ISSN 2347-1921

DGJ method for fractional initial-value problems

Huan Li, Yue Hu
College of Mathematics and Information Science, Henan polytechnic University, Jiaozuo 454000, China
College of Mathematics and Information Science, Henan polytechnic University, Jiaozuo 454000, China

Abstract

In this paper, a new iterative method (DGJM) is used to solve the nonlinear fractional initial-value problems(fIVPs). The fractional derivative is described in the Caputo sense. Approximate analytical solutions of the fIVPs are obtained. The results of applying this procedure to the studied cases show the high accuracy and efficiency of the approach.

Keywords: Caputo's fractional derivative; fractional calculus; Fractional IVPs; DGJM.

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol.11, No.4
www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

1 INTRODUCTION

In the last few years, fractional differential equations (FDEs) are gaining importance owing to their applications in the field of visco-elasticity, feed-back amplifiers, electrical circuits, electroanalytical chemistry, fractional multipoles, neuron modelling, encompassing different branches of Physics, Chemistry and Biological Sciences[1-5]. The Fractional differential equations have recently been addressed by several researchers for a variety of problems[6, 7]. Some analytic methods for solving nonlinear problems were invented, including the Adomian decomposition method (ADM)[8-11, 27], Homotopy-perturbation method (HPM)[12, 13], Variational iteration method (VIM)[14, 15], Homotopy analysis method (HAM)[16-20, 28] and a new iterative method-DGJM [23-25].

In the present paper, DGJM is applied to solve the nonlinear fractional initial-value problems (fIVPs). A nonlinear example shall be presented to show the efficiency and accuracy of DGJM. Furthermore, the Taylor series expansion shall be employed to avoid the difficulties with radical nonlinear terms.

2 Preliminaries

In this section, we give some definition and properties of the fractional calculus[21].

Definition 1. A real function $g(x)$, $x>0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p>\mu$, such that $g(x)=x^p g_1(x)$, where $g_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $g^{(n)}\in C_\mu$, $n\in N$.

Definition 2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha 0$, of a function $g\in C_\mu$, $\mu 1$, is defined as

$$\begin{aligned} J^\alpha g(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} g(\tau)d\tau (\alpha>0), \\ J^0 g(x) &= g(x) \end{aligned} \tag{1}$$

where $\Gamma(z)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

For $g\in C_\mu$, $\mu -1$, $\alpha,\beta 0$ and $\gamma 1$:

(1) $J^\alpha J^\beta g(x) = J^{\alpha+\beta} g(x)$,

(2) $J^\alpha J^\beta g(x) = J^\beta J^\alpha g(x)$,

(3) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D^α) of $g(x)$ in the Caputo's sense is defined as

$$D^\alpha g(x) = J^{n-\alpha} D^n g(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\tau)^{n-\alpha-1} g^{(n)}(\tau)d\tau, \tag{2}$$

for $n-1<\alpha,, n$, $n\in N$, $x>0$, $g\in C_{-1}^n$.

The following are two basic properties of the Caputo's fractional derivative[22]:

(1)Let $g\in C_{-1}^n$, $n\in N$. Then $D^\alpha g$, $0,, \alpha,, n$ is well defined and $D^\alpha g\in C_{-1}$.

(2)Let $n-1<\alpha,, n$, $n\in N$, and $g\in C_\mu^n$, $\mu 1$. Then

$$(J^\alpha D^\alpha) g(x) = g(x) - \sum_{k=0}^{n-1} g^k(0^+)\frac{x^k}{k!}. \tag{3}$$

ISSN 2347-1921

3 New iterative method-DGJM

Daftardar-Gejji and Jafari[23] first proposed a New Iterative Method (DGJM) to solve non-linear functional equations in 2006. They have considered the following functional equation:

$$u = f + L(u) + N(u), \tag{4}$$

where L is a linear operator, N is a nonlinear operator and f is a known function.

We are looking for a solution u of Eq.(4) having the series form:

$$u = \sum_{i=0}^{\infty} u_i. \tag{5}$$

Since L is a linear operator,

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L u_i. \tag{6}$$

The nonlinear operator N is decomposed as[23-25]

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} u_i\right) &= N(u_0) + \{N(u_0+u_1) - N(u_0)\} + \{N(u_0+u_1+u_2) - N(u_0+u_1)\} \\ &\quad + \{N(u_0+u_1+u_2+u_3) - N(u_0+u_1+u_2)\} + \cdots \\ &= N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \end{aligned} \tag{7}$$

From Eqs.(5)–(7), Eq.(4) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + L\left(\sum_{i=0}^{\infty} u_i\right) + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{8}$$

We define the recurrence relation

$$\begin{aligned} u_0 &= f, \\ u_1 &= L(u_0) + N(u_0), \\ u_{n+1} &= L(u_n) + \{N(u_0+\cdots+u_n) - N(u_0+\cdots+u_n-1)\} \\ &= L(u_n) + \left\{N\left(\sum_{j=0}^{n} u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right)\right\}, \quad n = 1,2,\ldots. \end{aligned} \tag{9}$$

Then,

$$\sum_{i=1}^{n+1} u_i = L\left(\sum_{i=0}^{n} u_i\right) + N\left(\sum_{i=0}^{n} u_i\right) \tag{10}$$

and

$$\sum_{i=0}^{\infty} u_i = f + L\left(\sum_{i=0}^{\infty} u_i\right) + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{11}$$

It is clear from Eq.(11) that $\Sigma_{i=0}^{\infty} u_i$ is solution of Eq.(4). Where $u_i, i = 0,1,2,\ldots,$ are given by algorithm (9). Also, the k-term approximate solution of Eq.(4) can be given by

$$\sum_{i=0}^{k-1} u_i.$$

4 Convergence of DGJM

In[26], S. Bhalekar and V. Daftardar-Gejji presented the following theorems for convergence of DGJM.

Theorem 4.1. If N is $C^{(\infty)}$ in a neighborhood of u_0 and

$$\|N^{(n)}(u_0)\| = Sup\{N^{(n)}(u_0)(h_1,...,h_n) : \|h_i\| \leq 1, 1 \leq i \leq n\} \leq L, \tag{12}$$

for any n and for some real $L > 0$ and $\|u_i\| \leq M < 1/e, i = 1,2,...,$ then the series $\Sigma_{n=0}^{\infty} G_n$ is absolutely convergent, and moreover,

$$\|G_n\| \leq LM^n e^{n-1}(e-1), \quad n = 1,2,.... \tag{13}$$

where G_n is such that $G_n = N(\Sigma_{i=0}^{n} u_i) - N(\Sigma_{i=0}^{n-1} u_i), n = 1,2,....$

Theorem 4.2. If N is $C^{(\infty)}$ and $\|N^{(n)}(u_0)\| \leq M \leq e^{-1}$, for all n, then the series $\Sigma_{n=0}^{\infty} G_n$ is absolutely convergent.

5 Numerical Example

In this section, we shall illustrate the applicability of DGJM to nonlinear fIVPs. We consider the following nonlinear fIVPs:

$$\begin{cases} D^{\alpha} u = \dfrac{9}{4}\sqrt{u} + u, \quad 1 < \alpha \leq 2, \quad t > 0, \\ u(0) = 1, \quad u'(0) = 2. \end{cases} \tag{14}$$

The exact solution of the initial-value problem (14) for $\alpha = 2$, is

$$u(t) = \frac{9}{4}\left[\frac{3e^{0.5t}}{2} + \frac{e^{-0.5t}}{6} - 1\right]^2. \tag{15}$$

In the case, Behiry obtained the series solution of the fIVPs by using differential transformation [27]. In general circumstances, Hashim etl. obtained the HAM series solution of the fIVPs by using homotopy analysis method [28]. Now we solve the fIVPs with DGJM. Firstly, expanding the nonlinear term, $\sqrt{u}$, in (14) by using the Taylor series, we get

$$\sqrt{u} \approx 1 + \frac{1}{2}(u-1) - \frac{1}{8}(u-1)^2 + \frac{1}{16}(u-1)^3. \tag{16}$$

Then, the fIVPs (14) can be approximated by

$$D^{\alpha} u = \frac{45}{64} + \frac{199}{64}u - \frac{45}{64}u^2 + \frac{9}{64}u^3. \tag{17}$$

According to the DGJM, in view of the algorithm(9), we construct the following recurrence relation:

$$\begin{aligned}
u_0(t) &= 1,\\
u_1(t) &= 2t,\\
u_2(t) &= \frac{61t^{\alpha}}{16\Gamma(\alpha+1)}+\frac{199t^{\alpha+1}}{32\Gamma(\alpha+2)},\\
u_3(t) &= -\frac{63t^{\alpha+1}}{32\Gamma(\alpha+2)}+\frac{70925391\Gamma(3\alpha+4)t^{4\alpha+3}}{2097152[\Gamma(\alpha+2)]^{3}\Gamma(4\alpha+4)}+\frac{1647\Gamma(\alpha+3)t^{2\alpha+2}}{256\Gamma(\alpha+1)\Gamma(2\alpha+3)}\\
&+\frac{1069227\Gamma(2\alpha+4)t^{3\alpha+3}}{32768[\Gamma(\alpha+2)]^{2}\Gamma(3\alpha+4)}-\frac{356409\Gamma(2\alpha+3)t^{3\alpha+2}}{32768[\Gamma(\alpha+2)]^{2}\Gamma(3\alpha+3)}+\frac{3383t^{2\alpha+1}}{256\Gamma(2\alpha+2)}\\
&-\frac{1791\Gamma(\alpha+3)t^{2\alpha+2}}{256\Gamma(\alpha+2)\Gamma(2\alpha+3)}+\frac{5373\Gamma(\alpha+4)t^{2\alpha+3}}{512\Gamma(\alpha+2)\Gamma(2\alpha+4)}-\frac{549\Gamma(\alpha+2)t^{2\alpha+1}}{128\Gamma(\alpha+1)\Gamma(2\alpha+2)}\\
&-\frac{109251\Gamma(2\alpha+2)t^{3\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(3\alpha+2)}+\frac{327753\Gamma(2\alpha+3)t^{3\alpha+2}}{8192\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(3\alpha+3)}+\frac{27t^{\alpha+3}}{4\Gamma(\alpha+4)}\\
&+\frac{19992933\Gamma(3\alpha+2)t^{4\alpha+1}}{524288[\Gamma(\alpha+2)^{3}]\Gamma(4\alpha+2)}+\frac{2042829\Gamma(3\alpha+1)t^{4\alpha}}{262144[\Gamma(\alpha+1)]^{3}\Gamma(4\alpha+1)}-\frac{9t^{\alpha}}{16\Gamma(\alpha+1)}\\
&+\frac{1037t^{2\alpha}}{128\Gamma(2\alpha+1)}-\frac{33489\Gamma(2\alpha+1)t^{3\alpha}}{8192[\Gamma(\alpha+2)]^{2}\Gamma(3\alpha+1)}+\frac{100457\Gamma(2\alpha+2)t^{3\alpha+1}}{8192[\Gamma(\alpha+1)]^{2}\Gamma(3\alpha+2)}\\
&-\frac{9t^{\alpha+2}}{4\Gamma(\alpha+3)}+\frac{65222847\Gamma(3\alpha+3)t^{4\alpha+2}}{1048576[\Gamma(\alpha+2)]^{2}\Gamma(\alpha+1)\Gamma(4\alpha+3)}.
\end{aligned}$$

Then, the DGJM series solution of the initial-value problem (14) can be approximated as

$$\begin{aligned}
u(t) = {} & 1+2t+\frac{61t^{\alpha}}{16\Gamma(\alpha+1)}+\frac{17t^{\alpha+1}}{4\Gamma(\alpha+2)}+\frac{70925391\Gamma(3\alpha+4)t^{4\alpha+3}}{2097152[\Gamma(\alpha+2)]^{3}\Gamma(4\alpha+4)}-\frac{9t^{\alpha+2}}{4\Gamma(\alpha+3)}\\
&+\frac{1647\Gamma(\alpha+3)t^{2\alpha+2}}{256\Gamma(\alpha+1)\Gamma(2\alpha+3)}+\frac{1069227\Gamma(2\alpha+4)t^{3\alpha+3}}{32768[\Gamma(\alpha+2)]^{2}\Gamma(3\alpha+4)}-\frac{356409\Gamma(2\alpha+3)t^{3\alpha+2}}{32768[\Gamma(\alpha+2)]^{2}\Gamma(3\alpha+3)}\\
&+\frac{3383t^{2\alpha+1}}{256\Gamma(2\alpha+2)}-\frac{1791\Gamma(\alpha+3)t^{2\alpha+2}}{256\Gamma(\alpha+2)\Gamma(2\alpha+3)}+\frac{5373\Gamma(\alpha+4)t^{2\alpha+3}}{512\Gamma(\alpha+2)\Gamma(2\alpha+4)}+\frac{27t^{\alpha+3}}{4\Gamma(\alpha+4)}\\
&-\frac{109251\Gamma(2\alpha+2)t^{3\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(3\alpha+2)}+\frac{327753\Gamma(2\alpha+3)t^{3\alpha+2}}{8192\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(3\alpha+3)}+\frac{1037t^{2\alpha}}{128\Gamma(2\alpha+1)}\\
&+\frac{19992933\Gamma(3\alpha+2)t^{4\alpha+1}}{524288[\Gamma(\alpha+2)^{3}]\Gamma(4\alpha+2)}+\frac{65222847\Gamma(3\alpha+3)t^{4\alpha+2}}{1048576[\Gamma(\alpha+2)]^{2}\Gamma(\alpha+1)\Gamma(4\alpha+3)}-\frac{9t^{\alpha}}{16\Gamma(\alpha+1)}\\
&-\frac{33489\Gamma(2\alpha+1)t^{3\alpha}}{8192[\Gamma(\alpha+2)]^{2}\Gamma(3\alpha+1)}+\frac{2042829\Gamma(3\alpha+1)t^{4\alpha}}{262144[\Gamma(\alpha+1)]^{3}\Gamma(4\alpha+1)}-\frac{549\Gamma(\alpha+2)t^{2\alpha+1}}{128\Gamma(\alpha+1)\Gamma(2\alpha+2)}\\
&+\frac{100457\Gamma(2\alpha+2)t^{3\alpha+1}}{8192[\Gamma(\alpha+1)]^{2}\Gamma(3\alpha+2)}+\cdots.
\end{aligned}\tag{18}$$

For the particular case $\alpha=2$,

$$\begin{aligned}
u(t) = {} & 1+2t+\frac{13t^{2}}{8}+\frac{17t^{3}}{24}+\frac{749t^{4}}{3072}+\frac{1817t^{5}}{30720}+\frac{63479t^{6}}{983040}+\frac{20227t^{7}}{229376}+\frac{8401277t^{8}}{117440512}\\
&+\frac{4542175t^{9}}{301989888}+\frac{2415661t^{10}}{251658240}+\frac{7880599t^{11}}{5536481280}+\cdots.
\end{aligned}\tag{19}$$

Table 1: Approximate solution of (12) for some values of α using the 4-term DGJM approximation

t	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$ (DGJM)	$\alpha = 2$ (HAM)Ref.[28]	$\alpha = 2$ (Exact)
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
.1	1.37836275	1.28266260	1.23784129	1.21698338	1.21288993	1.21697814
.2	1.87256490	1.65206467	1.53461631	1.47108115	1.45375932	1.47099037
.3	2.47577265	2.10126029	1.89032411	1.76756504	1.72601125	1.76704923
.4	3.20808013	2.63904388	2.31109936	2.11264153	2.03325161	2.11074085
.5	4.10959976	3.28316063	2.80697386	2.51389524	2.37939713	2.50828745
.6	5.24986919	4.06387321	3.39325199	2.98109536	2.76882583	2.96661654
.7	6.74211990	5.03100449	4.09358872	3.52757869	3.20658281	3.49343764
.8	8.76231359	6.26447112	4.94501176	4.17250918	3.69865784	4.09732727
.9	11.5736021	7.88903647	6.00549352	4.94442762	4.25235505	4.78782299
.0	15.5570562	10.0943002	7.36491769	5.88665211	4.87678065	5.57552765

Numerical results with comparison to Ref.[28] is given in Table 1 on the $[0,1]$. We can see that the numerical solution is in very good agreement with the exact solution when $\alpha = 2$. Therefore, we hold that the solution for $\alpha = 1.25$, $\alpha = 1.5$ and $\alpha = 1.75$ is also credible.

6 Conclusions

In this work, new iterative method-DGJM was used to derive approximate analytical solutions of the nonlinear fIVPs. The nonlinear terms involving radical powers were expanded by Taylor series. The DGJM is effective for the nonlinear fractional initial-value problems (fIVPs), and hold very great promise for its applicability to other nonlinear fractional differential equations.

References

[1] B.J.West, M. Bolognab, P. Grigolini, Physics of Fractal Operators, Springer, New York, 2003.

[2] Baleanu, O. Alipous, N. and Jafari, H., The Bernstein Operational Matrices for Solving the Fractional Quadratic Riccati Differential Equations with The Riemann-Liouville Derivative, Abstract and Applied Analysis, 2013.

[3] Luchko Y, Gorenflo R. The initial-value problem for some fractional differential equations with Caputo derivative. Preprint Series A08–98. Fachbereich Mathematik und Informatic, Berlin, Freie Universitat; 1998.

[4] Arikoglu, A. and Ozkol, I, Solution of Fractional Differential Equations by Using Differential Transform Method, Chaos, Solutons and Fractab, 34(5): 1473–1481, 2007.

[5] Odibat, Z. and Momani, S, Application of Variational Iteration Method to Nonlinear Differential Equations of Fractional Order, Int. J. Nonlinear Sci. Num. Simulation, 7(1): 27–34, 2006.

[6] Merdan, M., On the solutions of Fractional Riccati Differential Equations with Modified Riemann-Liouville Derivative, Int. J. Diff. Equations, 2012.

[7] Yuzbasi, S., Numerical Solutions of Fractional Riccati type Differential Equations by Means of the Bernstein Polynomials, Applied Math. Comput, 219(11); 6238–6343, 2013.

[8] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1994.

[9] Y. Hu, Y. Luo, Z. Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method, J. Comput. Appl. Math, doi:10.1016/j. cam. 2007.04.005.

[10] S. Momani, Z. Odibat, Numerical approach to differential of fractional order, J. Comput. Appl. Math, doi: 10. 1016/j. cam. 2006. 07. 015.

[11] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos, Solitons and Fractals, doi: 10. 1016/j. chaos. 2005. 10. 068.

[12] J.-H. He, Homotopy perturbation technique, Comput. Meth. Appl. Mech. Eng. 178 (1999) 257–262.

[13] J.-H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Nonlinear Mech. 35 (2000) 37–43.

[14] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Meth. Appl. Mech. Eng. 167 (1998) 57–68.

[15] Cang, J, Tan, Y, Xu, H and Liao, S.J, Series Solutions of Nonlinear Riccati Differential Equations with Fractional Order, Chaos, Solutons and Fractals, 40(1), 1–9, 2009.

[16] Liao SJ. The proposed homotopy analysis technique for the solution of nonlinear problems. Ph. D thesis, Shanghai Jiao Tong University; 1992.

[17] Liao SJ. An approximate solution technique which does not depend upon small parameters: a special example. Int J Nonlinear Mech 1995; 30: 371–80.

[18] Liao SJ, Campo A. Analytic solutions of the temperature distribution in Blasius viscous flow problems. J Fluid Mech 2002; 453: 411–25.

[19] Liao SJ. On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet. J Fluid Mech 2003; 488: 189–212.

[20] Liao SJ. On the homotopy analysis method for nonlinear problems. Appl Math Comput 2004; 147: 499–513.

[21] I. Podlubny, Fractional Differential Equations, Academic, New York, 1999.

[22] Gorenflo R, Mainardi F. Fractional calculus: integral and differential equations of fractional order. In: Fractals and fractional calculus in continuum mechanics. Wien and New York: Springer-Verlag; 1997. p. 223–76.

[23] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations J. Math. Anal. Appl, 316, pp. 752–763, 2006.

[24] S. Bhalekar, V. Daftardar-Gejji, New iterative method: Application to partial differential equations Appl. Math. Comput. 203, pp. 778–783, 2008.

[25] S. Bhalekar, V. Daftardar-Gejji, Solving evolution equations using a new iterative method, Numer. Methods Partial Differ. Equations, 26, pp. 906–916, 2010.

[26] S. Bhalekar, V. Daftardar-Gejji, Convergence of the New Iterative Method, International Journal of Differential Equations, 10, pp. 1155, 2011.

[27] S. H. Behiry, Differential transform method for nonlinear initial-value problems by Adomian polynomials, Applied & Computational Mathematics, 3(2012), 1–6.

[28] I. Hashim, O. Abdulaziz, S. Momani, Homotopy analysis method for fractional IVPs, Communications in nonlinear science and numerical simulation, 14(2009), 674–684.