



Role of conservation laws in the development of nonequilibrium and emergence of turbulence. Peculiarities of calculating nonequilibrium flows

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ABSTRACT

It turns out that the equations of mathematical physics, which consist equations of the conservation laws for energy, linear momentum, angular momentum, and mass, possess additional, hidden, properties that enables one to describe not only a variation of physical quantities (such as energy, pressure, density) but also processes such as origination of waves, vortices, turbulent pulsations and other ones. It is caused by the conservation laws properties.

In present paper the development of nonequilibrium in gasdynamic systems, which are described by the Euler and Navier-Stokes equations, will be investigated.

Under studying the consistence of conservation laws equations, from the Euler and Navier-Stokes equations it can be obtained the evolutionary relation for entropy (as a state functional). The evolutionary relation possesses a certain peculiarity, namely, it turns out to be nonidentical. This fact points out to inconsistency of the conservation law equations and noncommutativity of conservation laws.

Such a nonidentical relation discloses peculiarities of the solutions to the Navier-Stokes equations due to which the Euler and Navier-Stokes equations can describe the processes the development of nonequilibrium and emergence of vortices and turbulence.

It has been shown that such processes can be described only with the help of two nonequivalent coordinate systems or by simultaneous using numerical and analytical methods.

Keywords: duality of functionals, conservation laws, nonidentical evolutionary relation, connection of physical fields with material media.

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1. Introduction

It is well known that the the equations of mathematical physics, which consist equations of the conservation law equations for energy, linear momentum, angular momentum, and mass, are used for describing material systems such as thermodynamical, gas-dynamical, cosmical, and other ones. The Navier-Stokes and Euler equations are examples of such equations.

However, under the description of development of nonequilibrium processes by means of these equations, some difficulties arise.

It turns out that these equations possess additional, hidden, properties that enables one to describe not only a variation of physical quantities but also the development of nonequilibrium and emergence of various structures and formations. This is due to the properties of conservation laws.

In present paper the Euler and Navier-Stokes equations for gas-dynamical system that describe the processes of development of nonequilibrium and the emergence of vorticity and turbulence will be investigated.

As it is known, the Euler and Navier-Stokes equations, which describe fluid and gas flows, consist of the conservation law equations for energy, linear momentum, angular momentum, and mass. It turns out that the conservation law equations are inconsistent. This fact points out to the noncommutativity of the conservation law equations that brings into existence of internal forces, development of nonequilibrium and emergence of various formations examples of which are waves, vortices, turbulent pulsations.

This follows from the relation for entropy which is the state functional for hydrodynamical and gasdynamic systems. Such relation, which describes nonequilibrium processes, is obtained from the Euler and Navier-Stokes equations.

[The concept of "state functional" relates to the properties of physical quantities of material medium (like temperature, energy, pressure or density) and the conservation law properties. Since the physical quantities relates to a single material medium, a connection between them should exist. Such a connection defines the material system state (nonequilibrium or locally equilibrium one.) This means that there exists a characteristics of material system that defines its state. The state functional is just a functional that is a characteristics of material media. It should be noted that every material medium has state functional of its own. So the entropy is a state functional of thermodynamical, hydrodynamic and gasdynamic systems. However, the entropy that depends on thermodynamic variables is the state functional of thermodynamic system, the entropy that depends on space-time variables are the state functional of hydrodynamic and gasdynamic systems. Such functionals as the action functional, Poynting's vector, Einstein's tensor, wave function, and others are state functionals.]

2. Evolutionary relation for entropy considered as a state

functional of gasdynamic system

Evolutionary relation for entropy is obtained under investigating the consistence of conservation law equations that made up the set of Euler and Navier-Stokes equations. (Equations are consistent ones if they can be convolved into an identical relation for differentials.)

For investigation of the consistency of the conservation law equations it is necessary to use two nonequivalent frames of reference: the inertial frame of reference (the Euler frame of reference is an example of such frame) and the accompanying one. The accompanying frame of reference is a frame of reference connected with the manifold made up by the trajectories of the material system elements (the Lagrange frame of reference is an example of such a frame).



Let us analyze the correlation of the equations that describe conservation laws for energy and linear momentum

In the inertial frame of reference the energy equation can be reduced to the form:

$$\frac{Ds}{Dt} = A_1 \quad (1)$$

Here s is the entropy.

In the case of viscous heat-conducting gas described the Navier-Stokes equations the expression A_1 can be written as (see [1], Chapter 6, formula (6.2.4))

$$A_1 = \frac{1}{\rho} \frac{\partial}{\partial x_i} \left(-\frac{q_i}{T} \right) - \frac{q_i}{\rho T} \frac{\partial T}{\partial x_i} + \frac{\tau_{ki}}{\rho} \frac{\partial u_i}{\partial x_k} \quad (2)$$

Here q_i is the heat flux and τ_{ki} is the viscous stress tensor.

In the case of ideal gas described by the Euler equations we have $A_1 = 0$.

Since the total derivative with respect to time is that along the trajectory, in the accompanying frame of reference *the equation of the conservation law for energy takes the form:*

$$\frac{\partial s}{\partial \xi^1} = A_1 \quad (3)$$

where ξ^1 is the coordinate along the trajectory.

In the accompanying frame of reference *the equation of conservation law for linear momentum can be presented as*

$$\frac{\partial s}{\partial \xi^v} = A_v \quad (4)$$

where ξ^v is the coordinate in the direction normal to the trajectory. In the case of two-dimensional flow of ideal gas one can obtain the following expression for the coefficient A_v (see [1], Chapter 6, formula (6.7.12)):

$$A_v = \frac{\partial h_0}{\partial \xi^v} + (u_1^2 + u_2^2)^{1/2} \zeta - F_v + \frac{\partial U_v}{\partial t} \quad (5)$$

where $\zeta = \partial u_2 / \partial x - \partial u_1 / \partial y$.



In the case of viscous gas the expression A_ν includes additional terms related to viscosity and heat-conductivity.

One can see that in the accompanying frame of reference the equations for energy and linear momentum are reduced to the equations for derivatives of entropy s . In this case equation (3) obtained from the energy equation defines the derivative of entropy along the trajectory, and equation (4), assigned to the equation for linear momentum, defines the derivatives of entropy in the direction normal to trajectory.

Equations (3) and (4) can be convoluted into the relation

$$ds = \omega \quad (6)$$

where $\omega = A_\mu d\xi^\mu$ is the first degree skew-symmetric differential form [2] and $\mu = 1, \nu$. (A summing over repeated indices is carried out.) Since the conservation law equation are evolutionary ones, the relation obtained is also an evolutionary relation. In this case the skew-symmetric form ω is evolutionary one as well.

Relation (6) has been obtained from the conservation law equation for energy and linear momentum. In this relation the form ω is that of the first degree. Taking into account the conservation law equations for angular momentum and mass, the evolutionary relation may be written as

$$ds = \omega^p \quad (7)$$

where the form degree p takes the values $p = 1, 2, 3$.

3. Nonidentity of the evolutionary relation. Noncommutativity of the conservation law

Evolutionary relation (6) has a certain peculiarity. This relation appears to be nonidentical. This relation involves the skew-symmetric differential form ω , which is unclosed and cannot be a differential like the left-hand side of this relation. The evolutionary form ω is not closed since the differential of evolutionary form ω and its commutator are nonzero.

The differential of evolutionary form ω is expressed as $d\omega = \sum K_{1\nu} d\xi^1 d\xi^\nu$, where $K_{1\nu}$ are components of the form commutator. Without accounting for terms that are connected with the deformation of the manifold made up by the trajectories, the commutator can be written as

$$K_{1\nu} = \frac{\partial A_\nu}{\partial \xi^1} - \frac{\partial A_1}{\partial \xi^\nu} \quad (8)$$

The coefficients A_μ of the form ω have been obtained either from the equations of the conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. The



commutator of the form ω constructed of the derivatives of such coefficients is nonzero. Since the commutator of the form ω is nonzero, this means that the differential of the form ω is nonzero as well. Thus, the form ω proves to be unclosed and is not a differential. In the left-hand side of relation (6) it stands a differential, whereas in the right-hand side it stands an unclosed form that is not a differential. Such a relation cannot be an identical one.

The nonidentity of the evolutionary relation points to the fact that the conservation law equations for energy and linear momentum appear to be inconsistent. This means that the conservation laws are noncommutative. Below it will be shown that the noncommutativity of conservation laws leads to the fact that, due to inconsistency, the external actions cannot be directly transformed into the quantities of gasdynamic system of its own and acts as internal forces leading to development of nonequilibrium. The evolutionary relation, which turns out to be nonidentical, describes such processes.

The evolutionary relation, firstly, discloses specific features of the solutions to Euler and Navier-Stokes equations describing nonequilibrium processes and, secondly, describes the processes of nonequilibrium development and emergence of observable formations (such as waves, vortices, turbulent pulsations, and so on).

4. Peculiarities of the solutions to Euler and Navier-Stokes

equations. Processes of nonequilibrium development and emergence of observable formations

From the evolutionary relation it follows that the Euler and Navier-Stokes equations possess solutions of two types, namely, the solutions that are not functions (they depend not only on the variables) and the solutions that are discrete functions.

Inexact solutions to the Euler and Navier-Stokes equations (the solutions that are not functions)

The nonidentity of the evolutionary relation points to the fact that the conservation law equations for energy and linear momentum (entered into the set of Euler and Navier-Stokes equations) appear to be inconsistent. They cannot be contracted into an identical relation (which is built by differentials) and integrated directly. This means that the solutions to equations are not functions, which depend only on variables. Such solutions will depend on the commutator of the form ω which enters into the evolutionary relation. (If the commutator be equal to zero, the evolutionary relation would be identical and the equations would be integrated directly).

Physical meaning of inexact solutions. Nonequilibrium state of gasdynamic system

Inexact solutions have a physical meaning. They describe a nonequilibrium state of gasdynamic system. This follows from the evolutionary relation.

Evolutionary relation (see (6) and (7)) includes a differential of entropy, which is a state functional.

If from evolutionary relation the differential of entropy could be obtained, this would point to the fact that entropy is a state function. And this would mean that the state of a gasdynamic system is a



equilibrium one. But, since evolutionary relation is a nonidentical relation, from that one cannot obtain the differential of entropy and find the state function. This means that the gasdynamic system is in a non-equilibrium state.

One can see that *the solutions of the Euler and Navier-Stokes equations, which are not functions, describe a nonequilibrium state of gasdynamic system.*

The nonequilibrium means that in a gasdynamic system an internal force acts. It is evident that the internal force is described by the commutator of skew-symmetric form \mathcal{O} . Everything that gives a contribution into the commutator of the evolutionary form \mathcal{O} leads to emergence of internal force that causes the nonequilibrium state of a gasdynamic system. [From the analysis of the expression A_{μ} in formulas (2) and (5) one can see that the terms, which are related to the multiple connectedness of the flow domain, the nonpotentiality of the external forces and the nonstationarity of the flow contribute into the commutator. In the case of a viscous non-heat-conducting gas, the terms related to the transport processes will contribute to the commutator (see, formula (2)). All these factors lead to emergence of internal forces, that is, to nonequilibrium, and to development of various types of instability.]

Here it can be noted that the nonidentity of the evolutionary relation is connected with a noncommutativity of conservation laws. And this points out to the fact that the noncommutativity of conservation laws is a cause of nonequilibrium state of a gas-dynamic system.

Evolutionary relation is a selfvarying relation. Since one of the objects of evolutionary relation is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot be terminated.

The process of the evolutionary relation selfvariation describes the process of selfvariation of nonequilibrium state of gasdynamic system. This process proceeds under the internal force action and is described by inexact solutions to the Euler and Navier-Stokes equations.

Realization of exact solutions of the Euler and Navier-Stokes equations

The Euler and Navier-Stokes equations can have exact solutions (which are functions) in the case if from the evolutionary skew-symmetric form \mathcal{O} in the right-hand side of nonidentical evolutionary relation it is realized a closed skew-symmetric form, which is a differential. In this case the identical relation is obtained from the nonidentical relation, and this will point out to a consistency of the conservation law equations and an integrability of the Euler and Navier-Stokes equations.

However, from the evolutionary unclosed skew-symmetric form, which differential is nonzero, one can obtain a closed exterior form with a differential being equal to zero only under degenerate transformation (a transformation that does not conserve differential). This is possible only under additional conditions. The vanishing of such functional expressions as determinants, Jacobians, Poisson's brackets, residues and others corresponds to these additional conditions. This is due to realization of some degrees of freedom.

The conditions of degenerate transformation specify the integrable structures (pseudostructures) on which the solutions become exact ones. The characteristics (the determinant of coefficients at the normal derivatives vanishes), the singular points (Jacobian is equal to zero), potentials of simple and double layers, and others are such integrable structures.

The conditions of degenerate transformation can be realized under selfvariation of nonidentical evolutionary relation.



If the conditions of degenerate transformation be realized, from the unclosed evolutionary form ω (see evolutionary relation (6)) with nonvanishing differential $d\omega \neq 0$, one can obtain the differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$$d\omega \neq 0 \rightarrow (\text{degenerate transformation}) \rightarrow d_{\pi}\omega = 0, d_{\pi}^*\omega = 0$$

The realization of the conditions of $d_{\pi}^*\omega = 0, d_{\pi}\omega = 0$ means that it is realized a closed dual form $^*\omega$, which describes some an integrable structure π , and the closed exterior form ω_{π} which basis is an integrable structure obtained.

Since the form ω_{π} is a closed on pseudostructure form, this form turns out to be a differential. (It should be emphasized that such a differential is an interior one: it asserts only on pseudostructure, which is defined by the condition of degenerate transformation).

On the pseudostructure π , which is an integrable structure, from evolutionary relation (6) it is obtained the relation

$$d_{\pi}(s) = \omega_{\pi} \tag{9}$$

which occurs to be an identical one, since the form ω_{π} is a differential.

On integrable structures the desired quantities of gasdynamic system (such as the temperature, pressure, density) become functions of only independent variables and do not depend on the commutator (and on the path of integrating). These are generalized solutions, which are the discrete functions, since they are realized only under additional conditions (on the integrable structures). Such solutions may be found by means of integrating (on integrable structures) the Euler and Navier-Stokes equations.

Physical meaning of exact solutions. Transition of gasdynamic system into locally-equilibrium state. Emergence of observable formations

From identical relation one can obtain the differential of entropy ds and find entropy s as a function of space-time coordinates. It is precisely the entropy that will be a gasdynamic function of state. The availability of gasdynamic function of state would point out to equilibrium state of a gasdynamic system. However, since the identical relation is satisfied only under additional conditions, such a state of gasdynamic system will be a locally-equilibrium one.

Since the non-equilibrium state has been induced by an availability of internal force and in the case of locally-equilibrium state there is no internal force (in local domain of gasdynamic system), it is evident that under transition of gasdynamic system from non-equilibrium state into locally-equilibrium state the nonmeasurable quantity, which acts as internal force, changes to a measurable quantity. This manifests itself in the form of arising a certain observable measurable formation. Waves, vortices, turbulent pulsations and so on are examples of such formations.

Exact generalized solutions to the Euler and Navier-Stokes equations describe such observable formations arisen.



It can be noted that under calculating the streamline flow around solids by ideal gas described by the Euler equations, the contribution to commutator is due to a multiply connectedness. In this case, at $U > a$ (U is the velocity of the gas particle, a is the speed of sound) the transition to the locally equilibrium state is possible on the characteristics and on the envelopes of characteristics [3] as well, and weak shocks and shock waves are observable formations. If $U < a$ such a transition is possible only at singular points. The formations emerged due to a convection are of the vortex type. (One can see that in gasdynamic system, even in the case of ideal gas, it can emerge physical structures and relevant formations that lead to emergence of vorticity.)

Under calculating the streamline flow around solids by viscous gas described by the Navier-Stokes equations, the contribution to commutator is due to a multiply connectedness and viscosity. The transition to the locally-equilibrium state is allowed at singular points, because in this case $\partial s / \partial \xi^1 = A_1 \neq 0$, that is, the external exposure acts onto the gas particle separately. The development of instability and the transitions to the locally equilibrium state are allowed only in an individual fluid particle. Hence, the formations emerged behave as pulsations. These are turbulent pulsations.

5. On the problem of numerical solving the Euler and Navier-Stokes

equations

Problems of numerical solving the Euler and Navier-Stokes equations relate to the fact that these solutions are defined on different spatial objects. The solutions of one type are defined on initial coordinate space whereas the solutions of another type are defined on integrable structures. Such solutions cannot be obtained by a continuous numerical simulation of derivatives.

The solutions of first type can be obtained only by numerical modeling the equations on the original nonintegrable manifold (it is impossible to find such a solution by analytical method). To obtain the generalized solutions by numerical simulation, one must use second systems of reference (on integrable structure). The generalized solutions can be obtained also by analytical methods if the integrability conditions are imposed on the equations. The methods of characteristics, symmetries, eigen-functions and others are examples of such methods.

Therefore, to describe the emergence of vorticity and turbulence by numerical simulation, one must use two systems of reference, or by using simultaneously numerical and analytical methods.

CONCLUSION

In present paper it is shown that the Euler and Navier-Stokes equations possess additional, hidden, properties that enables one to describe not only a variation of physical quantities (such as energy, pressure, density) but also processes such as origination of waves, vortices, turbulent pulsations and other ones. When studying the consistence of conservation law equations, from these equations it follows the evolutionary relation for entropy as state functional, that describe the development of nonequilibrium.

A evolutionary relation discloses peculiarities of the solutions to the Euler and Navier-Stokes equations. From the evolutionary relation it follows that the Euler and Navier-Stokes equations possess solutions of two types, namely, the solutions that are not functions (they depends not only on the variables) and the solutions that are discrete functions. The solutions of the first type describe a nonequilibrium state of a gasdynamic system. And the solutions of the second type describe a locally-equilibrium state of a gasdynamic system. The transition from the solutions of the first type to ones of the second type



describe a transition of gasdynamic system from a nonequilibrium state to a locally-equilibrium state, and this process is accompanied by emergence of vorticity or turbulence. Such a process relates to the transition from tangent manifold to integral structures and can be performed only with the help of two nonequivalent coordinate systems or by simultaneous using numerical and analytical methods.

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