The Reproducing Kernel Hilbert Space Method for Solving System of Linear Weakly Singular Volterra Integral Equations

Hameeda O. AL-Humedi and Ameer S. Abdul-hasan

Department of Mathematics, University of Basrah, College of Education for Pure Sciences, Basrah, Iraq.

hameedao@yahoo.com, amrstrabd@gmail.com

Abstract

The exact solutions of a system of linear weakly singular Volterra integral equations (VIE) have been a difficult to find. The aim of this paper is to apply reproducing kernel Hilbert space (RKHS) method to find the approximate solutions to this type of systems. At first, we used Taylor's expansion to omit the singularity. From an expansion the given system of linear weakly singular VIE is transform into a system of linear ordinary differential equations (LODEs). The approximate solutions are represent in the form of series in the reproducing kernel space$W_1[0,1]$. By comparing with the exact solutions of two examples, we saw that RKHS is a powerful, easy to apply and full efficiency in scientific applications to build a solution without linearization and turbulence or discretization .

Keywords: Linear system, weakly singular kernel, Volterra integral equations, Taylor's expansion, reproducing kernel Hilbert space.

Mathematics Subject Classifications 2010 **[MSC]:** 76S05, 65N99, 35Q35.

1- Introduction

In this paper, we consider the following system of linear weakly singular VIE of the following form [1]:

$$\left.\begin{array}{l} u_1(x) = f_1(x) + \int_a^x (k_{11}(x,y)u_1(y) + k_{12}(x,y)\, u_2(y))dy, \\ u_2(x) = f_2(x) + \int_a^x (k_{21}(x,y)u_1(y) + k_{22}(x,y)u_2(y)))dy \end{array}\right\} ,a \le x \le b \tag{1.1}$$

where a and b are real finite constant and the functions, $u_j(x) \in W_1[0,1]$, $j = 1,2$ are unknown functions to be determined, $f_j(x)$, $j = 1,2$ are continuous functions or square integrable on [a ,b], and the functions $u_j(y)$, are linear functions. The kernels k_{jl} are singular kernels of the generalized from given by $k_{jl} = \frac{1}{[g(x)-g(y)]^{\alpha_{jl}}}, g(x) = x$, $1 \le j,l \le 2$, $0 \le \alpha_{jl} \le 1$. Several valid methods for solving VIE have been developed in recent years, including power series method[2], Adomain's decomposition method [3], homotopy perturbation method[4,5], block by block method [6],and expansion method [7].

Reproducing kernel theory has important application in numerical analysis, differential equation, probability and statistics and so on [8,9], and the RKHS method to applied approximate method in the space $W_1[0,1]$

For obtaining the numerical solutions.

This paper organizing as follows: To transforming (1.1) into a system of LODEs using Taylor's expansion will be put forward in the section 2. In section 3, construction of RKHS. Discusses the implementation of method in section 4. In section 5, the numerical examples are present. Finally, the conclusions will give in section 6.

2- Taylor's Expansion

Consider the following system of linear weakly singular VIEs:

$$\left.\begin{aligned} u_1(x) &= f_1(x) + \int_a^x \left(\frac{1}{[g(x)-g(y)]^{\alpha_{11}}} u_1(y) + \frac{1}{[g(x)-g(y)]^{\alpha_{12}}} u_2(y)\right) dy, \\ u_2(x) &= f_2(x) + \int_a^x \left(\frac{1}{[g(x)-g(y)]^{\alpha_{21}}} u_1(y) + \frac{1}{[g(x)-g(y)]^{\alpha_{22}}} u_2(y)\right) dy \end{aligned}\right\} \tag{2.1}$$

rewriting (2.1) as follows:

$$\left.\begin{aligned} u_1(x) &= f_1(x) + \frac{(g(x)-g(a))^{1-\alpha_{11}} u_1(x)}{(1-\alpha_{11}) g'(a)} - \int_a^x \big(g(x)-g(y)\big)^{1-\alpha_{11}} \frac{u_1(y)-u_1(x)}{g(y)-g(x)} dy \\ &+ \frac{(g(x)-g(a))^{1-\alpha_{12}} u_2(x)}{(1-\alpha_{12}) g'(a)} - \int_a^x \big(g(x)-g(y)\big)^{1-\alpha_{12}} \frac{u_2(y)-u_2(x)}{g(y)-g(x)} dy \\ u_2(x) &= f_2(x) + \frac{(g(x)-g(a))^{1-\alpha_{21}} u_1(x)}{(1-\alpha_{21}) g'(a)} - \int_a^x \big(g(x)-g(y)\big)^{1-\alpha_{21}} \frac{u_1(y)-u_1(x)}{g(y)-g(x)} dy \\ &+ \frac{(g(x)-g(a))^{1-\alpha_{22}} u_2(x)}{(1-\alpha_{22}) g'(a)} - \int_a^x \big(g(x)-g(y)\big)^{1-\alpha_{22}} \frac{u_2(y)-u_2(x)}{g(y)-g(x)} dy \end{aligned}\right\} \tag{2.2}$$

by using Taylor's expansion about $y = x$ of degree n for $u_1(y)$ and $u_2(x)$ we get:

$$\left.\begin{aligned} u_1(y) - u_1(x) &\approx \big(g(y)-g(x)\big) u_1'(x) + \frac{\big(g(y)-g(x)\big)^2}{2!} u_1''(x) + \cdots + \frac{\big(g(y)-g(x)\big)^n}{n!} u_1^{(n)}(x) \\ u_2(y) - u_2(x) &\approx \big(g(y)-g(x)\big) u_2'(x) + \frac{\big(g(y)-g(x)\big)^2}{2!} u_2''(x) + \cdots + \frac{\big(g(y)-g(x)\big)^n}{n!} u_2^{(n)}(x) \end{aligned}\right\} \tag{2.3}$$

substituting (2.3) into the right hand side of (2.2) respectively, yields

$$\left.\begin{aligned} u_1(x) &= f_1(x) + \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{11}}}{i!(i+1-\alpha_{11}) g'(a)} u_1^{(i)}(x) + \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{12}}}{i!(i+1-\alpha_{12}) g'(a)} u_2^{(i)}(x) \\ u_2(x) &= f_2(x) + \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{21}}}{i!(i+1-\alpha_{21}) g'(a)} u_1^{(i)}(x) + \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{22}}}{i!(i+1-\alpha_{22}) g'(a)} u_2^{(i)}(x) \end{aligned}\right\} \tag{2.4}$$

suppose that

$$N_1(x, u_1(x)) = \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{11}}}{i!(i+1-\alpha_{11}) g'(a)} u_1^{(i)}(x), \quad N_2(x, u_1(x)) = \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{21}}}{i!(i+1-\alpha_{21}) g'(a)} u_1^{(i)}(x),$$

$$N_2(x, u_1(x)) = \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{21}}}{i!(i+1-\alpha_{21}) g'(a)} u_1^{(i)}(x), \quad \widetilde{N}_2(x, u_2(x)) = \sum_{i=0}^n (-1)^i \frac{\big(g(x)-g(a)\big)^{i+1-\alpha_{22}}}{i!(i+1-\alpha_{22}) g'(a)} u_2^{(i)}(x),$$

then (2.4) take the forms:

$$\left.\begin{aligned} u_1(x) &= f_1(x) + N_1(x, u_1(x)) + \widetilde{N}_1(x, u_2(x)) \\ u_2(x) &= f_2(x) + N_2(x, u_1(x)) + \widetilde{N}_2(x, u_2(x)) \end{aligned}\right\} \tag{2.5}$$

respectively. Therefore, (2.1) can be approximate by the nth-order system of LODEs (2.5).

3- Construction of RKHS

Definition 3.1 Let H be a Hilbert space of functions $u : X \to \mathbb{R}$. Denote by $\langle .,. \rangle$ the inner product and let $\|.\| = \sqrt{\langle .,. \rangle}$ be the induced norm in H the function $R : X \times X \to \mathbb{R}$ is called a reproducing kernel of H if the followings are satisfied [10]

(i) $R_x(.) = R(x,.) \in \mathrm{H}, \forall x \in X$,

(ii) $\langle u(.), R_x(.) \rangle = u(x)$, $\forall u \in \mathrm{H}$, for all $x \in X$ (the reproducing property).

The value of the function u at the point x is reproduce by the inner product of u with $R(x,.)$. Then H called a RKHS.

3.1 The reproducing kernel Hilbert space $W_1[0,1]$

The inner product space $W_1[0,1]$ is defined as $W_1[0,1]=\{u_1(x) \mid u_1(x)$ is an absolutely continuous real valued function in $[0,1]$, $u_1'(x) \in L^2[0,1]\}$. The inner product and norm in $W_1[0,1]$ are given by [11],

$$\langle u_1(x), u_2(x)\rangle_{W_1} = u_1(0)u_2(0) + \int_0^1 u_1'(x)u_2'(x)\,dx, \quad u_1(x), u_2(x) \in W_1[0,1] \tag{3.1}$$

$$\|u_1\|_{W_1} = \sqrt{\langle u_1(x), u_1(x)\rangle_{W_1}} \quad , \quad \text{where} \quad u_1(x) \in W_1[0,1] \tag{3.2}$$

respectively. It has been prove that $W_1[0,1]$is a complete reproducing kernel space.

Theorem 3.1 The reproducing kernel of $W_1[0,1]$ is

$$R_x(y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases} \tag{3.3}$$

Proof. Appling (3.1), we have

$$\langle u_1(y), R_x(y)\rangle_{W_1} = u_1(0)R_x(0) + \int_0^1 u_1'(y)R_x'(y)\,dy, \tag{3.4}$$

by integrating (3.4) by parts gives

$$\langle u_1(y), R_x(y)\rangle_{W_1} = u_1(0)\big(R_x(0) - R_x'(0)\big) + u_1(1)R_x'(1) - \int_0^1 u_1(y)R_x''(y)\,dy$$

in order to $\langle u_1(y), R_x(y)\rangle_{W_1} = u_1(x)$, it is enough to request the following equalities hold

$$-R_x''(y) = \delta(y-x) \tag{3.5}$$

$$R_x(0) = R_x'(0), \qquad R_x'(1) = 0, \tag{3.6}$$

where $\delta$is a Dirac delta function. From (3.5), if $y \neq x$ then $R_x''(y) = 0$, have a characteristic equation is $\lambda^2 = 0$. Then we obtain

$$R_x(y) = \begin{cases} a_1(x) + a_2(x)y\,, & y \le x \\ b_1(x) + b_2(x)y\,, & y > x \end{cases} \tag{3.7}$$

integrate both sides of (3.5) from $x-\varepsilon$ to $x+\epsilon$ with respect to y and let $\varepsilon \to 0$, one gets

$$R_x(x+0) = R_x(x-0), \tag{3.8}$$

$$R_x'(x-0) - R_x'(x+0) = 1, \tag{3.9}$$

to obtain (3.3) must be solve (3.5), (3.6), (3.8) and (3.9) then substituting the results of a_1, a_2, b_1 and b_2 into (3.7). ∎

4- The Method Implementation

To implement the RKHS method, we rewrite equations (2.5) are follows

$$Lu_1(x) = f_1(x) + N_1(x, u_1(x)) + \widetilde{N}_1(x, u_2(x)), \tag{4.1a}$$

$$Lu_2(x) = f_2(x) + N_2(x, u_1(x)) + \tilde{N}_2(x, u_2(x)), \tag{4.1b}$$

where, $L : W_1[0,1] \to W_1[0,1]$ is an invertible bounded linear operator, N_1, $\tilde{N}_1$, N_2 and $\tilde{N}_2 \in W_1[0,1]$ and $f_1(x), f_2(x)$ are continuous functions in $W_1[0,1]$. $W_1[0,1]$ is a reproducing kernel space defined according to the highest derivatives involved in (2.5). We choose a countable set of points $\{x_m\}_{m=1}^{\infty}$ in the interval $[0,1]$. Define

$$\varphi_m(x) = R(x, x_m), \qquad \psi_m(x) = L^*\varphi_m(x),$$

where L^* is the conjugate operator of L.

The orthonormal system of $\{\bar{\psi}_m\}_{m=1}^{\infty}$ from the space $W_1[0,1]$ can be derive from Gram-Schmidt orthogonal process of $\{\psi_m\}_{m=1}^{\infty}$,

$$\bar{\psi}_m(x) = \sum_{k=1}^{m} \beta_{mk}\, \psi_k(x)\,, \tag{4.2}$$

where β_{mk} are orthogonalization coefficients such that $\beta_{mm} > 0$, $m = 1,2,\ldots$

Theorem 4.1 If $\{x_m\}_{m=1}^{\infty}$ is dense on $[0,1]$ then $\{\psi_m\}_{m=1}^{\infty}$ is the complete function system of the space $W_1[0,1]$ and $\psi_m(x) = L_y R_x(y)|_{y=x_m}$, where the subscript y in the operator L indicates that the operator L applies to the function of y [12].

Proof. We have

$$\psi_m(x) = \langle \psi_m(y), R_x(y)\rangle_{W_1[0,1]} = \langle (L^*\varphi_m)(y), R_x(y)\rangle_{W_1[0,1]}$$

$$= \langle \varphi_m(y), L_y R_x(y)\rangle_{W_1[0,1]} = L_y R_x(y)|_{y=x_m}. \tag{4.3}$$

$\forall\, u(x) \in W_1[0,1]$, let $\langle u(x), \psi_m(x)\rangle_{W_1[0,1]} = 0, \quad m = 1,2,\ldots$

$$i.e. \quad \langle u(x), (L^*\varphi_m)(x)\rangle_{W_1[0,1]} = \langle Lu(.), \varphi_m(.)\rangle_{W_1[0,1]}$$

$$= Lu(x_m) = 0. \tag{4.4}$$

$Lu(x) = 0$ when $\{x_m\}_{m=1}^{\infty}$ is dense on $[0,1]$, if L^{-1} exists then $u(x) = 0$. ■

Theorem 4.2 If $\{x_m\}_{m=1}^{\infty}$ is dense on $[0,1]$ and $u_1(x), u_2(x) \in W_1[0,1]$ are the solutions of equations (4.1), then these solutions are satisfy the following form,

$$\left.\begin{aligned} u_1(x) &= \sum_{m=1}^{\infty}\sum_{k=1}^{m} \beta_{mk}\,(f_1(x) + N_1(x, u_1(x)) + \tilde{N}_1(x, u_2(x)))\bar{\psi}_m(x) \\ u_2(x) &= \sum_{m=1}^{\infty}\sum_{k=1}^{m} \beta_{mk}\,(f_2(x) + N_2(x, u_1(x)) + \tilde{N}_2(x, u_2(x)))\bar{\psi}_m(x), \end{aligned}\right\} \tag{4.5}$$

respectively.

Proof. By definition (4.2), for $u_1(x)$ in (4.5) we have

$$u_1(x) = \sum_{m=1}^{\infty}\langle u_1(x), \bar{\psi}_m(x)\rangle_{W_1[0,1]}\, \bar{\psi}_m(x)$$

$$= \sum_{m=1}^{\infty}\langle u_1(x), \sum_{k=1}^{m} \beta_{mk}\, \psi_k(x)\rangle_{W_1[0,1]}\, \bar{\psi}_m(x)$$

$$= \sum_{m=1}^{\infty}\sum_{k=1}^{m} \beta_{mk}\,\langle u_1(x), L^*\varphi_k(x)\rangle_{W_1[0,1]}\, \bar{\psi}_m(x)$$

$$= \sum_{m=1}^{\infty}\sum_{k=1}^{m} \beta_{mk}\,\langle Lu_1(x), \varphi_k(x)\rangle_{W_1[0,1]}\, \bar{\psi}_m(x)$$

$= \sum_{m=1}^{\infty} \sum_{k=1}^{m} \beta_{mk} L u_1(x_k)\, \bar{\psi}_m(x)$

$= \sum_{m=1}^{\infty} \sum_{k=1}^{m} \beta_{mk} \big(f_1(x_k) + N_1\big(x_k, u_1(x_k)\big) + \tilde{N}_1\big(x_k, u_2(x_k)\big)\big) \bar{\psi}_m(x),$

further, $u_1(x) \in W_1[0,1]$ and $u_1(x) = \sum_{m=0}^{\infty} \langle u_1(x), \bar{\psi}_m(x) \rangle\, \bar{\psi}_m(x)$, the Fourier series expansion about orthonormal system $\{\bar{\psi}_m\}_{m=1}^{\infty}$ and $W_1[0,1]$ is a Hilbert space. Thus, the series $\sum_{m=0}^{\infty} \langle u_1(x), \bar{\psi}_m(x) \rangle\, \bar{\psi}_m(x)$ is convergent in the sense of $\|.\|_{W_1}$. Similarly, for $u_2(x)$ in (4.5) we get

$$u_2(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{m} \beta_{mk} \big(f_2(x_k) + N_2\big(x_k, u_1(x_k)\big) + \tilde{N}_2\big(x_k, u_2(x_k)\big)\big) \bar{\psi}_m(x).$$

■

By taking, a limited number of terms in the series solutions $u_1(x)$and $u_2(x)$ to get the approximate solutions $u_{1n}(x)$ and $u_{2n}(x)$, respectively, as follows:

$$\left.\begin{aligned} u_{1n}(x) &= \sum_{m=1}^{n} \sum_{k=1}^{m} \beta_{mk} \big(f_1(x_k) + N_1\big(x_k, u_1(x_k)\big) + \tilde{N}_1\big(x_k, u_2(x_k)\big)\big) \bar{\psi}_m(x) \\ u_{2n}(x) &= \sum_{m=1}^{n} \sum_{k=1}^{m} \beta_{mk} \big(f_2(x_k) + N_2\big(x_k, u_1(x_k)\big) + \tilde{N}_2\big(x_k, u_2(x_k)\big)\big) \bar{\psi}_m(x) \end{aligned}\right\} \tag{4.6}$$

Theorem 4.3 Consider $u_1(x)$ and $u_2(x) \in W_1[0,1]$ are the solutions of (4.1). Let $E_{1n}(x) = u_1(x) - u_{1n}(x)$ and $E_{2n}(x) = u_2(x) - u_{2n}(x)$. Then $E_{1n}(x)$and $E_{2n}(x)$ are a monotone decreasing in the sense of the norm of $W_1[0,1]$. i.e., $E_{1n} \to 0$ and $E_{2n} \to 0$ as $n \to \infty$.

Proof. From $u_1(x)$ and $u_{1n}(x)$ in (4.5) and (4.6) respectively.

$\|E_{1n}\|^2_{W_1[0,1]} = \|u_1(x) - u_{1n}(x)\|^2_{W_1[0,1]}$

$= \left\|\sum_{m=n+1}^{\infty} \sum_{k=1}^{n} \beta_{mk} \big(f_1(x_k) + N_1\big(x_k, u_1(x_k)\big) + \tilde{N}_1\big(x_k, u_2(x_k)\big)\big) \bar{\psi}_m(x)\right\|^2_{W_1[0,1]} = \left\|\sum_{m=n+1}^{\infty} A_m \bar{\psi}_m(x)\right\|^2_{W_1[0,1]} = \sum_{m=n+1}^{\infty} (A_m)^2,$

and, $\|E_{1n-1}\|^2_{W_1[0,1]} = \sum_{m=n}^{\infty} (A_m)^2.$

Thus, $\|E_{1n}\|^2_{W_1[0,1]} \le \|E_{1n-1}\|^2_{W_1[0,1]}$. Similarly from $u_2(x)$ and $u_{2n}(x)$ in (4.5) and (4.6) respectively,

$\|E_{2n}\|^2_{W_1[0,1]} = \|u_2(x) - u_{2n}(x)\|^2_{W_1[0,1]}$

$= \left\|\sum_{m=n+1}^{\infty} \sum_{k=1}^{n} \beta_{mk} \big(f_2(x_k) + N_2\big(x_k, u_1(x_k)\big) + \tilde{N}_2\big(x_k, u_2(x_k)\big)\big) \bar{\psi}_m(x)\right\|^2_{W_1[0,1]} = \left\|\sum_{m=n+1}^{\infty} B_m \bar{\psi}_m(x)\right\|^2_{W_1[0,1]} = \sum_{m=n+1}^{\infty} (B_m)^2,$

and $\|E_{2n-1}\|^2_{W_1[0,1]} = \sum_{m=n}^{\infty} (B_m)^2.$

Thus, $\|E_{2n}\|^2_{W_1[0,1]} \le \|E_{2n-1}\|^2_{W_1[0,1]}$. Where the coefficients A_m, B_m of $\bar{\psi}_m(x)$, $m = 1, \dots$.

Consequently, $E_{1n}(x)$and $E_{2n}(x)$are a monotone decreasing in the sense of the norm of $W_1[0,1]$. ■

Remark Since $W_1[0,1]$ is a Hilbert space, it is clear that $\sum_{m=n}^{\infty} (A_m)^2 < \infty$ and$\sum_{m=n}^{\infty} (A_m)^2 < \infty$. Therefore, the sequence u_{1n} and u_{2n} are convergent.

5-Numerical Examples

The RKHS method will be apply to solve two Examples. To illustrate the applicability and effectiveness of this method comparing with the exact solutions by find the absolute error (Abs. err.) between the exact and approximate solutions, by taking $x_m = \frac{m-1}{n-1}$, where $m = 1, \dots, n$, at some choices $n = 50$ and100. Symbolic and numerical computations performed by using MATLAB17.

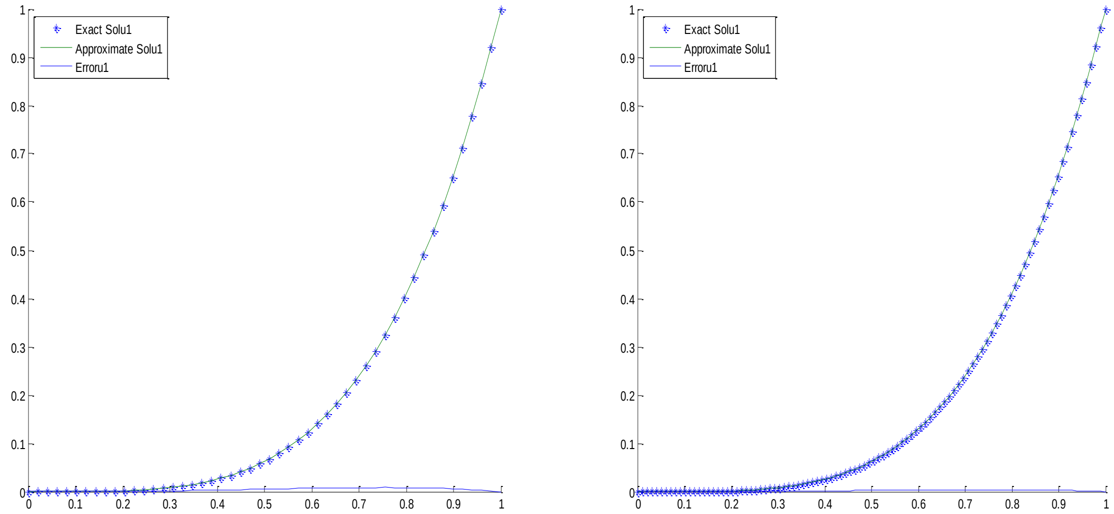
Example 5.1 [1] Consider the following system of linear weakly singular VIEs:

$$\left.\begin{aligned} u_1(x) &= f_1(x) + \int_0^x \Big(\frac{3}{(x-y)^{\frac{1}{2}}} u_1(y) + \frac{2}{(x-y)^{\frac{1}{2}}} u_2(y)\Big)dy \\ u_2(x) &= f_2(x) + \int_0^x \Big(\frac{2}{(x-y)^{\frac{1}{2}}} u_1(y) - \frac{3}{(x-y)^{\frac{1}{2}}} u_2(y)\Big)dy, \end{aligned}\right\} \quad 0 \le x \le 1, \tag{5.1}$$

The exact solution is $u_1(x) = x^4, u_2(x) = x^3$, let $f_1(x) = x^4 - \frac{64}{105} x^{\frac{7}{2}}(4x+3)$ and $f_2(x) = x^3 - \frac{32}{315} x^{\frac{7}{2}}(16x - 27)$. The numerical results are displayed in Tables 1 and 2.

Table 1. Results of $u_1(x)$ for Example 5.1

x_m	Exact sol. $u_1(x)$	Approximate sol. $u_{1,50}(x)$	Abs. err. $u_{1,50}(x)$	Approximate sol. $u_{1,100}(x)$	Abs. err. $u_{1,100}(x)$
0.1	$1.0000e-04$	$4.4407e-05$	$5.5593e-05$	$6.8301e-05$	$3.1699e-05$
0.2	$1.6000e-03$	$1.1381e-03$	$4.6189e-04$	$1.3567e-03$	$2.4333e-04$
0.3	$8.1000e-03$	$6.6639e-03$	$1.4361e-03$	$7.3629e-03$	$7.3706e-04$
0.4	$2.5600e-02$	$2.2606e-02$	$2.9937e-03$	$2.4083e-02$	$1.5166e-03$
0.5	$6.2500e-02$	$5.7552e-02$	$4.9480e-03$	$6.0013e-02$	$2.4872e-03$
0.6	$1.2960e-01$	$1.2269e-01$	$6.9104e-03$	$1.2614e-01$	$3.4558e-03$
0.7	$2.4010e-01$	$2.3181e-01$	$8.2904e-03$	$2.3597e-01$	$4.1307e-03$
0.8	$4.0960e-01$	$4.0130e-01$	$8.2954e-03$	$4.0548e-01$	$4.1217e-03$
0.9	$6.5610e-01$	$6.5017e-01$	$5.9308e-03$	$6.5316e-01$	$2.9405e-03$
1.0	$1.0000e+00$	$1.0000e+00$	0	$1.0000e+00$	0

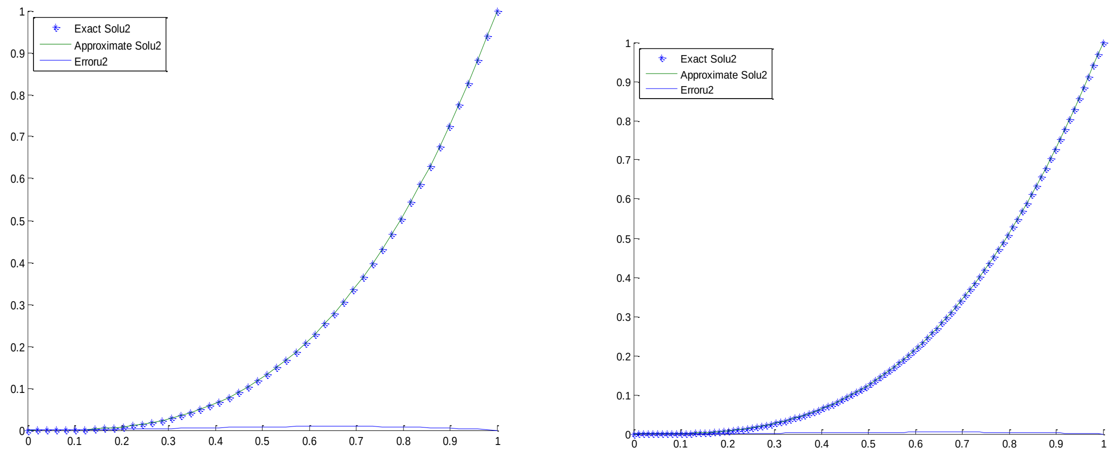


(a) (b)

Figure 1: Solution of RKHS method for $u_1(x)$ for $n = 50{,}100$ in (a) and (b) respectively.

Table 2. Results $u_2(x)$ for Example 5.1

x_m	Exact sol. $u_2(x)$	Approximate sol. $u_{2,50}(x)$	Abs. err. $u_{2,50}(x)$	Approximate sol. $u_{2,100}(x)$	Abs. err. $u_{2,100}(x)$
0.1	$1.0000e-03$	$5.4399e-04$	$4.5601e-04$	$7.5131e-04$	$2.4869e-04$
0.2	$8.0000e-03$	$6.1964e-03$	$1.8036e-03$	$7.0690e-03$	$9.3104e-04$
0.3	$2.7000e-02$	$2.3324e-02$	$3.6764e-03$	$2.5136e-02$	$1.8644e-03$
0.4	$6.4000e-02$	$5.8301e-02$	$5.6995e-03$	$6.1135e-02$	$2.8652e-03$
0.5	$1.2500e-01$	$1.1750e-01$	$7.4979e-03$	$1.2125e-01$	$3.7497e-03$
0.6	$2.1600e-01$	$2.0730e-01$	$8.6969e-03$	$2.1167e-01$	$4.3343e-03$
0.7	$3.4300e-01$	$3.3408e-01$	$8.9215e-03$	$3.3856e-01$	$4.4353e-03$
0.8	$5.1200e-01$	$5.0420e-01$	$7.7968e-03$	$5.0813e-01$	$3.8690e-03$
0.9	$7.2900e-01$	$7.2405e-01$	$4.9479e-03$	$7.2655e-01$	$2.4518e-03$
1.0	$1.0000e+00$	$1.0000e+00$	0	$1.0000e+00$	0



(a) (b)

Figure 2: Solution of RKHS method for $u_2(x)$ for $n = 50,100$ in (a) and (b) respectively.

Example 5.2 [1] Consider the following system of linear weakly singular VIEs:

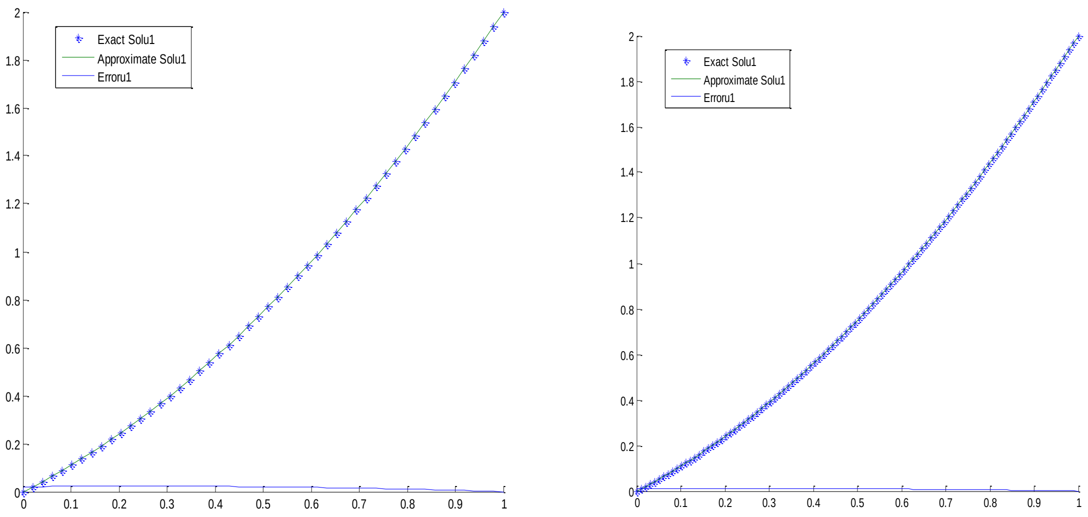
$$\left.\begin{aligned} u_1(x) &= f_1(x) + \int_0^x \Big(\frac{1}{(x-y)^{\frac{1}{5}}} u_1(y) + \frac{1}{(x-y)^{\frac{2}{5}}} u_2(y)\Big)dy \\ u_2(x) &= f_2(x) + \int_0^x \Big(\frac{1}{(x-y)^{\frac{3}{5}}} u_1(y) + \frac{1}{(x-y)^{\frac{4}{5}}} u_2(y)\Big)dy, \end{aligned}\right\} 0 \le x \le 1, \tag{5.2}$$

The exact solution is $u_1(x) = x + x^2$, $u_2(x) = x - x^2$, let $f_1(x) = x + x^2 - \frac{25}{6552} x^{\frac{8}{5}}(130x^{\frac{6}{5}} + 182x^{\frac{1}{5}} - 210x + 273)$ and $f_2(x) = x - x^2 - \frac{25}{924} x^{\frac{6}{5}}(55x^{\frac{6}{5}} + 66x^{\frac{1}{5}} - 140x + 154)$. The numerical results are displayed in Tables 3 and 4.

Table 3. Results of $u_1(x)$ for Example 5.2

x_m	Exact sol. $u_1(x)$	Approximate sol. $u_{1,50}(x)$	Abs. err. $u_{1,50}(x)$	Approximate sol. $u_{1,100}(x)$	Abs. err. $u_{1,100}(x)$
0.1	$1.1000e-01$	$8.8297e-02$	$2.1703e-02$	$9.9174e-02$	$1.0826e-02$
0.2	$2.4000e-01$	$2.1741e-01$	$2.2591e-02$	$2.2875e-01$	$1.1248e-02$
0.3	$3.9000e-01$	$3.6735e-01$	$2.2653e-02$	$3.7874e-01$	$1.1263e-02$
0.4	$5.6000e-01$	$5.3811e-01$	$2.1891e-02$	$5.4913e-01$	$1.0872e-02$
0.5	$7.5000e-01$	$7.2970e-01$	$2.0304e-02$	$7.3992e-01$	$1.0076e-02$
0.6	$9.6000e+01$	$9.4211e-01$	$1.7893e-02$	$9.5113e-01$	$8.8726e-03$
0.7	$1.1900e+00$	$1.1753e+00$	$1.4656e-02$	$1.1827e+00$	$7.2635e-03$
0.8	$1.4400e+00$	$1.4294e+00$	$1.0596e-02$	$1.4348e+00$	$5.2484e-03$
0.9	$1.7100e+00$	$1.7043e+00$	$5.7101e-03$	$1.7072e+00$	$2.8273e-03$

1.0	$2.0000e+00$	$2.0000e+00$	0	$2.0000e+00$	0

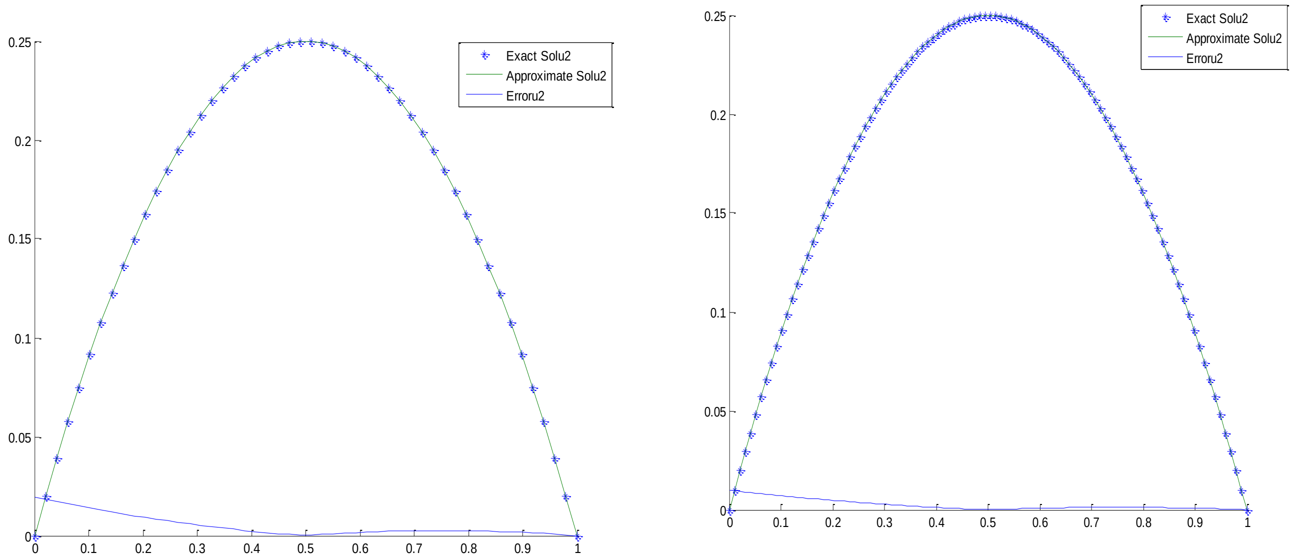


(a) (b)

Figure 3: Solution of RKHS method for $u_1(x)$ for $n = 50{,}100$ in (a) and (b) respectively.

Table 4. Results of $u_2(x)$ for Example 5.2

x_m	Exact sol. $u_2(x)$	Approximate sol. $u_{2,50}(x)$	Abs. err. $u_{2,50}(x)$	Approximate sol. $u_{2,100}(x)$	Abs. err. $u_{2,100}(x)$
0.1	$9.0000e-02$	$7.4969e-02$	$1.5031e-02$	$8.2645e-02$	$7.3554e-03$
0.2	$1.6000e-01$	$1.4994e-01$	$1.0062e-02$	$1.5509e-01$	$4.9138e-03$
0.3	$2.1000e-01$	$2.0408e-01$	$5.9184e-03$	$2.0712e-01$	$2.8783e-03$
0.4	$2.4000e-01$	$2.3740e-01$	$2.5989e-03$	$2.3875e-01$	$1.2489e-03$
0.5	$2.5000e-01$	$2.4990e-01$	$1.0412e-04$	$2.4997e-01$	$2.5508e-05$
0.6	$2.4000e-01$	$2.4157e-01$	$1.5660e-03$	$2.4079e-01$	$7.9176e-04$
0.7	$2.1000e-01$	$2.1241e-01$	$2.4115e-03$	$2.1120e-01$	$1.2029e-03$
0.8	$1.6000e-01$	$1.6243e-01$	$2.4323e-03$	$1.6121e-01$	$1.2080e-03$
0.9	$9.0000e-02$	$9.1628e-02$	$1.6285e-03$	$9.0807e-02$	$8.0706e-04$
1.0	0	0	0	0	0



(a) (b)

Figure 4: Solution of RKHS method for $u_2(x)$ for $n = 50{,}100$ in (a) and (b) respectively.

6. Conclusions

In this paper, we have deduced the solution of system of linear weakly singular VIE by the solution system of LODE by using Taylor's expansion to get rid of the singularity. Then we have portray the solution of this system of LODE by RKHS method, which compared with the exact solution we found that the method is easy to apply and full efficiency in scientific applications to build a solution without linearization, turbulence or discretization. Therefore, it can concluded that the proposed method is very strong and powerful.

References

1. M. Wazwaz, Linear and nonlinear integral equations, Methods and Applications, HEP and Springer, Peking and Berlin, (2011).
2. Tahmasbi and O.S. Fard, Numerical solution of linear Volterra integral equations system of the second kind, Applied Mathematics and Computation, vol. 201. No. 1-2, pp.547-552, (2008).
3. E. Babolian, J. Biazar, and A. R. Vahidi, On the decomposition method for system of linear equations and system of linear Volterra integral equations, Applied Mathematics and Computation, vol. 147, no.1, pp. 19-27, (2004).
4. J. Biazar and H. Ghazvini, He's homotopy perturbation method for solving systems of Volterra integral equations of the second kind, Chaos, Solutions and Fractals, vol. 39, no 2,pp. 770-777, (2009).
5. E. Yusufoglu , A homotopy perturbation algorithm to solve a system of Fredholm- Volterra type integral equations, Mathematical and Computer Modelling, vol.47, no. 11-12, pp. 1099–1107, (2008).
6. R. Katani and S. Shahmorad, Block by block method for the systems of nonlinear Volterra integral equations, Applied Mathematical Modelling, vol. 34, no. 2, pp. 400–406,(2010).

7. M. Rabbani, K. Maleknejad, and N. Aghazadeh, "Numerical computational solution of the Volterra integral equations system of the second kind by using an expansion method," Applied Mathematics and Computation, vol. 187, no. 2, pp. 1143–1146, (2007).
8. H. Yao, The research of algorithms for some singular differential equations of higher even-order, Department of mathematics, Harbin Institute of Technology, 2008.
9. M. Cui and Y. Lin, Nonlinear numerical Analysis in the Reproducing kernel space, Nova Science Publisher, New York, NY, USA, 2008.
10. A. Berlinet and C. Thomas-Agnan, Reproducing kernel Hilbert space in probability and statistics, Kluwer Academic Publishers, 2004.
11. Z. Chen and Y. Z. Lin, The exact solution of a linear integral equation with weakly Singular kernel, Journal of Mathematical Analysis and Applications, vol. 344, pp.726- 734, (2008).
12. S. Javadi, E. Babolian and E. Moradi, New implementation of reproducing kernel Hilbert space method for solving a class of functional integral equations, International Scientific Publications and consulting Services, vol. 2014, pp. 1-7, 2014.