DOI: https://doi.org/10.24297/jam.v16i0.8136

Intransitive Permutation Groups with Bounded Movement Having Maximum Degree

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Abstract

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for each subset Γ of Ω the size $|\Gamma^g \backslash \Gamma|$ is bounded, for $g \epsilon G$, we define the movement of g as the $max|\Gamma^g \backslash \Gamma|$ over all subsets Γ of Ω. In this paper we classified all of permutation groups on set Ω of size $3m+1$ with 2 orbits such that has movement m .
2000 AMS classification subjects: 20B25

1 Introduction

Let G be a transitive permutation group on a set Ω such that G is not 2-group and let m be a positive integer. In [ ], C.E.Oraeger shown that if $|\Gamma^g \setminus \Gamma| \leq m$ for every subset Γ of Ω and all $g \in G$, $\mid \Omega \mid \leq \lfloor \frac{2mp}{p-1} \rfloor$, where p is the least odd prime dividing $\mid G \mid$. If $p = 3$ the upper bounded for $\mid \Omega \mid$ is $3m$, and the groups G attaining this bound where classified in the work of $Gardiner$([2]), $Mann$ and the $C.E.Praeger$([3]). Here we show that if G be a *intrasitve* permutation group on set Ω of size $3m+1$ with 2 orbits such that has movement m, and let B is the semi-direct product of $Z_2^2.Z_3$. Then G is satisfy one of the following : $G_1 = B \times H^d$ or $G_2 = A_4 \times H^d$, where $H = Z_3$ or S_3, $d = m-2$, and A_4 is the permutation group on 4 elements. Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for a subset Γ of Ω the size $|\Gamma^g \setminus \Gamma|$ is bounded, for $g \in G$, we define the movement of Γ as move(Γ) = $\max_{g\in G}|\Gamma^g \setminus \Gamma|$. If move($\Gamma$) $\le m$ for all $\Gamma \subseteq \Omega$,then G is said to have *bounded movement* and the *movement* of G is define as the maximum of move(Γ) over all subsets Γ, that is,

$$m := move(G) := sup\{|\Gamma^g \setminus \Gamma||\Gamma \subseteq \Omega, g \in G\}.$$

This notion was introduced in [3]. By [3,Theorem 1],if G has bounded movement m,then Ω is finite. Moreover both the number of G-orbits in Ω and the length of each G-orbit are bounded above by linear functions of m.In particular it was shown that the number of G-orbits is at most $2m$-1. 1.The main result is the following theorem.

Theorem 1.1. Let G a permutation group on set Ω of size $3m + 1$ with 2 orbits such that has movement m, and let B is the semi-direct product of $Z_2^2.Z_3$. Then G is $G_1 = B \times H^d$ or $G_2 = A_4 \times H^d$, where $H = Z_3$ or S_3, $d = m - 2$, and A_4 is the permutation group on 4 elements.

Note that an orbit of a permutation group is non trivial if its length is greater than 1. The groups described below are examples of permutation groups with bounded movement equal to m which have exactly $\frac{1}{2}(3m-1)+\frac{1}{p}$ nontrivial orbits.

2 Examples and Preliminaries

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle
representations has t nontrivial cycles of lengths $l_1, ..., l_t, say$. We might represent g as
$g = (a_1a_2...a_{l_1})(b_1b_2...b_{l_2})...(z_1z_2...z_{l_t})$. Let $\Gamma(g)$ denote a subset of Ω consisting $\lfloor l_i/2\rfloor$ points from the ith cycle , for each i, chosen in such a way that $\Gamma(g)^g \bigcap \Gamma(g)$ = $\emptyset$. For example ,we could choose
$\Gamma(g) = \{a_2, a_4, ..., a_{k_1}, b_2, b_4, ..., b_{k_2}, ..., z_2, z_4, ..., z_{k_t}\}$, where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even . Note that $\Gamma(g)$is not uniquency determined as it depends on the way each cycle is written . For any set $\Gamma(g)$ consists of every point of very cycle of g. From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g \setminus \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t}\lfloor l_i/2\rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g \setminus \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1. [5, Lemma 2.1]. Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g \setminus \Gamma| \leq \sum_{i=1}^{t} \lfloor l_i/2 \rfloor$, where l_i is the length of the ith cycle of g and t is the number of nontrivial cycles of g in its disjoint cycle representation . This upper bound is attained for $\Gamma = \Gamma(g)$ defined above .

Now we will show that there certainly is an infinite family of 3-groups for which the maximum bound obtained in Theorem 1.1 holds .

Example 2.2 . Let d be a positive integer $\Omega = \Omega_1 \cup \Omega_2$ be a set of size 7, such that $\Omega_1 = \{1, 2, 3\}$ and $\Omega_2 = \{1\acute{}, 2\acute{}, 3\acute{}, 4\acute{}\}$. Moreover, suppose that $Z_2^2 \cong \langle (1\acute{}2\acute{})(3\acute{}4\acute{}), (1\acute{}3\acute{})(2\acute{}4\acute{}) \rangle$ and $Z_3 \cong \langle (123)(1\acute{}2\acute{}3\acute{}) \rangle$. Then the semi-direct product $G = Z_2^2 Z_3$ with normal subgroup $G = Z_2^2$ is a permutation group on a set Ω with 2-orbits which movement 2, since each non-identity element of G has two cycle of length 2 or two cycle of length 3.

Example 2.3 . Let $Z_2^2 = \langle x \rangle$ and $Z_3 = \langle y \rangle$, and write $G = \{x^i y^j z | z \in Z_3^d\}$. Note that y lies in G. If x lies in G, then $G = (Z_3.Z_2^2) \times Z_3$. If $x \notin G$,x^2 lies in G. We then consider a subgroup $T = \{z \in Z_3^d | z \in G\}$ and a subset $S = \{z \in Z_3^d | yz \in G\}$ of Z_3^d. Let $\Omega_1, ..., \Omega_d$, $d\ G - orbits$ and $\Delta = \bigcup_{i=1}^{d} \Omega_i$, $\Delta\acute{} = \Omega \backslash \Delta$ and K the pointwise stabilizer on Δ. Since the permutation group induced by G/K on is an elementary abelian 3-group Z_3^d, we have $T \cap S =$ and $T \cup S = Z_3^d$. If $z\acute{}$ and $z\acute{}\acute{}$ lie in S, then $yz\acute{}yz\acute{}\acute{} \in G$ and so does $z\acute{}z\acute{}\acute{} \in G$. This means $S \subset \alpha T$ for some $\alpha \in Z_3^d \setminus T$, and $Z_3^d = T \cup \alpha T$. Hence $G = \{x^i y^{3j+1} \alpha t | t \in T\} \cup \{x^i y^{3j} t | t \in T\} = \{x^i (y\alpha)^j t | t \in T\}$. Let $H = \{x^i (y\alpha)^j\}$. Then $T \cap H = \{1\}$ and $HT = G$. Since T and H are normal subgroups of G, we have $G = H \times T$. Since $H = \{x^i (y\alpha)^j\} \simeq Z_3.Z_2^2$ and $T \simeq Z_3^{(d-1)}$, we have $G \simeq (Z_3.Z_2^2) \times Z_3^d$. This is complete the proof of Theorem 1.1.

Corrolary For every $m > 2$, the theorem of this paper has answers .

3 Proof of Theorem 1.2.

In this section we prove Theorem 1.2, we show first that a minimal counterexample to Theorem 1.2, must be a nonabelian simple group acting primitively on Ω. If a group G has bounded movement equal to m for convenience we shall say that G satisfies BM(m).

3.1.Proposition : Suppose that Theorem 1.2, is false and let m be the least integer for which Theorem 1.2 false. Further let G be a counterexample to Theorem 1.2, with $|\ G\ |$ minimal. Then G is a nonabelian simple group

acting primitively on Ω.
Proof : Since G is a counterexample to Theorem 1.2 with $\mid G \mid$ minimal, it follows that G is not a 2-group ,G is intransitive on Ω, G satisfies BM(m), and $\mid \Omega \mid = 3m+1$. The proof proceeds in five steps.

Let $\Omega_1, \ldots, \Omega_t$ be t orbits of G of lengths $n_1, \ldots, n_t$. Choose $\alpha_i \in \Omega$ and let $H_i := G_{\alpha_i}$, so that $|G : H_i| = n_i$. For $g \in G$, let $\Gamma(g) = \{\alpha_i | \alpha_i^g \neq \alpha_i\}$ be every second point of every cycle of g and let $\gamma(g) := |\Gamma(g)|$. Since $\Gamma(g) \cap \Gamma(g)^g = \emptyset$ it follows that $\gamma(g) \leq m$ for all $g \in G$. Let $\bar{\Omega} := \Omega_1 \cup \ldots \cup \Omega_t$, and let $\bar{G}$ and $\bar{H}_1, \ldots, \bar{H}_t$ denote the finite permutation groups on $\bar{\Omega}$ induced by G and $H_1, \ldots, H_t$ respectively. Then $n_i = |\bar{G}_1 : \bar{H}_i|$.

For $g \in G$, let $\bar{g} \in \bar{G}$ denote the permutation of $\bar{\Omega}$ induced by g. Then as $\gamma(1_G) = 0$, we have $\sum_{\bar{g} \in \bar{G}} \gamma(g) < m|\bar{G}|$.
Now, Counting the pairs $(\bar{g}, i)$ such that $\bar{g} \in \bar{G}$ and $\alpha_i^g \neq \alpha_i$ gives

$$\sum_{\bar{g} \in \bar{G}} \gamma(g) = \sum_i |\{\bar{g} \in \bar{G} | \alpha_i^g \neq \alpha_i\}| = \sum_i |\{\bar{g} \in \bar{G} | g \notin H_i\}| = \sum_i (|\bar{G}| - |\bar{H}_i|) = |\bar{G}| \sum_i (1 - \frac{1}{n_i}).$$

It follows that $\sum_i (1 - \frac{1}{n_i}) < m$. Since $n_i \geq 3, p$ for each i, it follows that $\sum_i (1 - \frac{1}{n_i}) \geq \frac{p-1}{p} + \frac{2}{3}(t-1)$ and hence $\frac{p-1}{p} + \frac{2}{3}(t-1) < m$, that is, $t \leq \frac{1}{2}(3m-1) + \frac{1}{p}$.

Consequently G has at most $\frac{1}{2}(3m-1) + \frac{1}{p}$ orbits in Ω. Now Let m be a positive integer greater than 1. Suppose that $G \leq Sym(\Omega)$ with orbits,$\Omega_2, ..., \Omega_t$, where t=$\frac{1}{2}(3m-1) + \frac{1}{p}$. Suppose further that $\Gamma \subseteq \Omega$ has move $(\Gamma) = m$ and that cuts across each of the G-orbits Ω_i. For each i set $n_i = |\Omega_i|$ and $\Gamma_i = \Gamma \cap \Omega_i$. Note that $0 < |\Gamma_i| < n_i$.

Claim 3.1 If Theorem 2.3 holds for the special case in which $|\Gamma_i| = 1$ for $i = 1, ..., \frac{1}{2}(3m-1) + \frac{1}{p}$, then it holds in general .
Proof : Suppose that Theorem 2.3 holds for the case where each $|\Gamma_i| = 1$. For $i = 1, ..., t$, define $\sum_i := \{\Gamma_i^g | g \in G\}$, and note that $| \sum_i | \geq 3$ since Γ cuts across Ω_i. Set $\Sigma = \cup_{i \geq 1} \sum_i$. Then G induces a natural action on Σ for which the G-orbits are $\Sigma_1, ..., \Sigma_t$. Let G^{Σ} denote the permutation group induced by G on Σ , and let K denote the kernel of this action.

We claim that the t-element subset $\Gamma_{\Sigma} = \{\Gamma_1, ..., \Gamma_t\} \subseteq \Sigma$ has movement equal to m relative to G^{Σ}, and that Γ_{Σ} cuts across each Γ^{Σ}-orbit Σ_i. For each $g \in G$, $|\Gamma^g - \Gamma| \leq m$ and hence $|\Gamma_\Sigma^g - \Gamma_\Sigma| \leq m$. Thus move $(\Gamma_\Sigma) \leq m$. Also, Since $|\Sigma_i| \geq 3$ and $\Gamma_\Sigma \cap \Sigma_i$ Consists of the single element Γ_i of Σ_i , the set Γ_Σ cuts across each of the $\frac{1}{2}(3m-1) + \frac{1}{p}$ orbits Σ_i. However, it follows that the number of G^Σ- orbits is at most $\frac{1}{2}(3.move(\Gamma_\Sigma) - 1) + \frac{1}{p}$, and hence move $(\Gamma_\Sigma) = m$.

Thus the hypotheses of theorem 2.3 hold for the subset $\Gamma_\Sigma \subseteq \Sigma$ relative to G^Σ, and Γ_Σ meets each G^Σ-orbit in exactly one point. By our assumption it follows that $t = \frac{1}{2}(p3^r - 1)\frac{1}{p} = \frac{1}{2}(3m-1) + \frac{1}{p}$ for some $r > 1$, and that $G^\Sigma = Z_3^r$ and each $|\Sigma_i| = 3$. Further, the subgroups H_i of G fixing Γ_i setwise range over the $\frac{1}{2}(p3^r - 1) + \frac{1}{p}$ distinct subgroups which have index 3 in G and which contain K. In particular, for each i, H_i is normal in G and hence the H_i-orbits in Ω_i are blocks of imprimitivity for G, and their number is at most $|G : H| = 3$. Since H_i fixes Γ_i setwise it follows that Γ_i is an H_i -orbit and $n_i = 3|\Gamma_i|$.

Let $g \in G \setminus K$. Then in its action on Σ , g moves exactly m of the Γ_i. Since the Γ_i are blocks of imprimitivity for G, each Γ_i^g is equal to either Γ_i or $\Omega_i - \Gamma_i$. It follows that $|\Gamma^g \setminus G|$ is equal to the sum of the sizes of the m subsets Γ_i moved by g. However, since move $(\Gamma) = m$, each of these m subsets Γ_i must have size 1. Since for each i we may choose an element g which moves Γ_i , we deduce that each of the Γ_i has size 1, and that K is the identify subgroup. It follows that theorem 2.3 hold for G . Thus the claim is proved .

From now on we may and shall assume that each $|\Gamma_i| = 1$. Let $\Gamma_i = \{\Omega_i\}$. Further we may assume that $n_1 \leq n_2 \leq ... \leq n_t$. For $g \in G$ let $c(g)$ denote the number of integers I such that $\omega_i^g = \omega_i$. Note that since move $(\Gamma) = m$, we have $c(g) > t - m = \frac{1}{2}(3m-1) + \frac{1}{p} - m = \frac{m-1}{2} + \frac{1}{p}$ and also $c(1_G) = t > \frac{m-1}{2} + \frac{1}{p}$.

Lemma 3.2. If one of the orbits of G has length equal to p, then the rest orbits of G has size 3.

Proof : Let X denote the number of pairs (g,i) such that $g \in G$, $1 \leq i \leq t$, and $\omega_i^g = \omega_i$. Then $X = \sum_{g \in G} c(g)$, and by our observations, $X > |G|.(\frac{m-1}{2} + \frac{1}{p})$. On the other hand, for each i, the number of elements of G which fix ω_i is $|G_{\omega_i}| = \frac{|G|}{n_i}$, and hence $X = |G| \sum_{i=1}^{t} n_i^{-1}$ If all the $n_i \geq 3$, and one of n_i is equal to p, then $X \leq |G|.(\frac{1}{p} + \frac{t-1}{3}) = |G|(\frac{1}{p} + \frac{3m-1}{6} + \frac{1}{3p} + \frac{1}{3}) \leq$

$|G|.(\frac{m-1}{2}+\frac{1}{p})$ (since $m \geq 3$) which is a contradiction. Hence n=3.

A similar argument to this enables us to show that except one of n_i the rest of n_i is $n_i = 3$, and hence that G is an $3-group$.

Lemma 3.3. The group $G = Z_p.Z_3^r$ for some $r \geq 2$. Moreover for each $n_i = 3$, except one , the stabilizers $G_{\omega_i}(2 \leq i \leq t)$ are pair wise distinct subgroups of index 3 in G, and for each $g \neq 1, c(g) = (\frac{m-1}{2}+\frac{1}{p})$.

Proof: By Lemma 3.2, except one of n_i the rest of n_i is $n_i = 3$. Thus $H := G_{\omega_i}$ is a subgroup of index 3. This time we compute the number Y of pairs (g,i) such that $g \in G \setminus H, 2 \leq i \leq t$, and $\omega_i^g = \omega_i$. For each such g, $\omega_1^g \neq \omega_1$ and hence there are $c(g)$ of these pairs with first entry g. Thus $Y = \sum_{g \in G\setminus H} c(g) \geq |G\setminus H|(\frac{3(m-1}{2}+\frac{3}{p}) = |G|(\frac{m-1}{2}+\frac{1}{p})$.

On the other hand, for each $i \geq 2$, the number of elements of G, which fix ω_i is $|G_{\omega_i}\setminus H|$. If $H = G_{\omega_i}$ then $|G_{\omega_i}\setminus H| = 0$, while if $G_{\omega_i} \neq H$, then $|G_{\omega_i}\setminus H| = \frac{|G_{\omega_i}|}{3} = \frac{|G|}{3n_i} \leq \frac{|G|}{9}$. Hence

$$\begin{aligned} Y &= \textstyle\sum_{i=2}^{t} |G_{\omega_i}\setminus H| \leq] \frac{|G|}{3} \sum_{i=2}^{t} \frac{1}{n_i} \leq \frac{|G|}{3}(\frac{1}{p}+\frac{t-1}{3}) \\ &= \frac{|G|}{3}(\frac{3+p(t-1)}{3p}) < |G|(\frac{m-1}{2}+\frac{1}{p}) \end{aligned}$$

It follows that equality holds in both of the displayed approximations for Y. This means in particular that each $n_i = 2$, Whence $G = Z_p.Z_3^r$ for some r. Further, for each $i \geq 3, G_{\omega_i} \neq H$ and so $r \geq 2$. Arguing in the same way with H replaced by G_{ω_i}, for some $i \geq 2$, we see that $G_{\omega_i} \neq G_{\omega_j}$ if $j \neq i$, and also if $g \in G_{\omega_i}$ then $c(g) = (\frac{m-1}{2}+\frac{1}{p})$. Thus the stabilizers $G_{\omega_i}(1 \leq i \leq t)$ are pairwise distinct , and if $g \leq 1$ then $c(g) = (\frac{m-1}{2}+\frac{1}{p})$. Finally we determine m.

Lemma 3.4.. $m = 3^{r-2}$

Proof: We use the information in lemma3.3 to determine precise the quantity $X = \sum_{g\in G} c(g) : X = t + (|G|-1).(\frac{m-1}{2}+\frac{1}{p}) = \frac{1}{2}(3m-1)+\frac{1}{p}+(p.3^{r-1}-1)(\frac{m-1}{2}+\frac{1}{p})$. On the other hand, from the proof of lemma 2.1,

$$X = |G|\sum_{i=1}^{t} n_i^{-1} = |G|.(\frac{1}{p}+\frac{t-1)}{3}) = p.3^{r-1}.(\frac{1}{p}+\frac{3m-1}{6}+\frac{1}{3p}-\frac{1}{3}).$$

Thus implies that $m = 3^{r-2}$.

The proof of theorem 2.3 now follows from lemmas 3.2-3.4.

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