following question: How do we choose the function T to ensure that the generated values are converged from the repeated formula x_{n+1} = T (x_n)?

To answer the question , we can prove the existence and uniqueness of fixed point under some new conditions by constructing a new algebra called $\Delta - ordered\ Banach\ algebra$.

1- Δ- Ordered Banach Algebra

We start this section by a definition of Banach algebra.

"Definition (2.1)[2] :let E is a linear space over field of real numbers .E is called Banach algebra if E is Banach space with an operation of multiplication is defined as following :for $x, y, z \in A$, for all $\alpha \in R$

1) $(xy)z = x(yz)$

2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$

3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

4) $\| xy \| \leq \| x \| \| y \|$

We consider a Banach algebra has an identity, that is $ex = xe$ for all $x \epsilon E$. (Multiplicative identity)

If there is an element $y \epsilon A$ such that $= yx = e$, $y \epsilon E$ is called inverse of x and denoted by x^{-1}."

"Proposition 2.2 [19]: Let E be Banach algebra has a unite e, $x\, \epsilon E$. If the condition spectral radius $\sigma_\epsilon(x)$<1$(for\ all\ \varepsilon > 0)$, then

$$(e-x)^{-1} = \sum_{i=0}^{\infty} a^i$$

"Remark 2.3 [19]: Let E be Banach algebra with spectral radius $\sigma_\epsilon(x)$ of x satisfy $\sigma_\epsilon(x) \leq ||x||$."

"Remark 2.4 [2]: If $\sigma_\epsilon(x) < 1$, then $\| x^n \| \to 0$ as $n \to \infty$".

"Lemma 2.5 [2]: If E is a real Banach algebra with cone C and if $o \preccurlyeq u \preccurlyeq c$ for each $o \preccurlyeq c$, therefore $u = o$."

"Lemma 2.6 [2]: Let C be a cone and $a \preccurlyeq b + c$ for $c \epsilon C$, then $a \preccurlyeq b$.

A sub set C of E is called a algebra cone of E if

1) C non- empty closed and $\{ o, e\} \subset C$

2) $\alpha a + \beta b \in C$ for all $\alpha, \beta > 0$

3) $x.y\epsilon C$

4) $C \cap (-C) = \{o\}$."

" We can define a preference ordering $\preccurlyeq$ with respect to C by $x \preccurlyeq y$ iff$y - x\epsilon C$. $x \prec y$ with stand for $x \preccurlyeq y$ and $x \npreccurlyeq y$ the cone C is called normal if there exist $N > 0$ such that, for all $x, y\epsilon E$

$o \preccurlyeq x \preccurlyeq y \Longrightarrow \| x \| \preccurlyeq N \| y \|$."

Now, we define a new construction called Δ- ordered Banach algebra.

Definition 2.7: Let X be a non-empty. A function $\Delta_\lambda: [0, \infty) \times X \times X \to E$ is called an Δ- metric on X if

1) $\Delta(\lambda, x, y) \succcurlyeq o$ for $x, y\epsilon X$, $\lambda \geq 0$

2) $x = y$ if and only if $\Delta(\lambda, x, y) = o$

3) $\Delta(\lambda, x, y) = \Delta(\lambda, y, x)$

4)$\Delta(\lambda, x, y) \preccurlyeq \Delta(\mu, y, a)$ for ,$\mu > \lambda > 0$ and $x, y, a\epsilon X$

5))$\Delta(\lambda + \mu, x, y) \preccurlyeq \Delta(\lambda, x, y) + \Delta(\mu, y, a)$

The triple (X, E, Δ) is called Δ- ordered Banach algebra.

Example 2.8: Let X be locally compact Housdorff space , $C(X) = \{f | f: X \to R, continuous\ function\}$, and $C^+(X) = \{f \in C(X): f(x) \geq 0\ for\ all\ x \in X\}$, define multiplication in the natural way. Therefore C(X) with supermom norm is ordered Banach algebra. It is obvious $that$ (C(x), X, Δ) is $\Delta -$ $ordered\ Banach\ algebra$ where

$\Delta: [0, \infty) \times X \times X \to C(X)$ by $\Delta(\lambda, a, b) = sup\left| f(a) - f(b) \right| e^\lambda$

2- Topological structure on $\Delta -$ ordered Banach Algebra

Definition 3.1: Let (X, E, C) be Δ –ordered Banach algebra . For all $x\epsilon X$, for all $c \succ o$, $the\ set$ $B_\Delta(\lambda, x, c) = \{y\epsilon X: \Delta(\lambda, x, y) \prec c\}$ is called Δ-ball with and radius $c \succ o$ and admits x .

And put $\beta = \{B_\Delta(\lambda, x, c): x\epsilon X, and\ c \succ o\}$.

Theorem 3.2: Let (E, C) be ordered Banach algebra, then (X, E, Δ) is a Housdorff space.

Proof:- Let (E, X, Δ) be a $\Delta -$ ordered Banach algebra. Let $x, y\ \epsilon X$ with $x \neq y$, $\lambda, \mu \geq 0$, $we\ take\ c = \Delta(\lambda + \mu, x, y), U = B\left(\lambda, x, \frac{c}{2}\right), V = B\left(\mu, y, \frac{c}{2}\right)$.

Then $x\epsilon U$ and $y\epsilon V$. We support $U \cap V \neq \emptyset$. There exist $a\epsilon U \cap V$.

But $\Delta(\lambda + \mu, x, a) \preccurlyeq \Delta(\lambda, x, a) + \Delta(\mu, y, a) \preccurlyeq \frac{c}{2} + \frac{c}{2} = c$.

That is $c \prec c$ and this contradiction.

Then, (X, E, Δ) is a Housdorff space.

Definition 3.3: Let (X, E, Δ) be $a\,\Delta$ −ordered Banach algebra. A sequence $\{x_n\}$ in (X, Δ) converges to a point x if for every $c \in E$ with $c \succ 0$, there exist a positive integer N_0 such that $\Delta(\lambda, x_n, x) \prec c$ for $n \geq N_0$, we denoted by $\lim_{n\to\infty} x_n = x$ ($x_n \to x$ as $(n \to \infty)$).

Definition 3.4: Let (C, E, Δ) be $a\,\Delta$ −ordered Banach algebra. A sequence $\{x_n\}$ is said to be Cauchy sequence if for each $c \succ 0$ there exists a positive integer N_0 such that $\Delta(\lambda, x_n, x_m) \prec c$ for all $n,\ m \geq N_0$.

Examples 3.5: *Let* $(X, C(X), \Delta)$ a Δ − ordered Banach algebra in example (3.2) , take the set of rational numbers Q.

Define $\Delta = [0, \infty) \times X \times X \to C(X)$ is in example. Let $\{x_t\}$ be a sequence defined by $\alpha_t = (1 + \frac{1}{t})^t$. We note that $x_t \epsilon \mathbb{Q}$ for each $t \epsilon \mathbb{Z}$, note that $\Delta(\lambda, x_t, x_k) = \left| f(x_t) - f(x_k) \right| e^{-\lambda}$

$= \left| \left(1 + \frac{1}{t}\right)^t - (1 + \frac{1}{k})^k \right| e^{\lambda}$ as $t, k \to \infty$

$\Delta(\lambda, x_t, x_k) \to 0$

That is for each $c \succ \theta$, there is $N_0 \in Z^+$ such that $\Delta(\lambda, a_t, a_k) \prec c$ for all $t, k \geq N_0$.

Thus, $\{a_t\}$ is a Cauchy sequence, but $a_t \to e$ as $\to \infty$, $e \notin \mathbb{Q}$. Hence, $\{a_t\}$ is not convergent.

Definition 3.7: Let (X, E, Δ) and (X', E', Δ') are Δ- ordered Banach algebra . A mapping $f: X \to X'$ is said to be continuous at $x \in X$ when ever $\{x_n\}$ convergent to x, then $\{f(x_n)\}$ is convergent to $f(x)$.

Definition 3.8: Let (X, E, Δ) be Δ −ordered Banach algebra , (X, E, Δ) is called complete if for each Cauchy sequence is convergent in X.

Definition 3.9: Let (X, E, Δ) be Δ −ordered Banach algebra. A map $\mathrm{T}: X \to X$ is called Lipchitz if for all $c \succ 0$, there exist a vector $N \epsilon C$ with $\sigma_\epsilon(N) \prec 1$ for each $x, y \epsilon X$,

$\Delta(\lambda, \mathrm{T}_a, \mathrm{T}_b) \preccurlyeq \mathrm{N}.\Delta(\mu, a, b)$ for all $x, y \epsilon X$ and $\lambda \leq \mu$

Example 3.10: Let $([0, \infty), C(X), \Delta)$ be a Δ- ordered Banach algebra. Define $T: X \to X$ as follows $T(a) = \frac{a}{2}$

$\Delta(\lambda, \mathrm{T}_a, \mathrm{T}_b) = sup \left| f(\mathrm{T}_a)_{-} f(\mathrm{T}_b) \right| e^{\lambda}$

$= sup \left| f \circ T(a) - f \circ T(b) \right| e^{\lambda} = sup \left| f\left(\frac{a}{2}\right) - f\left(\frac{b}{2}\right) \right| e^{\lambda}$

$= \frac{1}{2} sup \left| f(a) - f(b) \right| e^{\lambda}$

$\Delta(\mu, a, b) = sup \left| f(a) - f(b) \right| e^{\mu}$

That is T is a Lipschitz map in X

Definition 3.11: Let (X, E, Δ)be Δ −ordered Banach algebra. A sequence $\{x_t\}$ is said to be m- sequence if for all $m \succ 0$, there exists $t \epsilon x_t$ such that $x_t \prec m$ for all $n \geq t$.

Lemma 3.12: Let (X, E, Δ)be Δ − ordered Banach algebra. $\{m x_t\}$ is a m- sequence for all $c \succ 0$ if the sequence $\{x_t\}$ is a m- sequence in$C$.

Proof:- Suppose $\{x_t\}$ is a m- sequence for all $c \succ 0$, there exists $t \epsilon \mathbb{Z}^+$ such that $x_t \prec c$ for $n > t$. For all $c \succ 0$, $m x_t \preccurlyeq mc$ by take $\frac{c}{m} = t$.

3- Main Results

Definition 4.1: Let (X, E, Δ) be Δ- ordered Banach algebra. $T: X \to X$ holds the contradiction condition if $\Delta(\lambda, \mathrm{T}_x, \mathrm{T}_y) \preccurlyeq t^n \Delta(\frac{\lambda}{2^n}, x_1, x_0)$

Theorem 4.2: Let (X, E, Δ) be Δ- ordered Banach algebra. Suppose $T: X \to X$ holds the Δ-contradiction condition

$$\Delta(\lambda, \mathrm{T}_x, \mathrm{T}_y) \preccurlyeq m_1 \Delta\left(\left(\frac{\lambda}{4}\right), x, \mathrm{T}_x\right) + m_2 \Delta\left(\left(\frac{\lambda}{4}\right), \mathrm{T}_x, y\right) + m_3 \Delta\left(\left(\frac{\lambda}{4}\right), x, \mathrm{T}_y\right) + m_4 \Delta(\left(\frac{\lambda}{4}\right), y, \mathrm{T}_y)$$

where $0 < \sum_{i=1}^{4} m_i \leq 1$, for i=1,2,3,4 Then T is a unique fixed point in X.

Proof: choose $x_0 \epsilon X$, $x_1 = T_{x_0}$ and $x_{n+1} = \mathrm{T}_{x_n}$

Take $0 < m_i \leq 1 \,, for\ i = 1{,}2{,}3{,}4$

First we see,

$$\Delta(\lambda, x_{n+1}, x_n) = \Delta(\lambda, \mathrm{T}_{x_n}, \mathrm{T}_{x_{n-1}}) \preccurlyeq m_1 \Delta\left(\left(\frac{\lambda}{4}\right), x_n, \mathrm{T}_{x_n}\right) + m_2 \Delta\left(\left(\frac{\lambda}{4}\right), x_{n-1}, T_{x_n}\right) + m_3 \Delta\left(\left(\frac{\lambda}{4}\right), x_n, \mathrm{T}_{x_{n-1}}\right) + m_4 \Delta\left(\left(\frac{\lambda}{4}\right), x_{n-1}, \mathrm{T}_{x_{n-1}}\right) +$$

$$\preccurlyeq m_1 \Delta\left(\left(\frac{\lambda}{4}\right), x_n, x_{n+1}\right) + m_2 \Delta\left(\left(\frac{\lambda}{4}\right), x_{n-1}, x_{n+1}\right) + m_3 \Delta\left(\left(\frac{\lambda}{4}\right), x_n, x_n\right) + m_4 \Delta\left(\left(\frac{\lambda}{4}\right), x_{n-1}, x_n\right)$$

$$\preccurlyeq m_1 \Delta((\lambda/4), x_n, x_{n+1}) + m_2 \Delta\left(\left(\frac{\lambda}{4}\right), x_{n-1}, x_n\right) + m_3 \Delta((\lambda/4), x_n, x_{n+1}) + m_4 \Delta\left(\left(\frac{\lambda}{4}\right), x_{n-1}, x_n\right)]$$

$$\preccurlyeq (m_1 + m_3) \Delta\left(\left(\frac{\lambda}{4}\right), x_n, x_{n+1}\right) + (m_2 + m_4)[(\Delta\left(\left(\frac{\lambda}{4}\right), x_n, x_{n-1}\right)]$$

$$\preccurlyeq t_1 \Delta\left(\left(\frac{\lambda}{4}\right), x_n, x_{n+1}\right) + t_2 \Delta(\left(\frac{\lambda}{4}\right), x_n, x_{n-1})$$

$$\text{............} \preccurlyeq t_1^n \Delta\left(\left(\frac{\lambda}{2^n}\right), x_0, x_1\right) + t_2^n \Delta\left(\left(\frac{\lambda}{2^n}\right), x_1, x_0\right)$$

if follows that

$$\Delta(\lambda, x_{n+1}, x_n) \preccurlyeq t_1^n \Delta\left(\left(\frac{\lambda}{2^n}\right), x_0, x_1\right) + t_2^n \Delta\left(\left(\frac{\lambda}{2^n}\right), x_1, x_0\right)$$

$$\Delta(\lambda, x_{n+1}, \mathrm{x}_n) = (t_1^n + t_2^n)\, \Delta\left(\left(\frac{\lambda}{4}\right), x_0, x_1\right)$$

Put $k = (t_1^n + t_2^n)$,

$\frac{1-\varepsilon}{1+\varepsilon} \leq |k| \leq \frac{1+\varepsilon}{1-\varepsilon}$. For $0 < \epsilon < 1$

It is clearly see that $\sigma_\epsilon(k) \prec 1$

$$\Delta(\lambda, x_{n+1}, x_n) \preccurlyeq k \Delta\left(\frac{\lambda}{2^{n-1}}, x_n, x_{n-1}\right) + \cdots + k^n \Delta(\frac{\lambda}{2^{n-1}}, x_1, x_0)$$

$$\Delta(\lambda, x_n, x_{n+m}) \preccurlyeq k^m \Delta\left(\frac{\lambda}{2^{n-1}}, x_n, x_{n-1}\right) + \cdots + k^{n+m} \Delta(\frac{\lambda}{2^{n-1}}, x_1, x_0)$$

When n, $m \to \infty$ we have $\lim_{n,m\to\infty} \Delta(\lambda, x_n, x_{n+m}) = 0$

Thus $\{x_n\}$ is Cauchy sequence in (X, E, Δ)

Since (X, E, Δ) is Banach algebra

That is $(X, E, \Delta) is$ complete.

Then $\{x_n\}$ is convergent to $x^* \epsilon X$ such that $x_n \to x^*$

Next, we claim that x^* is a fixed point of T

Actually,

$\Delta(\frac{\lambda}{4}, \mathrm{T}_{x^*}, x^*) \preccurlyeq \Delta(\lambda, \mathrm{T}_{x^*}, x^*) \preccurlyeq K[\Delta\left(\frac{\lambda}{2}, x^*, \mathrm{T}_{x_n}\right) + \Delta\left(\frac{\lambda}{2}, \mathrm{T}_{x_n}, \mathrm{T}_{x^*}\right)]$

$= k\Delta\left(\frac{\lambda}{2}, x^*, x_{n+1}\right) + k\Delta(\frac{\lambda}{2}, \mathrm{T}_{x_n}, \mathrm{T}_{x^*})$

$\preccurlyeq k[m_1\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{x^*}\right) + m_2\Delta\left(\frac{\lambda}{4}, x_{n+1}, \mathrm{T}_{x^*}\right)$

$+m_3\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{x_{n+1}}\right) + m_4\Delta(\frac{\lambda}{4}, x_{n+1}, \mathrm{T}_{x_{n+1}})] + k\Delta(\frac{\lambda}{2}, x_{n+1}, x^*)$

$\preccurlyeq km_1\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{x^*}\right) + k^2m_2\Delta\left(\frac{\lambda}{4}\lambda, x_{n+1}, Tx^*\right) + k^2m_2\Delta\left(\frac{\lambda}{4}\lambda, x^*, x_{n+1}\right) + km_3\Delta_\lambda\left(\frac{\lambda}{4}\lambda, x^*, x_{n+1}\right) +$
$k^2m_4\Delta\left(\frac{\lambda}{4}, Tx^*, x^*\right) + k^2m_2\Delta(\frac{\lambda}{4}, x^*, x_n)] + k\Delta(\frac{\lambda}{2}, x_{n+1}, x^*)$

$= (km_1 + k^2m_4)\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{x^*}\right) + (k^2m_2 + k^2m_4)\Delta(\frac{\lambda}{4}, x_n, x^*)$

$+(k^2m_2 + km_3)\Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) + k\Delta(\frac{\lambda}{2}, x^*, x_{n+1})$

then

$(1 - km_1 - k^2m_4)\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{x^*}\right) \preccurlyeq (k^2m_2 + k^2m_4)\Delta\left(\frac{\lambda}{4}, x_n, x^*\right) + (k^2m_2 + km_3 + k)\Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) +$
$k\Delta(\frac{\lambda}{2}, x^*, x_{n+1})$ ....(4.3)

$\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{x^*}\right) \preccurlyeq \frac{(k^2m_2+k^2m_4)\Delta\left(\frac{\lambda}{4}, x_n, x^*\right) + (k^2m_2+km_3+k)\Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) + k\Delta(\frac{\lambda}{2}, x^*, x_{n+1})}{(1-km_1-k^2m_4)} \preccurlyeq c$

We can see easily $\Delta(\lambda, x^*, \mathrm{T}_{x^*}) = 0$ is the mapping T $which$ has a fixed point x^*

At last, for uniqueness, if there is y^*other fixed point, then

$\Delta(\lambda, x^*, y^*) = \Delta(\lambda, \mathrm{T}_{x^*}, \mathrm{T}_{y^*})$

$\preccurlyeq m_1\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{x^*}\right) + m_2\Delta\left(\frac{\lambda}{4}, x^*, \mathrm{T}_{y^*}\right) + m_3\Delta\left(\frac{\lambda}{4}, y^*, \mathrm{T}_{x^*}\right) + m_4\Delta(\frac{\lambda}{4}, x^*, \mathrm{T}_{y^*})$

$$\preccurlyeq m_1\Delta\left(\frac{\lambda}{4}, x^*, Tx^*\right) + m_2\Delta\left(\frac{\lambda}{4}, x^*, y^*\right) + m_4\Delta\left(\frac{\lambda}{4}, y^*, x^*\right) + m_3\Delta\left(\frac{\lambda}{4}, y^*, \mathrm{T}_{y^*}\right)$$

$\preccurlyeq (m_2 + m_4)\Delta(\lambda, x^*, y^*)$

Since $0 < (m_2 + m_4) < 1$, we deduce from lemma that $x^* = y^*$

Definition 4.3: Let (E, C) be ordered Banach algebra with algebra cone C. Take Φ be the set of all functions $\varphi: E^3 \to E$ satisfying the following properties:

1) $\varphi(e, e, e) = c \preccurlyeq e$

2) Let $a, b \epsilon E$ be such that if either $a \preccurlyeq \varphi(a, b, b)$ or $a \preccurlyeq \varphi(b, a, b)$ or$a \preccurlyeq \varphi(b, a, a)$

Definition 4.3: A self-mapping T on Δ- ordered Banach algebra $(X, E, \Delta)$is called φ-contraction, if there exists a map $\varphi \epsilon \Phi$ satisfy

$\Delta(\lambda, \mathrm{T}_x, \mathrm{T}_y) \preccurlyeq \varphi(\Delta(\lambda, x, y), \Delta(\lambda, x, \mathrm{T}_x), \Delta(\lambda, y, \mathrm{T}_y))$..................................(4.1)

Theorem 4.4: Let (X, E, Δ)be Δ- ordered Banach algebra and T a φ- contraction. If there exists $\lambda > 0$ such that for all $x \epsilon X$.

$\Delta(\lambda, x_0, \mathrm{T}_{x_o}) = \sup \{\Delta(\lambda, x, \mathrm{T}_x): x \epsilon X\}$, then T has a unique fixed point

Proof: Suppose $x_0 \neq \mathrm{T}_{x_o}$. We take $x = x_0$, $y = \mathrm{T}_{x_o}$ in (5.1). then $\Delta(\lambda, \mathrm{T}_{x_o}, \mathrm{T}^2{}_{x_o}) \preccurlyeq \varphi(\Delta(\lambda, x_0, \mathrm{T}_{x_0}), \Delta(\lambda, x_0, \mathrm{T}^2{}_{x_o}), \Delta(\lambda, \mathrm{T}_{x_o}, \mathrm{T}^2{}_{x_o}))$

Since $\Delta(\lambda, \mathrm{T}_{x_0}, \mathrm{T}^2{}_{x_o}) \preccurlyeq k. [\Delta(\lambda, x_0, \mathrm{T}_{x_0})]$. But given that

$\Delta(\lambda, x_0, \mathrm{T}_{x_0}) = \sup\{\Delta(\lambda, x, \mathrm{T}_x) : x \epsilon X\}$

Hence $\mathrm{T}_{x_0} = x_0$

For uniqueness, let y_0 be other fixed point of T that is $\mathrm{T}_{y_o} = y_0$

Now, $\Delta(\lambda, x_0, y_0) =$

$\preccurlyeq \varphi(\Delta_{\frac{\lambda}{3}}(x_0, y_0), \Delta_{\frac{\lambda}{3}}(x_0, \mathrm{T}_{x_0}), \Delta_{\frac{\lambda}{3}}(y_0, \mathrm{T}_{y_0}))$

$\preccurlyeq \varphi(\Delta_{\frac{\lambda}{3}}(x_0, y_0), \Delta_{\frac{\lambda}{3}}(x_0, x_0), \Delta_{\frac{\lambda}{3}}(y_0, y_0))$

$\preccurlyeq \varphi\left(\Delta\left(\frac{\lambda}{3}, x_0, y_0\right), 0,0\right)$

There for $\Delta_\lambda(\lambda, x_0, y_0) \preccurlyeq 0$ or $\Delta_\lambda(x_0, y_0) = 0$. Implies $x_0 \neq y_0$

That is the fixed point is unique and this complete the proof

Theorem 4.5: Let S and T be self-mapping on Δ- Banach algebra (X, E, Δ)satisfy the condition

$\Delta_\lambda(\lambda, \mathrm{T}_x, \mathrm{S}_y) \preccurlyeq \varphi((\Delta(\lambda, x, y), \Delta(\lambda, x, \mathrm{T}_x), \Delta(\lambda, y, \mathrm{S}_y))$ for all $x, y \epsilon X$

If there exists $y \epsilon X$ such that

$\Delta(\lambda, y, \mathrm{T}_y) \preccurlyeq \Delta(\lambda, z, S_z)$ ....(4.2)

Then there exist a unique common fixed point of S and T

Proof: Let $\mathrm{T}_{y_0} = x_0$, put $x = x_0$, $y = \mathrm{T}_{x_0}$,we obtain

$\Delta_\lambda(\lambda, T_{x_0}, S(T_{x_0})) \preccurlyeq \varphi(\Delta(\lambda, x_0, T_{x_0}), \Delta(\lambda, x_0, T_{x_0}), \Delta(\lambda, x_0, S(T_{x_0})$

By (3) we get

$\Delta(\lambda, T_{x_0}. S(T_{x_0})) \preccurlyeq k\Delta(\lambda, x_0, T_{x_0}) \preccurlyeq \Delta(\lambda, x_0, T_{x_0})$

This contradict of (4.1)

To prove that x_0 is also a fixed point of S, let $S_{x_0} = x_0$, therefore.

$\Delta(\lambda, x_0, S_{x_0}) = \Delta(\lambda, T_{x_0}, S_{x_0}) \preccurlyeq \varphi[\Delta(\lambda, x_0, x_0), \Delta(\lambda, x_0, T_{x_0}), \Delta(\lambda, x_0, S_{x_0})]$

Or $\Delta(\lambda, x_0, S_{x_0}) \preccurlyeq \varphi(0,0, \Delta(\lambda, x_0, S_{x_0}))$ that is $\Delta(\lambda, x_0, S_{x_0}) \preccurlyeq 0$ or $S_{x_0} = x_0$

For uniqueness, let y_0 be another fixed point of S and T that is

$T_{y_0} = S_{y_0} = y_0$, then

$\Delta(\lambda, x_0, y_0) = \Delta(\lambda, T_{x_0}, T_{y_0}) \preccurlyeq \varphi[(\Delta(\lambda, x_0, y_0), \Delta(\lambda, x_0, T_{x_0}), \Delta(\lambda, y_0, T_{y_0})]$

Or $\Delta_\lambda(\lambda, x_0, y_0) \preccurlyeq \varphi(\Delta(\lambda, x_0, y_0), \Delta(\lambda, x_0, x_0), \Delta(\lambda, y_0, y_0))$

$\preccurlyeq \varphi(\Delta(\lambda, x_0, y_0), 0,0)$

That is $\Delta(\lambda, x_0, y_0) \preccurlyeq 0$ implies $x_0 = y_0$.

Corollary 4.6: Let S and T be self-mapping of Δ- ordered Banach algebra (X, E, Δ) satisfying the following conditions:

1) There exists integer n and m such that

$\Delta(\lambda, T^n{}_x, S^m{}_y) \preccurlyeq \varphi[\Delta(\lambda, x, y), \Delta(\lambda, x, T^n{}_x), \Delta(\lambda, y, S^m{}_y]$ for some $\varphi \epsilon \Phi$

2) If there exists a point $y \epsilon X$ such that $\Delta(\lambda, y, T^n{}_x) \preccurlyeq \Delta(\lambda, x, S^m{}_x)$

Then there exists a unique common fixed point of S and T

Theorem 4.7: Let (X, E, Δ) be a Δ- ordered Banach algebra such that

$\Delta(\lambda, T_x, T_y) \preccurlyeq \min \{\lambda\Delta(\lambda, x, T_x), \mu\Delta(\mu, y, T_y)\}$

If there exists function F defined by $F(x) = \lambda\Delta_\lambda(\lambda, x, T_x)$ for each $x \epsilon X$ such that $F(x) \preccurlyeq F(T(x))$, then, T has a unique fixed point

Proof: Suppose for some x_0, $x_0 \neq T_{x_0}$. Then $F(T_{x_0}) = \Delta(\lambda, T_{x_0}, T(T(x_0)) \preccurlyeq \min \{\lambda\Delta(\lambda, x_0, T_{x_0}), \mu\Delta(\mu, T_{x_0}, T_{x_0}\}$ since $\Delta(T_{x_0}, T_{x_0}) = \theta$

$\Delta(\lambda, T_{x_0}, T(T(x_0)) \preccurlyeq \lambda\Delta(\lambda, x_0, T_{x_0})$

$F(T_{x_0}) \preccurlyeq F(x_0)$ which is contradiction

Hence $T_{x_0} = x_0$

For uniqueness, let y be another point of X different from x_0 such that $y_0 = T_{y_0}$, then

$\Delta(\lambda, x_0, y_0) = \Delta(\lambda, \mathrm{T}_{x_0}, \mathrm{T}_{y_0}) \preccurlyeq \min\{\lambda\Delta(\lambda, x_0, \mathrm{T}_{x_0}), \lambda\Delta(\lambda, y_0, \mathrm{T}_{y_0})\}$

$= \min\{\lambda\Delta(\lambda, x_0, x_0), \lambda\Delta(\lambda, y_0, y_0)\} = \min\{\theta, \theta\}$

$\Delta(\lambda, x_0, y_0) \preccurlyeq \theta$

Hence $\Delta(\lambda, x_0, y_0) \preccurlyeq 0$ which implies that $\Delta(\lambda, x_0, y_0) = 0$ or $y_0 = x_0$

The proof is complete.

Theorem 4.8: Let T be a self-map on a compact Δ- ordered Banach algebra(E, A, C) satisfy Lipsctiz condition

Then, T has a unique fixed point.

Proof: Suppose T satisfy Lipsctiz condition. Then, T is a continuous map on X we define a function from X into x as $F(x) = \Delta(\lambda, x, \mathrm{T}_x)$ for all $x\epsilon X$.

Since T and Δ are continuous, it follow F is continuous on X . Since X is compact there exists a point $y\epsilon X$ such that $F(y) = \inf\{\Delta(\lambda, x, \mathrm{T}_x): x\epsilon X\}$.

We support that $y \neq \mathrm{T}_y$.

Otherwise, that a fixed point by Lipscitiz condition

We have$\Delta(\lambda, \mathrm{T}_y, \mathrm{T}^2{}_y) \preccurlyeq k\Delta(\mu, y, \mathrm{T}_y)$. $0 < \lambda \leq \mu$

So that $F(\mathrm{T}_y) \preccurlyeq T(y)$ which contradiction.

Then, $y = \mathrm{T}_y$

Uniqueness follows from Lipscitz condition.

Proposition 4.9: Let (X, E, Δ)be a complete Δ- ordered Banach algebra. Assume that the mapping $\mathrm{T}: X \to X$ satisfy.

$\Delta(\lambda, \mathrm{T}^n{}_x, \mathrm{T}^n{}_y) \preccurlyeq k\Delta(\lambda, x, y)$, For each $x, y\epsilon X$, for $n \in Z^+$,where k a vector with is $\sigma_\varepsilon(k) < 1$. Then, T has a unique fixed point

Proof: $\mathrm{T}^n(\mathrm{T}_{x^*}) = \mathrm{T}(\mathrm{T}^n x^*) = \mathrm{T}^n x^* = \mathrm{T}(\mathrm{T}^{n-1}x^*) = \mathrm{T}^{n-1}(x^*) = \cdots = \mathrm{T}x^*$.

So, $\mathrm{T}x^*$ is also has fixed point of T^n then $Tx^* = x^*$

x^* is a fixed point of T.

Theorem 4.10: Let (X, E, Δ) be a compact Δ- ordered Banach algebra. Suppose the mapping satisfy $\Delta - \mathit{lipshtiz\ condition}$ in the following:

$\Delta(\lambda, \mathrm{T}_x, \mathrm{T}_y) \preccurlyeq k[\beta\Delta(\beta, \mathrm{T}_x, y) + \mu\Delta(\mu, \mathrm{T}_y, x)]$, for all $x, y\epsilon X$, where k is a vector with $k\epsilon(0,1)$. Then, T has a unique fixed point in X. Another sequence $\{T_x^t\}$ converge to the fixed point.

Proof: choose $x_0\epsilon X$ and set $x_t = T_x^t$,$t \geq 1$, we have for $t < m$

$\Delta(\lambda, x_{t+1}, x_m) \preccurlyeq \beta\Delta(\beta, x_t, x_{t+1}) + \mu\Delta(\mu, x_{t+1}, x_m)$

$\preccurlyeq \beta\Delta(\beta, x_t, x_{t+1}) + \mu[\beta\Delta(\beta, x_{t+1}, x_{t+2}) + \mu\Delta(\mu, x_{t+2}, x_m)]$

$\preccurlyeq \beta\Delta(x_t, x_{t+1}) + \mu\beta\Delta(x_{t+1}, x_{t+2}) + \mu^2[\beta\Delta(x_{t+2}, x_{t+1}) + \mu\Delta(x_{t+3}, x_m)]$

$\preccurlyeq \beta\Delta(x_t, x_{t+1}) + \mu\beta\Delta(x_{t+1}, x_{t+2}) + \mu^2\beta\Delta(x_{t+1}, x_{t+2}) + \mu^3\beta\Delta(x_{t+3}, x_{t+1}) + \mu^4\Delta(x_{t+4}, x_m)$

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$\preccurlyeq \beta[\Delta(\lambda, x_t, x_{t+1}) + \mu\Delta(\lambda, x_{t+1}, x_{t+2}) + \mu^2\Delta(\lambda, x_{t+2}, x_{t+3}) + \ldots +$

$\mu^n\Delta(\lambda, x_1, x_0)] + \mu^n[\Delta(\lambda, x_{t+m}, x_m)]$

$\preccurlyeq \beta[k^t\Delta(\lambda, x_1, x_0) + \mu k^{t+1}\Delta(\lambda, x_1, x_0) + \cdots + \mu^m k^{t+m}\Delta(\lambda, x_1, x_0)$

$+\mu^{t+1}[k^{t+m+1}\Delta(\lambda, x_1, x_0)]$

$\preccurlyeq \beta k^t[1 + \mu k + \cdots + \mu^m k^m]\Delta(\lambda, x_1, x_0) + \mu^t k^{t+m}\Delta(\lambda, x_1, x_0)$

$\preccurlyeq \beta k^t[\sum_{i=1}^{m} \mu^i k^i]\Delta(\lambda, x_1, x_0) + \mu^{m+1}k^{t+m}\Delta(\lambda, x_1, x_0)$

$\preccurlyeq \beta k^t[\sum_{i=1}^{m+1} \mu^i k^i\Delta(\lambda, x_1, x_0)]$

$\preccurlyeq \beta k^t[\sum_{i=1}^{\infty} \mu^i k^i]\Delta(\lambda, x_1, x_0)$

$\preccurlyeq \beta k^t[\sum_{i=1}^{0} \mu^i k^i]\Delta(\lambda, x_1, x_0)$

$\preccurlyeq \beta k^t(e - \mu k)^{-1}\Delta(\lambda, x_1, x_0)$

$\|\Delta(\lambda, x_{n+1}, x_m)\| \leq \|\beta k^t\|.\|(e - \mu k)^{-1}\|.\|\Delta(\lambda, x_1, x_0)\|$

Since $\|k^n\| \to 0$ as $n \to \infty$, where $\|\Delta(\lambda, x_n, x_m)\| \to 0$ as $n \to \infty$

Which implies $\Delta(\lambda, x_t, x_m) \to 0$ as $(t, m \to 0)$

Hence, $\{x_t\}$ is a Cauchy sequence. *since X is complete*, there exists $x^* \epsilon X$ such that $x_t \to x^*$ as $n \to \infty$, therefore

$\lim \Delta(\lambda, T_{x^*}, x^*) \preccurlyeq k[\beta\Delta(\lambda, T_{x_t}, T_{x^*}) + \mu\Delta(\lambda, T_{x^*}, x_t)]$

$\preccurlyeq \beta k[\Delta(\lambda, x_t, x^*) + \Delta(\lambda, x^*, x_{t+1})] + \mu\Delta(\lambda, T_{x^*}, x_t)$

$\|\Delta(\lambda, T_{x^*}, x^*)\|$

$\leq \|\lambda\|.\|k\|.[\|\Delta(\lambda, x_t, x^*)\| + \|\Delta(\lambda, x^*, x_{t+1})\|] + \|\mu\|.\|\Delta(\lambda, x_t, x^*)\|$

Which implies $T_{x^*} = x^*$ and so x^* is fixed point

To prove uniqueness, let b be another fixed point of T.

Then $\Delta(\lambda, x^*, b) = \Delta(\lambda, T_{x^*}, T_b) \leq k[\beta\Delta(\lambda, T_{x^*}, b) + \mu\Delta(\lambda, T_b, x^*)]$

$= k[\beta\Delta(\lambda, x^*, b) + \mu\Delta(\lambda, b, x^*)] = k[\lambda + m]\Delta(\lambda, x^*, b)$

Then, $[1-k(\beta+\mu]\Delta(\lambda,x^*,b)\leq 0.$

Since $k\epsilon(0,1)$ and $\beta,\mu>0 \Longrightarrow \Delta(\lambda,x^*,b)=0$ so $x^*=b$

The proof is complete.

5-Conclusion

In this paper, we introduce a new concept which is called Δ- ordered Banach algebra. Also, we define $lipshtiz\ condition\ in\ this\ pace\ ,\varphi-contraction,\ \Delta-contraction$ and $\Delta-lipshtiz\ condition$**.** In the new work, we prove fixed point theorems satisfying these maps in Δ- ordered Banach algebra. Our conditions and results are new in comparison with those of the results of cone metric space .These results can be extended to other spaces.

References

1. Akbar Zada, Rahim Shah and Shahid Saifullah,"Some Fixed Point Theorems in Multiplicative Cone b- Matric Spaces", Journal of Analysis & Number Theory, 5, No. 1, 35-39, (2017).

2. B. Y.Hussein. and S. F.Hassan ,"Some properties of Characterization on ordered Banach algebra",Ms.C. thesis ,Al-Qadisiyah university,2017.

3. Arun Gary, Zaheer K. Ansari," Cone Metric Spaces and Fixed Point Theorems of T- Contractive Mapping", International journal of Mathematics and Statistics Invention (IJMSI), Vol.2, Iss. 2, PP. 72-76, (2014)

4. D. Dey and M.Saha," Partial Cone Metric Space and Some Fixed Point Theorems", TWMS J.App. Math. U_3, N.1, PP. 1-9, (2013).

5. Dhage, B.C.,"A study of some fixed point theorem", Ph. Thesis, Marath Wada University, Aurangabad, (1984).

6. Gahler, S., " 2- Metriche Raume and Ihre Topologisch Structure", Mathematiche Nachrichten, 26, 115-148, (1963).

7. G. S. Sulugq and B. P. Tripathi, "Some Common Fixed Point Theorems of Generalized Contractive Mappings in Cone Metric Spaces", Functional Analysis Approximation and Computation, 5:1, 33-41, (2013).

8. Hao Liu and Shaoman XU, " Conemetric Spaces with Banach Algebras and Fixed Point Theorems of Generalized Lipschitz Mapping, Springre, Fixed Point and Application, 320, (2013).

9. Huang, L., G., & Zhang X, "Cone Metric Space and Fixed Point Theorem for Contractive Mapping", Journal of Mathematical Analysis and Applications, 332, (1468- 1476).

10. H. K Nashine and I. Altun, "A Common Fixed Point Theorem on Ordered Metric Spaces", Bulletin of The Iranian Mathematical Society, Vol. 38 No. 4, PP. 925- 934, (2012).

11. H. Liu and S. Xu," Cone Metric Spaces with Banach Algebra and Fixed Point Theorems of Generalized Lipchitz Mappings," Fixed Point Theory and Applications, 320. Dio:10.118611687-1812-320 (2013).

12. H. Rahimi and G. Soleimani Rad, "Fixed Point Theorems Under C. Distance in Ordered Cone Metric Space", Int. J. Industrial Mathematics, Vol. 6, No. 2, Article IDISIM-00253, 9 Page, (2014).

13. Jerolina Fernardcz, Sompob Sale, Kalpana Saxena, Neeraj Malviya and Poom Kuman, " The A- Cone Metric Space Over Banach Algebra with Applications ", Fernardczct. at. Cogent Mathematics, 4:1828690, (2017).

14. K. P. R Sastry, Ch-Srinivasa Rao, A. Chandra Sekhar and M. Balaiah,"A Fixed Point Theorem in Cone Metric Spaces Under Weak Contractions", Gen. Math. Notes, Vol. 7, No. 2, PP.38-44, (2011).

15. K. Ha, Y. J. Choand Awhile, " Striclly Convex and 2- Convex 2- Normd Spacy." Mathematica Japonica, 33, 375-384, (1988).

16. L. G. Huang and X. Zhand, " Cone Metric Spaces and Fixed Point Theorems for Contractive Mappings ", Journal of Mathematics Analysis and Applications, 332, 1468-1476, (2007).

17. Mustafa and B. Sims, " A New Approach to Generalized Metris spaces", Journal of Nonlinear and Convex Analysis, 7, 289- 297, (2006).

18. Sedghi S., Shobe, N., & Aliouche, A. " A Generalization of Fixed Point Theorems in S- Metric Spaces ", Mathematichki Vesnik, 64, 258-266, (2012).

19. R. De Jong, "Ordered Banach Algebras", Ms.C. Thesis, Leiden University. Holland, 2010.

20. Sedghi S., Shobe, N., & Zhou, H., "A Common Fixed Point Theorem in D^*- Metric Space", Fixed Point Theory and Applications, 29, 631-636, (2007).

21. Sevng Hyun Kim and Byvng Soo Lee, "Generalized Fixed Point Theorems in Cone Metric Spaces", Korean J. Math 20, No. 3, PP.353-360, (2012).

22. S. K. Tiwari and R. P. Dubey, "Cone Metric Spaces and Fixed Point Theorems for Generalized Y. Reich and T Rhoadg Contractive Mapping", Asin Journal of Mathematics and Applications Article Idama 0077, 9 Page, vol. (2013).

23. V. H. Badshah, P. Bhagatand S. Shukla, "Some Fixed Point Theorems for α-ϕ- Outmetric Mapping in 2-Metric Spaces", Scientific Publications of State University of Novipazar, Ser. A: ppl. Math. INF or M. and Mech, Vol 802, 181-199, (2016).

24. Zhenhua Ma, Lining Jiang, and Hongkaisun, " C^*- Algebra- Valued Metric Spaces and Related Fixed Point Theorems", Ma et al. Fixed Point and Applications Springer: 206., (2014).

25. Yan Han, Shaoyuan Yu, " New Common Fixed Point Results for Four Maps on Cone Metric Spaces" Applied Mathematics, 2, 1114-1118,(2011).