SOME NEW FORMULAS ON THE K-FIBONACCI NUMBERS

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ABSTRACT

In this paper, we find some formulas for finding some special sums of the k-Fibonacci or the k-Lucas numbers. We find also some formulas that relate the k-Fibonacci or the k--Lucas numbers to some sums of these numbers..

Keywords

k-Fibonacci numbers, k-Lucas numbers, Binomial transform, Geometric sum.

SUBJECT CLASSIFICATION

MSC2000: 15A36; 11C20; 11B39.

1 INTRODUCTION

There exist generalizations of the classical Fibonacci numbers given by many researchers as Horadam [4] and recently by Falcon and Plaza[3].

1.1 Definition of the k-Fibonacci numbers

For any positive real number k, the k–Fibonacci sequence, say $F_k = \{F_{k,n}\}_{n\in N}$, is defined by the recurrence relation

$$F_{k,n+1} = k\,F_{k,n} + F_{k,n-1} \qquad (1)$$

with initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$.

For k = 1, classical Fibonacci sequence is obtained and for k = 2, Pell sequence appears.

We define the negative k–Fibonacci numbers as $F_{k,-n} = (-1)^{n+1} F_{k,n}$.

In similar formFalcon [2], the k–Lucas numbers are defined as $L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}$ with initial conditions $L_{k,0} = 2, L_{k,1} = k$.

The well–known Binet formula in the Fibonacci numbers theory [5,3,1] allows us to express the k–Fibonacci and the k–Lucas numbers by mean of the roots σ_1 and σ_2 of the characteristic equation associated to the recurrence relation $r^2 - k\cdot r - 1 = 0$. If $\sigma_{1,2} = \dfrac{k \pm \sqrt{k+4}}{2}$, then

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \qquad (2)$$

$$L_{k,n} = \sigma_1^n + \sigma_2^n \qquad (3)$$

As a result of the above, $L_{k,n} = F_{k,n-1} + F_{k,n+1}$.

An interesting formula that we will apply next is the Convolution formula: $F_{k,n+m} = F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1}$.

Some properties of σ_1 and σ_2 that we will use in this paper are the following:

$$\sigma_1 + \sigma_2 = k,\ \ \sigma_1 - \sigma_2 = k^2 + 4,\ \ \sigma_1 \cdot \sigma_2 = -1,\ \ \sigma^2 = k\sigma + 1,\ \ k - \sigma_1 = -\frac{1}{\sigma_1} = \sigma_2$$

2 SOME NEW FORMULAS

First we will find a formula that we will use for obtain other simpler formulas.

2.1 Theorem

For $a, r, p \in R$

$$\sum_{j=0}^{n} \frac{F_{k,a\,j+r}}{p^j} = \frac{1}{p\,L_{k,a} - (-1)^a - p^2}\left[\frac{1}{p^n}\left(pF_{k,a(n+1)+r} - (-1)^a F_{k,an+r}\right) + (-1)^a pF_{k,r-a} - p^2 F_{k,r}\right]$$

Proof.

$$\sum_{j=0}^{n}\frac{F_{k,aj+r}}{p^j}=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\left(\frac{\sigma_1^{aj+r}}{p^j}-\frac{\sigma_2^{aj+r}}{p^j}\right)=\frac{1}{\sigma_1-\sigma_2}\left[\sigma_1^r\sum_{j=0}^{n}\left(\frac{\sigma_1^a}{p}\right)^j-\sigma_2^r\sum_{j=0}^{n}\left(\frac{\sigma_2^a}{p}\right)^j\right]$$

$$=\frac{1}{\sigma_1-\sigma_2}\left[\sigma_1^r\frac{\left(\frac{\sigma_1^a}{p}\right)^{n+1}-1}{\frac{\sigma_1^a}{p}-1}-\sigma_2^r\frac{\left(\frac{\sigma_2^a}{p}\right)^{n+1}-1}{\frac{\sigma_2^a}{p}-1}\right]=\frac{1}{\sigma_1-\sigma_2}\frac{1}{p^n}\left[\sigma_1^r\frac{\sigma_1^{a(n+1)}-p^{n+1}}{\sigma_1^a-p}-\sigma_2^r\frac{\sigma_2^{a(n+1)}-p^{n+1}}{\sigma_2^a-p}\right]$$

$$=\frac{1}{p^n}\frac{1}{(-1)^a-p(\sigma_1^a+\sigma_2^a)+p^2}\frac{1}{\sigma_1-\sigma_2}$$

$$\cdot\left[\sigma_1^r\left((-1)^a\sigma_1^{an}-p\sigma_1^{a(n+1)}-p^{n+1}\sigma_2^a+p^{n+2}\right)-\sigma_2^r\left((-1)^a\sigma_2^{an}-p\sigma_2^{a(n+1)}-p^{n+1}\sigma_1^a+p^{n+2}\right)\right]$$

$$=\frac{1}{p^n}\frac{1}{(-1)^a-p\,L_{k,a}+p^2}\frac{1}{\sigma_1-\sigma_2}$$

$$\cdot\left[(-1)^a\left(\sigma_1^{an+r}-\sigma_2^{an+r}\right)-p\left(\sigma_1^{a(n+1)+r}-\sigma_2^{a(n+1)+r}\right)-p^{n+1}(-1)^a\left(\sigma_1^{r-a}-\sigma_2^{r-a}\right)+p^{n+2}\left(\sigma_1^r-\sigma_2^r\right)\right]$$

$$=\frac{1}{p^n}\frac{1}{(-1)^a-p\,L_{k,a}+p^2}\left[(-1)^aF_{k,an+r}-pF_{k,a(n+1)+r}-(-1)^ap^{n+1}F_{k,r-a}+p^{n+2}F_{k,r}\right]$$

$$=\frac{1}{p\,L_{k,a}-(-1)^a-p^2}\left[\frac{1}{p^n}\left(pF_{k,a(n+1)+r}-(-1)^aF_{k,an+r}\right)+(-1)^a\,p\,F_{k,r-a}-p^2F_{k,r}\right]$$

If a=1 and r=0, then

$$\sum_{j=0}^{n}\frac{F_{k,j}}{p^j}=\frac{1}{1+kp-p^2}\left(\frac{1}{p^n}\left(pF_{k,n+1}+F_{k,n}\right)-p\right)\qquad(4)$$

Next we will apply this formula to the k–Lucas numbers. Because $L_{k,n}=\sigma_1^n+\sigma_2^n$, we can prove of a similar form

2.2 Corollary

$$\sum_{j=0}^{n}\frac{L_{k,aj+r}}{p^j}=\frac{1}{pL_{k,a}-(-1)^a-p^2}\left[\frac{1}{p^n}\left(pL_{k,a(n+1)+r}-(-1)^aL_{k,an+r}\right)+(-1)^a\,pL_{k,r-a}-p^2L_{k,r}\right]$$

If a=1 and r=0, then

$$\sum_{j=0}^{n}\frac{L_{k,j}}{p^j}=\frac{1}{1+k\,p-p^2}\left(\frac{1}{p^n}\left(pL_{k,n+1}+L_{k,n}\right)+k\,p-2p^2\right)\qquad(5)$$

2.3 Particular cases

From Equation (4) and Equation (5),

a) $p=k:\ \sum_{j=0}^{n}\frac{F_{k,j}}{k^j}=\frac{F_{k,n+2}}{k^n}-k\rightarrow\sum_{j=0}^{n}F_j=F_{n+2}-1$

$\sum_{j=0}^{n}\frac{L_{k,j}}{k^j}=\frac{L_{k,n+2}}{k^n}-k^2\rightarrow\sum_{j=0}^{n}L_j=L_{n+2}-1$

b) $p=\frac{1}{2}:\ \sum_{j=0}^{n}2^jF_{k,j}=\frac{1}{2k+3}\left[2^{n+1}(F_{k,n+1}+2F_{k,n})-2\right]\rightarrow\sum_{j=0}^{n}2^jF_j=\frac{1}{5}\left(2^{n+1}L_{n+1}-2\right)$

$\sum_{j=0}^{n}2^jL_{k,j}=\frac{1}{2k+3}\left[2^{n+1}(L_{k,n+1}+2L_{k,n})+2k-2\right]\rightarrow\sum_{j=0}^{n}2^jL_j=2^{n+1}F_{n+1}$ [1]

c) For p=1 we obtain the classical formulas

$$\sum_{j=0}^{n}F_{k,j}=\frac{1}{k}(F_{k,n+1}+F_{k,n}-1)\rightarrow\sum_{j=0}^{n}F_j=F_{n+2}-1$$

$$\sum_{j=0}^{n}L_{k,j}=\frac{1}{k}(L_{k,n+1}+L_{k,n}+k-2)\rightarrow\sum_{j=0}^{n}L_j=L_{n+2}-1$$

2.4 Corollary (Forthe alternated sums)

If p=-r, the equations (4) and (5) become

$$\sum_{j=0}^{n}(-1)^j\frac{F_{k,j}}{r^j}=\frac{1}{1-k\,r-r^2}\left(\frac{(-1)^n}{r^n}\left(-r\,F_{k,n+1}+F_{k,n}\right)+r\right)$$

$$\sum_{j=0}^{n}(-1)^j\frac{L_{k,j}}{r^j}=\frac{1}{1-k\,r-r^2}\left(\frac{(-1)^n}{r^n}\left(-r\,L_{k,n+1}+L_{k,n}\right)-r\,k-2r^2\right)$$

Then

a) If $r=1$: $\sum_{j=0}^{n}(-1)^j F_{k,j}=\frac{1}{k}\left((-1)^n\left(F_{k,n+1}-F_{k,n}\right)-1\right)\rightarrow\sum_{j=0}^{n}(-1)^j F_j=(-1)^n F_{n-1}-1$

$\sum_{j=0}^{n}(-1)^j l_{k,j}=\frac{1}{k}\left((-1)^n\left(L_{k,n+1}-L_{k,n}\right)+k+2\right)\rightarrow\sum_{j=0}^{n}(-1)^j L_j=(-1)^n L_{n-1}+3$

b) If $r=2$: $\sum_{j=0}^{n}(-1)^j\frac{F_{k,j}}{2^j}=\frac{1}{2k+3}\left(\frac{(-1)^n}{2^n}\left(2F_{k,n+1}-F_{k,n}\right)-2\right)\rightarrow\sum_{j=0}^{n}(-1)^j\frac{F_j}{2^j}=\frac{1}{5}\left((-1)^n\frac{L_n}{2^n}-2\right)$

$\sum_{j=0}^{n}(-1)^j\frac{L_{k,j}}{2^j}=\frac{1}{2k+3}\left(\frac{(-1)^n}{2^n}\left(2L_{k,n+1}-L_{k,n}\right)+2k+8\right)\rightarrow\sum_{j=0}^{n}(-1)^j\frac{L_j}{2^j}=(-1)^n\frac{F_n}{2^n}+2$

c) If $r=k$: $\sum_{j=0}^{n}(-1)^j\frac{F_{k,j}}{k^j}=\frac{1}{2k^2-1}\left(\frac{(-1)^n}{k^n}\left(k\,F_{k,n+1}-F_{k,n}\right)-k\right)$ (6)

$\sum_{j=0}^{n}(-1)^j\frac{L_{k,j}}{k^j}=\frac{1}{2k^2-1}\left(\frac{(-1)^n}{k^n}\left(k\,L_{k,n+1}-L_{k,n}\right)+3k^2\right)$

d) If $r=\frac{1}{p}$: $\sum_{j=0}^{n}(-1)^j p^j F_{k,j}=\frac{1}{p^2-k\,p-1}\left((-1)^n p^{n+1}\left(p\,F_{k,n}-F_{k,n+1}\right)+p\right)$

$\sum_{j=0}^{n}(-1)^j p^j L_{k,j}=\frac{1}{p^2-k\,p-1}\left((-1)^n p^{n+1}\left(p\,L_{k,n}-L_{k,n+1}\right)-k\,p-2\right)$

2.5 Corollary

By summing up the formulas (5) and (4),

$$\sum_{j=0}^{n}\frac{L_{k,j}+F_{k,j}}{p^j}=\frac{1}{p^n(1+k\,p-p^2)}\left((p\,k+p+2)F_{k,n+1}+(2p+1-k)(F_{k,n}-p^{n+1})\right) \quad (7)$$

Proof. Taking into account $L_{k,n}=F_{k,n+1}+F_{k,n-1}=2F_{k,n+1}-k\,F_{k,n}$, it is

$$\sum_{j=0}^{n}\frac{L_{k,j}+F_{k,j}}{p^j}=2\sum_{j=0}^{n}\frac{F_{k,j+1}}{p^j}+(1-k)\sum_{j=0}^{n}\frac{F_{k,j}}{p^j}.$$

On the auther hand, $\sum_{j=0}^{n}\frac{F_{k,j+1}}{p^j}=\frac{1}{1+k\,p-p^2}\left(\frac{1}{p^n}\left(p\,F_{k,n+2}+F_{k,n+1}\right)-p^2\right)$ because $\sum_{j=0}^{n}\frac{F_{k,j+1}}{p^j}=p\sum_{j=0}^{n}\frac{F_{k,j}}{p^j}+\frac{F_{k,n+1}}{p^n}$ and then we apply Formula (4). So,

$$\sum_{j=0}^{n}\frac{L_{k,j}+F_{k,j}}{p^j}=2\sum_{j=0}^{n}\frac{F_{k,j+1}}{p^j}+(1-k)\sum_{j=0}^{n}\frac{F_{k,j}}{p^j}=\frac{2}{1+k\,p-p^2}\left(\frac{1}{p^n}(pF_{k,n+2}+F_{k,n+1})-p^2\right)+\frac{1-k}{1+k\,p-p^2}\left(\frac{1}{p^n}(pF_{k,n+1}+F_{k,n})-p\right)$$

$$=\frac{1}{p^n(1+k\,p-p^2)}\left(2pF_{k,n+2}+2F_{k,n+1}+pF_{k,n+1}+F_{k,n}-kpF_{k,n+1}-kF_{k,n}\right)-\frac{2p^2+p-p\,k}{1+k\,p-p^2}$$

$$=\frac{1}{p^n(1+k\,p-p^2)}\left((pk+p+2)F_{k,n+1}+(2p+1-k)(F_{k,n}-p^{n+1})\right)$$

3 BINOMIAL TRANSFORM

In this section we study the binomial transform of some of the previous numbers. We find also the main theorem of this paper.

3.1 Theorem: First binomial transform.

$$\sum_{j=0}^{n}\binom{n}{j}k^j F_{k,an+r+j}=F_{k,(a+2)n+r}$$

*Proof.*Taking into account $1+k\sigma=\sigma^2$ it is

$$\sum_{j=0}^{n}\binom{n}{j}k^j F_{k,an+r+j}=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\binom{n}{j}\left(\sigma_1^{an+r}(k\sigma_1)^j-\sigma_2^{an+r}(k\sigma_2)^j\right)=\frac{1}{\sigma_1-\sigma_2}\left(\sigma_1^{an+r}(1+k\sigma_1)^n-\sigma_2^{an+r}(1+k\sigma_1)^n\right)$$

$$=\frac{1}{\sigma_1-\sigma_2}\left(\sigma_1^{an+r}\sigma_1^{2n}-\sigma_2^{an+r}\sigma_2^{2n}\right)=F_{k,(a+2)n+r}$$

In similar form, $\sum_{j=0}^{n}\binom{n}{j}k^j L_{k,an+r+j}=L_{k,(a+2)n+r}$

3.2 Theorem (Second binomial transform: Alternated binomial sums with the k-Fibonacci numbers)

Binomial transform.

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{F_{k,an+r+j}}{k^j}=(-1)^n\frac{F_{k,(a-1)n+r}}{k^n} \quad (8)$$

Proof. Taking into account $\sigma_1=-\frac{1}{\sigma_2}$ and $1+k\sigma=\sigma^2$, it is

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{F_{k,an+r+j}}{k^j}=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\binom{n}{j}\left(\sigma_1^{an+r}\left(\frac{-\sigma_1}{k}\right)^j-\sigma_2^{an+r}\left(\frac{-\sigma_2}{k}\right)^j\right)=\frac{1}{\sigma_1-\sigma_2}\left(\sigma_1^{an+r}\left(1-\frac{\sigma_1}{k}\right)^n-\sigma_2^{an+r}\left(1-\frac{\sigma_2}{k}\right)^n\right)$$

$$=\frac{1}{\sigma_1-\sigma_2}\left(\sigma_1^{an+r}\left(\frac{-1}{k\sigma_1}\right)^n-\sigma_2^{an+r}\left(\frac{-1}{k\sigma_2}\right)^n\right)=\frac{1}{\sigma_1-\sigma_2}\frac{(-1)^n}{k^n}\left(\sigma_1^{an+r-n}-\sigma_2^{an+r-n}\right)=\frac{(-1)^n}{k^n}F_{k,(a-1)n+r}$$

3.3 Corollary

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{L_{k,an+r+j}}{k^j}=(-1)^n\frac{L_{k,(a-1)n+r}}{k^n}$$

The proof is similar to the previous.

The following formula is an adaptation of the Formula (1.6) in [2]

4 MAIN THEOREM

Let t be an indeterminate. Then the following identity holds for n = 1, 2, 3,…

$$\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(tF_{k,j-r+2}-kF_{k,j-r+1}\right)=\left(\frac{t}{k}\right)^n tF_{k,n-r+2}-kF_{k,-r+1} \quad (9)$$

*Proof.*First, $1+(t-1)\sigma^2=1+t\sigma^2-\sigma^2=t\sigma^2-k\sigma=\sigma(t\sigma-k)$, and applying the Binet identity

$$\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(tF_{k,j-r+2}-kF_{k,j-r+1}\right)=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(t\left(\sigma_1^{j-r+2}-\sigma_2^{j-r+2}\right)-k\left(\sigma_1^{j-r+1}-\sigma_2^{j-r+1}\right)\right)$$

$$=\frac{1}{\sigma_1-\sigma_2}\left[\sigma_1^{-r+1}(t\sigma_1-k)\sum_{j=0}^{n}\left(\frac{t\sigma_1}{k}\right)^j-\sigma_2^{-r+1}(t\sigma_2-k)\sum_{j=0}^{n}\left(\frac{t\sigma_2}{k}\right)^j\right]$$

$$=\frac{1}{\sigma_1-\sigma_2}\left[\sigma_1^{-r+1}(t\sigma_1-k)\frac{\left(\frac{t\sigma_1}{k}\right)^{n+1}-1}{\frac{t\sigma_1}{k}-1}-\sigma_2^{-r+1}(t\sigma_2-k)\frac{\left(\frac{t\sigma_2}{k}\right)^{n+1}-1}{\frac{t\sigma_2}{k}-1}\right]$$

$$=\frac{k}{\sigma_1-\sigma_2}\left[\sigma_1^{-r+1}\frac{(t\sigma_1)^{n+1}-k^{n+1}}{k^{n+1}}-\sigma_2^{-r+1}\frac{(t\sigma_2)^{n+1}-k^{n+1}}{k^{n+1}}\right]$$

$$=\frac{k}{\sigma_1-\sigma_2}\left[\left(\frac{t}{k}\right)^{n+1}\sigma_1^{n-r+2}-\sigma_1^{-r+1}-\left(\frac{t}{k}\right)^{n+1}\sigma_2^{n-r+2}+\sigma_2^{-r+1}\right]=\left(\frac{t}{k}\right)^n tF_{k,n-r+2}-kF_{-r+1}$$

$$=\frac{1}{\sigma_1-\sigma_2}\left[\sigma_1^{-r+1}(t\sigma_1-k)\frac{\left(\frac{t\sigma_1}{k}\right)^{n+1}-1}{\frac{t\sigma_1}{k}-1}-\sigma_2^{-r+1}(t\sigma_2-k)\frac{\left(\frac{t\sigma_2}{k}\right)^{n+1}-1}{\frac{t\sigma_2}{k}-1}\right]$$

$$=\frac{k}{\sigma_1-\sigma_2}\left[\left(\frac{t}{k}\right)^{n+1}\sigma_1^{n-r+2}-\sigma_1^{-r+1}-\left(\frac{t}{k}\right)^{n+1}\sigma_2^{n-r+2}+\sigma_2^{-r+1}\right]=\left(\frac{t}{k}\right)^n tF_{k,n-r+2}-kF_{k,-r+1}$$

Taking into account $L_{k,n}=F_{k,n-1}+F_{k,n+1}$, this formula can be written as

$$\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(L_{k,j-r+1}+(t-2)F_{k,j-r+2}\right)=\left(\frac{t}{k}\right)^n tF_{k,n-r+2}-kF_{k,-r+1}$$

Particular cases:

- $t=k^2\rightarrow\sum_{j=0}^{n}k^j\left(kF_{k,j-r+2}-F_{k,j-r+1}\right)=k^{n+1}F_{k,n-r+2}-F_{k,-r+1}$
- $r=1\rightarrow\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(tF_{k,j+1}-kF_{k,j}\right)=\left(\frac{t}{k}\right)^n tF_{k,n+1}$
- $k=r=1\rightarrow\sum_{j=0}^{n}t^j\left(tF_{j+1}-F_j\right)=t^{n+1}F_{n+1}$
- $r=0\rightarrow\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(tF_{k,j+2}-kF_{k,j+1}\right)=\left(\frac{t}{k}\right)^n tF_{k,n+2}-k$

Is similar form we can prove the following corollary.

4.1Corollary. Formula (9) for the k-Lucas numbers.

$$\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(tL_{k,j-r+2}-kL_{k,j-r+1}\right)=\left(\frac{t}{k}\right)^n tL_{k,n-r+2}-kL_{k,-r+1} \qquad (10)$$

And then

- $r=1\rightarrow\sum_{j=0}^{n}\left(\frac{t}{k}\right)^j\left(tL_{k,j+1}-kL_{k,j}\right)=\left(\frac{t}{k}\right)^n tL_{k,n+1}-2k$

- $r=0 \to \sum_{j=0}^{n}\left(\frac{t}{k}\right)^{j}\left(t\,L_{k,j+2}-k\,L_{k,j+1}\right)=\left(\frac{t}{k}\right)^{n} t\,L_{k,n+2}-k^2$

4.2 Theorem. Formula (9) for the alternated sums.

$$\sum_{j=0}^{n}(-1)^{j}\left(\frac{t}{k}\right)^{j}\left(t\,F_{k,j-r+2}-k\,F_{k,j-r+1}\right)$$
$$=\frac{k}{t^2-k\,t-k^2}\left[(-1)^{n}\left(\frac{t}{k}\right)^{n+1}\left(\left(t^2+k^2\right)F_{k,n-r+2}-k\,t\,L_{k,n-r+2}\right)+\left(t^2+k^2\right)F_{k,-r+1}-k\,t\,L_{k,-r+1}\right]$$

To prove this theorem we must apply the following identities:

$$\sigma_1\cdot\sigma_2=-1,\quad \sigma_1+\sigma_2=k,\quad \sigma_1-\sigma_2=\sqrt{k^2+4},\quad 1+(t-1)\sigma^2=\sigma(t\,\sigma-k),\quad (t\,\sigma_1+k)(t\,\sigma_2+k)=-\left(t^2-k^2t-k^2\right),$$
$$(t\,\sigma_1-k)(t\,\sigma_2+k)=-\left(t^2-k\,t\sqrt{k^2+4}-k^2\right),\quad (t\,\sigma_1+k)(t\,\sigma_2-k)=-\left(t^2+k\,t\sqrt{k^2+4}-k^2\right)$$

Proof.

$$\sum_{j=0}^{n}(-1)^{j}\left(\frac{t}{k}\right)^{j}\left(t\,F_{k,j-r+2}-k\,F_{k,j-r+1}\right)=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\left(\frac{-t}{k}\right)^{j}\left(t\left(\sigma_1^{j-r+2}-\sigma_2^{j-r+2}\right)-k\left(\sigma_1^{j-r+1}-\sigma_2^{j-r+1}\right)\right)$$

$$=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\left(\frac{-t}{k}\right)^{j}\left(t\left(\sigma_1^{j-r+2}-\sigma_2^{j-r+2}\right)-k\left(\sigma_1^{j-r+1}-\sigma_2^{j-r+1}\right)\right)$$

$$=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\left(\left(\frac{-t\sigma_1}{k}\right)^{j}\sigma_1^{-r+1}\left(t\,\sigma_1-k\right)-\left(\frac{-t\sigma_2}{k}\right)^{j}\sigma_2^{-r+1}\left(t\,\sigma_2-k\right)\right)$$

$$=\frac{1}{\sigma_1-\sigma_2}\left(\sigma_1^{-r+1}\left(t\,\sigma_1-k\right)\frac{\left(\frac{-t\,\sigma_1}{k}\right)^{n+1}-1}{\frac{-t\,\sigma_1}{k}-1}-\sigma_2^{-r+1}\left(t\,\sigma_2-k\right)\frac{\left(\frac{-t\,\sigma_2}{k}\right)^{n+1}-1}{\frac{-t\,\sigma 2}{k}-1}\right)$$

$$=\frac{-k}{\sigma_1-\sigma_2}\left(\frac{t\,\sigma_1-k}{t\,\sigma_1+k}\left(\left(\frac{-t}{k}\right)^{n+1}\sigma_1^{n-r+2}-\sigma_1^{-r+1}\right)-\frac{t\,\sigma_2-k}{t\,\sigma_2+k}\left(\left(\frac{-t}{k}\right)^{n+1}\sigma_2^{n-r+2}-\sigma_2^{-r+1}\right)\right)$$

$$=\frac{-k}{\sigma_1-\sigma_2}\left(\left(\frac{-t}{k}\right)^{n+1}\left(\frac{t\,\sigma_1-k}{t\,\sigma_1+k}\sigma_1^{n-r+2}-\frac{t\,\sigma_2-k}{t\,\sigma_2+k}\sigma_1^{n-r+2}\right)-\left(\frac{t\,\sigma_1-k}{t\,\sigma_1+k}\sigma_1^{-r+1}-\frac{t\,\sigma_2-k}{t\,\sigma_2+k}\sigma_2^{-r+1}\right)\right)$$

$$=\frac{-k}{-t^2+k^2t+k^2}\frac{1}{\sigma_1-\sigma_2}\left(\left(\frac{-t}{k}\right)^{n+1}\left(\left(-t^2+k\,t\sqrt{k^2+4}-k^2\right)\sigma_1^{n-r+2}+\left(t^2+k\,t\sqrt{k^2+4}+k^2\right)\sigma_2^{n-r+2}\right)\right)$$

$$-\frac{-k}{-t^2+k^2t+k^2}\frac{1}{\sigma_1-\sigma_2}\left(\left(-t^2+k\,t\sqrt{k^2+4}-k^2\right)\sigma_1^{-r+1}+\left(t^2+k\,t\sqrt{k^2+4}+k^2\right)\sigma_2^{-r+1}\right)$$

$$=\frac{-k}{-t^2+k^2t+k^2}\left(\frac{t}{k}\right)^{n+1}\left(\frac{\left(-t^2-k^2\right)\sigma_1^{n-r+2}+\left(t^2+k^2\right)\sigma_2^{n-r+2}}{\sigma_1-\sigma_2}+k\,t\sqrt{k^2+4}\,\frac{\sigma_1^{n-r+2}+\sigma_2^{n-r+2}}{\sqrt{k^2+4}}\right)$$

$$+\frac{-k}{-t^2+k^2t+k^2}\left(\frac{\left(-t^2-k^2\right)\sigma_1^{-r+1}+\left(t^2+k^2\right)\sigma_2^{-r+1}}{\sigma_1-\sigma_2}+k\,t\sqrt{k^2+4}\,\frac{\sigma_1^{-r+1}+\sigma_2^{-r+1}}{\sqrt{k^2+4}}\right)$$

$$=\frac{k}{-t^2+k^2t+k^2}\left((-1)^{n+1}\left(\frac{t}{k}\right)^{n-r+1}\left(\left(t^2+k^2\right)F_{k,n-r+2}-k\,t\,L_{k,n-r+2}\right)-\left(\frac{t}{k}\right)^{-r}\left(\left(t^2+k^2\right)F_{k,-r+1}-k\,t\,L_{k,-r+1}\right)\right)$$

$$=\frac{k}{t^2-k^2t-k^2}\left((-1)^{n}\left(\frac{t}{k}\right)^{n+1}\left(\left(t^2+k^2\right)F_{k,n-r+2}-k\,t\,L_{k,n-r+2}\right)+\left(t^2+k^2\right)F_{k,-r+1}-k\,t\,L_{k,-r+1}\right)$$

$$r=1:\sum_{j=0}^{n}(-1)^{j}\left(\frac{t}{k}\right)^{j}\left(t\,F_{k,j+1}-k\,F_{k,j}\right)=\frac{k}{t^{2}-k^{2}t-k^{2}}\left[(-1)^{n}\left(\frac{t}{k}\right)^{n+1}\left(\left(t^{2}+k^{2}\right)F_{k,n+1}-k\,t\,L_{k,n+1}\right)-2k\,t\right]$$

$$t=k:\sum_{j=0}^{n}(-1)^{j}k\left((k-1)F_{k,j-r+1}+F_{k,j-r}\right)=(-1)^{n}\left(\left(k^{2}-2k+2\right)F_{k,n-r+1}+(k-2)F_{k,n-r}\right)+(-1)^{r}\left((k-2)F_{k,r-1}-2F_{k,r}\right)$$

If $t=k=1$, for the classical Fibonacci numbers it is $\sum_{j=0}^{n}(-1)^{j}F_{j-r}=(-1)^{n}F_{n-r-1}-(-1)^{r}F_{r+2}$

If $t=k=2$, for the Pell numbers it is $\sum_{j=0}^{n}(-1)^{j}\left(P_{j-r+1}+P_{j-r}\right)=(-1)^{n}P_{n-r+1}-(-1)^{r}P_{r}$

Similarly, for the alternated sum with k-Lucas numbers, one can prove this last corollary.

4.3 Corolllary.

$$\sum_{j=0}^{n}(-1)^{j}\left(\frac{t}{k}\right)^{j}\left(t\,L_{k,j-r+2}-k\,L_{k,j-r+1}\right)$$
$$=\frac{k}{t^{2}-k\,t-k^{2}}\left[(-1)^{n}\left(\frac{t}{k}\right)^{n+1}\left(\left(t^{2}+k^{2}\right)L_{k,n-r+2}-k\,t(k^{2}+4)F_{k,n-r+2}\right)+\left(t^{2}+k^{2}\right)L_{k,-r+1}-k\,t(t^{2}+4)F_{k,-r+1}\right]$$

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