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ON $I\alpha$- CONNECTED SPACES VIA IDEAL

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ABSTRACT

The aim of this paper is to study some properties of $I\alpha$- open set in ideal topological spaces, which was introduced by M.E. Abd El-Monsef [1], such as interior, exterior, functions and connectedness.

Keywords

$I\alpha$- Open set; $I\alpha$- neighborhood, $I\alpha$- interior; $I\alpha$- exterior; $I\alpha$- boundary; $I\alpha$- closure; $I\alpha$- connected space; $I\alpha$- separated space and $I\alpha$- separated connected space.

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1. INTRODUCTION

The subject of ideals in topological spaces has been studied by kuratowski [3] and vaidyanathaswamy [7]. An ideal I on a set X is a nonempty collection of subsets of X which satisfies: (1) $A \in I$ and $B \subset A$ implies $B \in I$ and (2) $A \in I$, $B \in A$ implies $A \cup B \in I$.

Given a topological space (X, τ) with ideal I on X and if $P(X)$ is the set of all subsets of X. A set operator $(\ )^*: P(X) \to P(X)$, called a local function [3] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X: A \cap U \notin I \ for\ every\ U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau: x \in U\}$. Kuratowski closure operator $cl^*(\ )$ for the topology $\tau^*(I, \tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I, \tau)$ [7]. When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal space.

In an ideal space (X, τ, I), if $A \subset X$, then $int^*(A)$ will denote the interior of A in (X, τ^*). The closed subsets of X in (X, τ^*) are called $*$- closed sets. A subset A of an ideal space (X, τ, I) is $*$- closed if and only if $A^* \subset A$ [2].

For any ideal space (X, τ, I), the collection $\{V - J:\ V \in \tau \ and\ J \in I\}$ is a basis for τ^*[2]. The elements of $\tau^*$are called $*$- open sets. A subset A of an ideal topological space (X, τ, I) is said to be $*$- dense set if $cl^*(A) = X$. It is clear that, in an ideal space (X, τ, I) if $A \subset B \subset X$, then $A^* \subset B^*$ and so $cl^*(A) \subset cl^*(B)$.

Recall that if (X, τ, I) is an ideal space and A is a subset of X, then (A, τ_A, I_A) where τ_A is the relative topology on A and $I_A = \{A\ \cap J: J \in I\}$ is an ideal topological subspace.

Given a topological space (X, τ). A subset A of a space X is said to be α- open set if $A \subset int\left(cl\left(int(A)\right)\right)$. The family of all α- open subsets of a space (X, τ) forms a topology on X , called the α- topology on X and denoted by τ_α and it is finer than τ. If every nowhere dense set in a space (X, τ) is closed, then $\tau_\alpha = \tau$ [5].

The concept of a set operator $(\ )^{\alpha*}: P(X) \to P(X)$ was introduced by A.A. Nasef [4] in 1992, which is called an α-local function of I with respect to τ and was defined as follows: for $A \subset X$, $A^{\alpha*}(I, \tau) = \{x \in X: A \cap U \notin I \ for\ every\ U \in$ $\tau\alpha x$ where $\tau\alpha x = U \in \tau\alpha : x \in U$. When there is no chance for confusion, we will simply write $A\alpha*$ for $A\alpha*I,\ \tau$. An $\alpha*$- closure operator, denoted by $cl^{\alpha*}(\ )$, for a topology $\tau^{\alpha*}(I, \tau)$ which is called the $*$-α -topology, finer than $\tau$and it is defined as follows: $cl^{\alpha*}(A)(I, \tau) = A \cup A^{\alpha*}(I, \tau)$. When there is no ambiguity, we will simply write $cl^{\alpha*}(A)$ for $cl^{\alpha*}(A)(I, \tau)$. A basis $\mathfrak{B}(I, \tau)$ for $\tau^{\alpha*}$ is described as follows: $\mathfrak{B}(I, \tau) = \{V - J:\ V \in \tau_\alpha \ and\ J \in I\}$. We will denote by $int^{\alpha*}(A)$ and $cl^{\alpha*}(A)$ the interior and closure of $A \subset\ (X, \tau, I)$ with respect to $\tau^{\alpha*}$. The elements of $\tau^{\alpha*}$ are called $\tau^{\alpha*}$- open sets. Subsets of X closed in $(X, \tau^{\alpha*})$ are called $\tau^{\alpha*}$- closed sets. A subset A of an ideal space (X, τ, I) is $\tau^{\alpha*}$- closed (respectively, $\tau^{\alpha*}$- dense) if and only if $A^{\alpha*} \subset A$ (respectively, $cl^{\alpha*}(A) = X$). In an ideal space (X, τ, I) if $A \subset B \subset X$, then $A^{\alpha*} \subset B^{\alpha*}$ and $cl^{\alpha*}(A) \subset cl^{\alpha*}(B)$. So, $A^{\alpha*} \subset A^*$ and $cl^{\alpha*}(A) \subset cl^*(A)$.

2. SOME PROPERTIES of $I\alpha$- OPEN SET

In this section, we will study some properties of $I\alpha$- open set in ideal topological space which was defined by Abd-Elmonsef and et, in 2013 [1].

Definition 2.1 [1] A subset A of an ideal topological space (X, τ, I) is said to be $I\alpha$- open if it satisfy that $A \subseteq int\left(cl^{\alpha*}\left(int(A)\right)\right)$. The family of all $I\alpha$- open sets in ideal topological space (X, τ, I) is denoted by $I\alpha O(X)$.

Remark 2.2 In an ideal topological space (X, τ, I), the following statements are hold;

i. both X and $\emptyset$ are $I\alpha$- open sets.

ii. the arbitrary union of $I\alpha$- open sets is $I\alpha$- open.

iii. every open set in topological space (X, τ) is $I\alpha$- open.

iv. Every $I\alpha$- open set in a topological space (X, τ, I) is $\tau^{\alpha*}$.

Remark 2.3 In an ideal topological space (X, τ, I), if A, B are two subsets of X such that A is α- open set then $A \cap cl^{\alpha*}(B) \subseteq cl^{\alpha*}(A \cap B)$.

Proof:- If $x \notin cl^{\alpha*}(A \cap B)$ then we have two cases let in the first case that $x \notin A$ then $x \notin cl^{\alpha*}(A \cap B)$. But in the second case we have $x \in A$ so there exist $u \in \tau^{\alpha*}_{(x)}\ such\ that\ u \cap (A \cap B) = \emptyset$. Thus there exist $u \in \tau^{\alpha*}_{(x)}\ such\ that\ (u \cap A) \cap B = \emptyset$. But $A \in \tau_\alpha$ then $A \in \tau_\alpha, x \in A$ and $x \in u$. So $x \in (u \cap A)$ which mean that $x \notin B$ for $(u \cap A) \cap B = \emptyset$. Therefore $x \notin cl^{\alpha*}(B)$ and finally $x \notin A \cap cl^{\alpha*}(B)$.

Proposition 2.4 In an ideal topological space (X, τ, I), if A, B are two subsets of X such that A is α- open set and B is $I\alpha$- open set then $A \cap\ B$ is $I\alpha$- open set.

Proof:- To prove that $A \cap\ B$ is $I\alpha$- open set in the case of A is α- open set and B is $I\alpha$- open set we must prove that $A \cap\ B \subseteq int\ (cl^{\alpha*}[int\ (A \cap\ B)])$. But B is $I\alpha$- open set then $B \subseteq int\ (cl^{\alpha*}[int\ (\ B)])$ so $A \cap\ B \subseteq A \cap\ int\ (cl^{\alpha*}[int\ (\ B)]) = int(A) \cap\ int\ (cl^{\alpha*}[int\ (\ B)]) = int\ (A \cap cl^{\alpha*}[int(B)])$ and from remark 2.3 we get $A \cap\ B \subseteq int\left(cl^{\alpha*}\left(A \cap int(B)\right)\right) = int\left(cl^{\alpha*}\left(int(A) \cap int(B)\right)\right) = int\ (cl^{\alpha*}[int\ (A \cap\ B)])$.

Definition 2.5 [6] A subfamily μ of the power set $\mathrm{P}(X)$ of a nonempty set X is called supra topology on X if it satisfies the following conditions:

i. μ contains $\emptyset, X$.

ii. μ is closed under the arbitrary union.

From Proposition 2.2, the family of all $I\alpha$- open set on X forms a supra topology.

Definition 2.6 Let (X,τ,I) be an ideal topological space and $x \in A \subseteq X$. Then A is said to be an $I\alpha$- neighborhood of x if there exist an $I\alpha$- open set U such that $x \in U \subseteq A$, and simply write as $I\alpha N(x)$. If A is $I\alpha$- open set then it is $I\alpha$- open neighborhood for any element $x \in A$.

proposition 2.7 Let (X,τ,I) be an ideal topological space. $A \subset X$ is $I\alpha$- open set if and only if for each $x \in A$ there exist an $I\alpha$- open set U such that $x \in U \subseteq A$.

Definition 2.8 Let (X,τ,I) be an ideal topological space and $x \in A \subseteq X$. Then x is said to be an $I\alpha$- interior point of A if A contain an $I\alpha$- open neighborhood set for x. The set of all $I\alpha$- interior points of A is called $I\alpha$- interior set and simply is denoted by $I\alpha$- $int(A)$.

Proposition 2.9 Let (X,τ,I) be an ideal topological space and A, B are two subsets of X. Then the following statements are hold;

i. $I\alpha\text{- } int(A) \subseteq A$.

ii. If $A \subseteq B, then\ I\alpha\text{- } int(A) \subseteq I\alpha\text{- } int(B)$.

iii. $A \in I\alpha\text{-}O(X)\ \ if\ and\ only\ if\ A = I\alpha\text{- } int(A)$.

iv. $I\alpha\text{- } int(A \cap B) \subseteq\ I\alpha\text{- } int(A) \cap I\alpha\text{- } int(B)$.

v. $I\alpha\text{- } int(A) \cup I\alpha\text{- } int(B) \subseteq I\alpha\text{- } int(A \cup B)$.

vi. $I\alpha\text{- } int(A) = \cup\{U \subseteq X: U \in I\alpha\text{-}O(X) and\ U \subseteq A\}$.

Definition 2.10 Let (X,τ,I) be an ideal topological space and $x \in A^c \subseteq X$. Then x is said to be an $I\alpha$- exterior point of A if A^c contain $I\alpha$- open neighborhood set for x. The set of all $I\alpha$- exterior points of A is called $I\alpha$- exterior set and simply is denoted by $I\alpha$- $ext(A)$.

Proposition 2.11 Let (X,τ,I) be an ideal topological space and A, B are two subsets of X. Then the following statements are hold;

i. $I\alpha\text{-}int(A) \cap I\alpha\text{-}ext(A) = \emptyset$.

ii. If $A \subseteq B$, then $I\alpha\text{-}ext(B)\ \subseteq I\alpha\text{-}ext(A)$.

iii. $I\alpha\text{-}ext(A \cup B) \subseteq I\alpha\text{-}ext(A) \cap I\alpha\text{-}ext(B)$.

iv. $I\alpha\text{-}ext(A) = A^c$ if and only if $A^c \in I\alpha\text{-}O(X)$.

Definition 2.12 Let (X,τ,I) be an ideal topological space and $A \subseteq X$. $x \in X$ is said to be an $I\alpha$- boundary point of A if for every $I\alpha$- open neighborhood set for x satisfies that the intersection with A and A^c is nonempty set. The set of all $I\alpha$- boundary points of A is called $I\alpha$-boundary set of A and simply is denoted by $I\alpha$- $b(A)$.

Proposition 2.13 Let (X,τ,I) be an ideal topological space and A, B are two subsets of X. Then the following statements are hold;

i. $I\alpha\text{-}b(A) \cap I\alpha\text{-}ext(A) = \emptyset$, $I\alpha\text{-}b(A) \cap I\alpha\text{-}int(A) = \emptyset$ and $I\alpha\text{-}b(A) = X -(I\alpha\text{-}int(A) \cup\ I\alpha\text{-}ext(A))$.

ii. $I\alpha\text{-}b(A)$ is $I\alpha$- closed set.

iii. $I\alpha\text{-}b(A^c) = I\alpha\text{-}b(A)$.

iv. $I\alpha\text{-}b(A \cup B) \subseteq I\alpha\text{-}b(A) \cup I\alpha\text{-}b(B)$.

v. $I\alpha\text{-}b(A) \subseteq A^c\ if\ and\ only\ if\ A \in I\alpha\ \text{-}O(X)$.

vi. $I\alpha\text{-}b(A) \subseteq A\ if\ and\ only\ if\ A^c \in I\alpha\ \text{-}O(X)$.

vii. $I\alpha\text{-}b(A) = \emptyset\ if\ and\ only\ if\ A, A^c \in I\alpha\ \text{-}O(X)$.

Definition 2.14 Let (X,τ,I) be an ideal topological space and $A \subseteq X$ and $x \in X$ is said to be an $I\alpha$- accumulation point of A if for every $I\alpha$- open neighborhood for x containing at least one element of A which is not x. The set of all $I\alpha$- accumulation points of A is called $I\alpha$-derived set of A and simply is denoted by $I\alpha$- $d(A)$.

Proposition 2.15 Let (X,τ,I) be an ideal topological space and A,B are two subsets of X. Then the following statements are hold;

i. If $A \subseteq B$ $, then$ $I\alpha\text{-}d(A) \subseteq I\alpha\text{-}d(B)$.

ii. $I\alpha\text{-}d(A \cup B) = I\alpha\text{-}d(A) \cup I\alpha\text{-}d(B)$.

iii. $I\alpha\text{-}d(A \cap B) \subseteq I\alpha\text{-}d(A) \cap I\alpha\text{-}d(B)$.

iv. $I\alpha\text{-}d(A) \subseteq A$ $if\ and\ only\ if$ $A^c \in I\alpha\text{-}O(X)$.

Definition 2.16 Let (X,τ,I) be an ideal topological space and $A \subseteq X$. Then $I\alpha$- closure set of A is defined by the union of A and $I\alpha$-derived set of A and simply is denoted by $I\alpha\text{-}cl(A)$.

Proposition 2.17 Let (X,τ,I) be an ideal topological space and A,B are two subsets of X. Then the following statements are hold;

i. $A \subseteq I\alpha\text{-}cl(A)$.

ii. If $A \subseteq B$ $, then$ $I\alpha\text{-}cl(A) \subseteq I\alpha\text{-}cl(B)$.

iii. $I\alpha\text{-}cl(A \cup B) = I\alpha\text{-}cl(A) \cup I\alpha\text{-}cl(B)$.

iv. $I\alpha\text{-}cl(A \cap B) \subseteq I\alpha\text{-}cl(A) \cap I\alpha\text{-}cl(B)$.

v. $I\alpha\text{-}cl(A) = \cap\{F \subseteq X: F^c \in I\alpha O(X)\ and\ A \subseteq F\}$.

vi. $I\alpha\text{-}cl(A) = A$ $if\ and\ only\ if$ $A^c \in I\alpha\ O(X)$.

vii. $I\alpha\text{-}cl(I\alpha\text{-}cl(A)) = I\alpha\text{-}cl(A)$.

Now we are ready to define some different types of functions.

Definition 2.18 Let $f: (X,\tau,I) \to (Y,\sigma,J)$ be a function. f is said to be $I\alpha$- irresolute function if the inverse image of every $J\alpha$- open set in Y is $I\alpha$- open set in X.

Proposition 2.19 A function $f: (X,\tau,I) \to (Y,\sigma,J)$ is $I\alpha$- irresolute function if and only if one of the following is satisfied;

i. The inverse image of every $J\alpha$- closed set in Y is $I\alpha$- closed set in X.

ii. The inverse image of $J\alpha$- open neighborhood set of every element in Y is $I\alpha$- open set in X.

iii. $I\alpha\text{-}cl[f^{-1}(u)] \subseteq f^{-1}[J\alpha\text{-}cl(u)]$, $\forall\, u \subseteq Y$.

iv. $f^{-1}[J\alpha\text{-}int(u)] \subseteq I\alpha\text{-}int[f^{-1}(u)]$, $\forall\, u \subseteq Y$.

Definition 2.20 Let $f: (X,\tau,I) \to (Y,\sigma,J)$ be a function. f is said to be strongly $I\alpha$- continuous function if the inverse image of every $J\alpha$- open set in Y is open set in X.

Proposition 2.21 A function $f: (X,\tau,I) \to (Y,\sigma,J)$ is strongly $I\alpha$- continuous function if and only if one of the following is satisfied;

i. The inverse image of every $J\alpha$- closed set in Y is closed set in X.

ii. The inverse image of $J\alpha$- open neighborhood set of every element in Y is open set in X.

iii. $cl[f^{-1}(V)] \subseteq f^{-1}[J\alpha\text{-}cl(V)]$, $\forall\, V \subseteq Y$.

iv. $f^{-1}[J\alpha\text{-}int(V)] \subseteq int[f^{-1}(V)]$, $\forall\, V \subseteq Y$.

Definition 2.22 Let $f: (X,\tau,I) \to (Y,\sigma,J)$ be a function. f is said to be $I\alpha$- continuous function if the inverse image of every open set in Y is $I\alpha$- open set in X.

Proposition 2.23 A function $f: (X,\tau,I) \to (Y,\sigma,J)$ is $I\alpha$- continuous function if and only if one of the following is satisfied;

i. The inverse image of every closed set in Y is $I\alpha$- closed set in X.

ii. The inverse image of every open neighborhood set for every element in Y is open set in X.

iii. $I\alpha\text{-}cl[f^{-1}(u)] \subseteq f^{-1}(cl(u))$, $\forall\, u \subseteq Y$.

iv. $f^{-1}(int(u)) \subseteq I\alpha\text{-}int[f^{-1}(u)]$, $\forall\, u \subseteq Y$.

Proposition 2.24 Let $f: (X,\tau,I) \to (Y,\sigma,J)$, $g: (Y,\sigma,J) \to (Z,\xi,K)$ be two functions. Then $g \circ f$ is $I\alpha$- irresolute function in the following cases;

i. If f,g are $I\alpha, J\alpha$- irresolute functions respectively.

ii. If f, g are $I\alpha$- irresolute, strongly $J\alpha$-continuous functions respectively.

iii. If f, g are $I\alpha$- continuous, strongly $J\alpha$- continuous functions respectively.

Proposition 2.25 Let $f: (X, \tau, I) \to (Y, \sigma, J)$, $g: (Y, \sigma, J) \to (Z, \xi, K)$ be two functions. Then $g \circ f$ is strongly $I\alpha$- continuous function if f, g are strongly $I\alpha$- continuous, $J\alpha$-irresolute functions respectively.

Proposition 2.26 Let $f: (X, \tau, I) \to (Y, \sigma, J)$, $g: (Y, \sigma, J) \to (Z, \xi, K)$ be two functions. Then $g \circ f$ is $I\alpha$- continuous function if f, g are $I\alpha$- irresolute, continuous functions respectively.

Proposition 2.27 Let $f: (X, \tau, I) \to (Y, \sigma, J)$, $g: (Y, \sigma, J) \to (Z, \xi, K)$ be two functions. Then $g \circ f$ is continuous function if f, g are strongly $I\alpha$- continuous, $J\alpha$-continuous functions respectively.

Remark 2.28

The following diagram shows the relationships between different types of continuous functions

$$\begin{tikzcd} & f \text{ is continuous} \arrow[dr] & \\ f \text{ is strongly } I\alpha\text{- continuous} \arrow[ur] \arrow[r] & f \text{ is } I\alpha\text{- irresolute} \arrow[r] & f \text{ is } I\alpha\text{- continuous} \end{tikzcd}$$

The implications in the above diagram are not reversible in general as the following example shows.

Example 2.29

Consider the ideal topological spaces (X, τ, I) and (Y, σ, J) such that $= \{x, y, z\}, \tau = \{X, \phi, \{x\}\}, I = \{\phi, \{y\}\}, Y = \{a, b, c\}, \sigma = \{Y, \phi, \{b\}\}$ and $J = \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Then $I\alpha O(X) = \{X, \phi, \{x\}, \{x, y\}, \{x, z\}\}$ and $J\alpha O(Y) = \{Y, \phi, \{b\}, \{a, b\}, \{b, c\}\}$.

Define a function $f: (X, \tau, I) \longrightarrow (Y, \sigma, J)$ such that $f(x) = f(y) = a, f(z) = c$. This function is $I\alpha$- continuous and continuous but it's not $I\alpha$- irresolute for $\{b, c\} \in J\alpha O(Y)$, but $f^{-1}(\{b, c\}) = \{z\} \notin I\alpha O(X)$. It is not strongly $I\alpha$- continuous since $\{a, b\} \in J\alpha O(Y)$, but $f^{-1}(\{a, b\}) = \{x, y\} \notin \tau$.

Example 2.30

Consider the ideal topological spaces (X, τ, I) and (Y, σ, J) which defined as the same in Example 2.27. Now we define the function $f: (X, \tau, I) \to (Y, \sigma, J)$ as $f(x) = f(z) = b$ and $f(y) = a$. This function is $I\alpha$- irresolute and $I\alpha$- continuous but not strongly $I\alpha$- continuous for $\{b\} \in \sigma \in J\alpha O(Y)$, but $f^{-1}(\{b\}) = \{x, z\} \notin \tau$ and for the same reason it's not continuous.

Definition 2.31 Let $f: (X, \tau, I) \to (Y, \sigma, J)$ be a function. f is said to be

i. $I\alpha$-almost open function if the image of every $I\alpha$-open set in X is $J\alpha$- open set in Y.

ii. strongly $I\alpha$- open function if the image of every $I\alpha$-open set in X is open set in Y.

iii. $I\alpha$- open function if the image of every open set in X is $J\alpha$- open set in Y.

Below we will discuss the relation between the different types of functions that have been mentioned previously in definitions 2.29.

Remark 2.32

The following diagram shows the relationships between different types of continuous functions

$$\begin{tikzcd} & f \text{ is open} \arrow[dr] & \\ f \text{ is strongly } I\alpha\text{- open} \arrow[ur] \arrow[r] & f \text{ is } I\alpha\text{- almost open} \arrow[r] & f \text{ is } I\alpha\text{- open} \end{tikzcd}$$

The implications in the above diagram are not reversible in general as the following example shows.

Example 2.33

Consider the ideal topological spaces (X, τ, I) and (Y, σ, J) such that $= \{x, y, z\}, \tau = \{X, \phi, \{x\}\}, I = \{\phi, \{y\}\}, Y = \{a, b, c\}, \sigma = \{Y, \phi, \{b\}\}$ and $J = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Then $I\alpha O(X) = \{X, \phi, \{x\}, \{x, y\}, \{x, z\}\}$ and $J\alpha O(Y) = \{Y, \phi, \{b\}\}$.

Define a function $f: (X, \tau, I) \longrightarrow (Y, \sigma, J)$ such that $f(x) = b, f(y) = a, f(z) = c$. This function is $I\alpha$- open and open but it's not $I\alpha$- almost open for $\{x, y\} \in I\alpha O(X)$ but $f(\{x, y\}) = \{a, b\} \notin \sigma, \{a, b\} \notin J\alpha O(Y)$. It is not strongly $I\alpha$- open for the same reason.

Example 2.34

Consider the ideal topological spaces (X, τ, I) and (Y, σ, J) such that $= \{x, y, z\}, \tau = \{X, \phi, \{x\}\}, I = \{\phi, \{y\}\}, Y = \{a, b, c\}, \sigma = \{Y, \phi, \{b\}\}$ and $J = \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Then $I\alpha O(X) = \{X, \phi, \{x\}, \{x, y\}, \{x, z\}\}$ and $J\alpha O(Y) = \{Y, \phi, \{b\}, \{a, b\}, \{b, c\}\}$.

Define a function $f:(X,\tau,I)\longrightarrow(Y,\sigma,J)$ such that $f(x)=b, f(y)=f(z)=a$. This function is $I\alpha$- open and $I\alpha$- almost open. It is not strongly $I\alpha$- open function for $X\in\tau\in I\alpha O(X)$ but $f(X)=\{a,b\}\notin\sigma$ and it's not open for the same reason.

Definition 2.35 Let $f:(X,\tau,I)\to(Y,\sigma,J)$ be a function. f is said to be

i. $I\alpha$-almost closed function if the image of every $I\alpha$-closed set in X is $J\alpha$- closed set in Y.

ii. strongly $I\alpha$- closed function if the image of every $I\alpha$-closed set in X is closed set in Y.

iii. $I\alpha$- closed function if the image of every closed set in X is $J\alpha$- closed set in Y.

Below we will discuss the relation between the different types of functions that have been mentioned previously in Definitions 2.33.

Remark 2.36

The following diagram shows the relationships between different types of continuous functions

$$\begin{tikzcd} & f \text{ is closed} \arrow[dr] & \\ f \text{ is strongly } I\alpha\text{- closed} \arrow[ur] \arrow[r] & f \text{ is } I\alpha\text{- almost closed} \arrow[r] & f \text{ is } I\alpha\text{- closed} \end{tikzcd}$$

The implications in the above diagram are not reversible in general as the following example shows.

Example 2.37

Consider the ideal topological spaces (X,τ,I) and (Y,σ,J) such that $=\{x,y,z\}, \tau=\{X,\phi,\{x\}\}, I=\{\phi,\{y\}\}, Y=\{a,b,c\}, \sigma=\{Y,\phi,\{b\}\}$ and $J=\{\phi,\{a\},\{b\},\{a,b\}\}$. Then $I\alpha O(X)=\{X,\phi,\{x\},\{x,y\},\{x,z\}\}$ and $J\alpha O(Y)=\{Y,\phi,\{b\}\}$ .Hence $F=\tau^c=\{X,\phi,\{y,z\}\}$, $I\alpha C(X)=\{X,\phi,\{y\},\{z\},\{y,z\}\}$ and $\delta=\sigma^c=\{Y,\phi,\{a,c\}\}, J\alpha C(Y)=\{Y,\phi,\{a,c\}\}$.

Define a function $f:(X,\tau,I)\longrightarrow(Y,\sigma,J)$ such that $f(x)=a, f(y)=a, f(z)=c$. This function is $I\alpha$- closed and closed. It is not $I\alpha$- almost closed for $\{y\}\in I\alpha C(X)$, but $f(\{y\})=\{a\}\notin\delta, \{a\}\notin J\alpha C(Y)$, and it's not strongly $I\alpha$- closed for the same reason.

Example 2.38

Consider the ideal topological spaces (X,τ,I) and (Y,σ,J) such that $=\{x,y,z\}, \tau=\{X,\phi,\{x\}\}, I=\{\phi,\{y\}\}, Y=\{a,b,c\}, \sigma=\{Y,\phi,\{b\}\}$ and $J=\{\phi,\{a\},\{c\},\{a,c\}\}$, then $I\alpha O(X)=\{X,\phi,\{x\},\{x,y\},\{x,z\}\}$ & $J\alpha O(Y)=\{Y,\phi,\{b\},\{a,b\},\{b,c\}\}$. Hence $F=\tau^c=\{X,\phi,\{y,z\}\}$, $I\alpha C(X)=\{X,\phi,\{y\},\{z\},\{y,z\}\}$ and $\delta=\sigma^c=\{Y,\phi,\{a,c\}\}, J\alpha C(Y)=\{Y,\phi,\{a\},\{c\}\{a,c\}\}$.

Define a function $f:(X,\tau,I)\longrightarrow(Y,\sigma,J)$ such that $f(x)=a, f(y)=f(z)=c$, this function is $I\alpha$- closed function and $I\alpha$- almost closed function, but it's not strongly $I\alpha$- closed function for $\{y,z\}\in F, \{y,z\}\in I\alpha C(X)$but $f(\{y,z\})=\{c\}\notin\delta$ and it's not closed function for the same reason.

Definition 2.39 Let $f:(X,\tau,I)\to(Y,\sigma,J)$ be one to one and onto function. f is said to be

i. $I\alpha$-irresolute homeomorphism function if f is an $I\alpha$- irresolute and f^{-1} is an $J\alpha$- almost irresolute function.

ii. Strongly $I\alpha$- homeomorphism function if f is strongly $I\alpha$- continuous and f^{-1} is almost $J\alpha$- continuous function.

iii. $I\alpha$- homeomorphism function if f is an $I\alpha$- continuous and f^{-1} is an $J\alpha$- continuous function.

Remark 2.40 Let $f:(X,\tau,I)\to(Y,\sigma,J)$ be one to one and onto function. f is said to be

i. $I\alpha$-irresolute homeomorphism function if f is an $I\alpha$- irresolute and f is an $I\alpha$- almost open function.

ii. Strongly $I\alpha$- homeomorphism function if f is strongly $I\alpha$- continuous and f is strongly $I\alpha$- open function.

iii. $I\alpha$- homeomorphism function if f is an $I\alpha$- continuous and f is an $I\alpha$- open function.

Below we will discuss the relation between the different types of functions that have been mentioned previously in Definitions 2.39.

Remark 2.41

According to remark 2.40, remark 2. 28 and remark 2.32 we get the following diagram, which shows the relationships between different types of continuous functions

$$\begin{tikzcd} & f \text{ is homeomorphism} \arrow[dr] & \\ f \text{ is strongly } I\alpha\text{- homeomorphism} \arrow[ur] \arrow[r] & f \text{ is } I\alpha\text{- irresolute homeomorphism} \arrow[r] & f \text{ is } I\alpha\text{- homeomorphism} \end{tikzcd}$$

The implications in the above diagram are not reversible in general as the following example shows.

Example 2.42

It is easy to get from example 2.29 and example 2.33.

3. $I\alpha$- CONNECTED and $I\alpha$- SEPARATED SPACES

In this section we will study $I\alpha$- connected, $I\alpha$- separated and $I\alpha$- separated connected using the concept $I\alpha$- open sets.

Definition 3.1 Let (X,τ,I) be an ideal topological space then it's called $I\alpha$- disconnected space if X can be write as a union of two non-empty disjoint $I\alpha$- open sets,

i.e. X is $I\alpha$- disconnected space $\Leftrightarrow X = A \cup B$ *such that* $A, B \in I\alpha O(X)$ *and* $A \neq \emptyset \neq B$ *such that* $A \cap B = \emptyset$.

Definition 3.2 Let (X,τ,I) be an ideal topological space then it's called $I\alpha$- connected space if X is not $I\alpha$- disconnected space,

i.e. X is $I\alpha$- connected space $\Leftrightarrow X \neq A \cup B$ *such that* $A, B \in I\alpha O(X)$ *and* $A \neq \emptyset \neq B$ *such that* $A \cap B = \emptyset$.

Proposition 3.3 An ideal topological space (X,τ,I) is $I\alpha$- connected space if and only if X cannot be written as a union of non-empty disjoint $I\alpha$- closed sets.

Proposition 3.4 An ideal topological space (X,τ,I) is $I\alpha$- connected space if and only if the only sets which are $I\alpha$- open sets and $I\alpha$- closed sets in the same time are $X, \emptyset$.

Proposition 3.5 An ideal topological space (X,τ,I) is $I\alpha$- connected space if and only if the only sets which have no $I\alpha$- boundary points are $X, \emptyset$.

Corollary 3.6 Every $I\alpha$- connected space is connected space but the inverse is not true.

Remark 3.7 The continuous image of $I\alpha$- connected space is not necessary to be $I\alpha$-connected.

Example 3.8 Consider the ideal topological spaces (X,τ,I) and (Y,σ,J) such that $X = \{x,y,z\}, \tau = \{X,\phi,\{x\}\}, I = \{\phi,\{y\}\}, Y = \{a,b,c\}, \sigma = \{Y,\phi,\{b\}\}$ *and* $J = \{\phi,\{c\}\}$. Then $I\alpha O(X) = \{X,\phi,\{x\},\{x,y\},\{x,z\}\}$ and $J\alpha O(Y) = \{Y,\phi,\{b\},\{a,b\},\{b,c\}\}$. Define a function $f:(X,\tau,I) \longrightarrow (Y,\sigma,J)$ such that $f(x) = f(y) = a, f(z) = c$. This function is continuous. Let $= f(X) = \{a,c\}$. Then $\sigma_B = \{B,\phi\}$, $J_B = \{\phi,\{c\}\}$ *and* $J_B\alpha O(B) = \{B,\phi,\{a\},\{c\}\}$. It is clear that X is $I\alpha$- connected space but $B = f(X)$ is $J\alpha$- disconnected space.

Remark 3.9 $I\alpha$- connected property is not hereditary property.

Example 3.10 Consider the ideal topological space (X,τ,I), such that $X = \{x,y,z\}, \tau = \{X,\phi,\{x\},\{x,y\},\{x,z\}\}$ *and* $I = \{\phi,\{y\}\}$. Then $I\alpha O(X) = \{X,\phi,\{x\},\{x,y\},\{x,z\}\}$. It is clear that X is $I\alpha$- connected space but if we take a subset $A = \{y,z\}$ of the space X such that $\tau_A = \{A,\phi,\{y\},\{z\}\}, I_A = \{\phi,\{y\}\}$ and $I_A\alpha O(A) = \{A,\phi,\{y\},\{z\}\}$ we get that A is $I\alpha$- disconnected space.

Corollary 3.11 $I\alpha$- connected property is $I\alpha$- topological property.

Remark 3.12 if $A, B \subseteq X$ such that A, B are $I\alpha$- connected sets over X then it is not necessary that $A \cup B$ to be $I\alpha$- connected set as in the following example.

Example 3.13 Consider the ideal topological space (X,τ,I), such that $X = \{x,y,z\}, \tau = \{X,\phi,\{x\}\}$ *and* $I = \{\phi,\{y\}\}$. Then $I\alpha O(X) = \{X,\phi,\{x\},\{x,y\},\{x,z\}\}$. Take $A = \{x\}$ *and* $B = \{y\}$ then $A \cup B = \{x,y\}$, $\tau_{A\cup B} = \{A \cup B,\phi,\{x\},\{y\}\}$, $I_{A\cup B} = \{\phi,\{y\}\}$ *and* $I\alpha O(A \cup B) = \{A \cup B,\phi,\{x\},\{y\}\}$. It is clear that A *and* B are $I\alpha$- connected sets over X but $A \cup B$ is $I\alpha$- disconnected set over X.

Remark 3.14 if $A, B \subseteq X$ such that A, B are $I\alpha$- disconnected sets over X then it is not necessary that $A \cap B$ to be $I\alpha$- disconnected set.

Example 3.15 Consider the ideal topological space (X,τ,I), such that $X = \{x,y,z\}, \tau = \{X,\phi,\{x\}\}$ *and* $I = \{\phi,\{y\}\}$. Then $I\alpha O(X) = \{X,\phi,\{x\},\{x,y\},\{x,z\}\}$. Take $A = \{x,y\}$ *and* $B = \{x,z\}$ then $A \cap B = \{x\}$, $\tau_{A\cap B} = \{A \cap B,\phi\}, I_{A\cap B} = \{\phi\}$ *and* $I\alpha O(A \cap B) = \{A \cap B,\phi\}$. It is clear that A *and* B are $I\alpha$- disconnected sets over X but $A \cap B$ is $I\alpha$- connected set over X.

Corollary 3.16 if $A, B \subseteq X$ such that A, B are $I\alpha$- connected sets over X then $A \cup B$ is $I\alpha$- connected set only in the case of being $A \cap B \neq \emptyset$.

Definition 3.17 A nonempty subsets A, B of an ideal space (X,τ,I) are said to be $\star$- separated sets if $cl^*(A) \cap B = \emptyset = A \cap\ cl(B)$.

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Definition 3.18 A subset A of an ideal space (X, τ, I) is called $\star_s$- separated connected if it cannot be written as a union of two $\star$- separated sets. An ideal space (X, τ, I) is called $\star_s$- separated connected if it is not the union of two $\star$- separated sets.

Definition 3.19 A nonempty subsets A, B of an ideal space (X, τ, I) are said to be $\alpha \star$- separated sets if $cl^{\alpha\star}(A) \cap B = \emptyset = A \cap \ cl_{\alpha}(B)$.

Definition 3.20 A subset A of an ideal space (X, τ, I) is called $I\alpha$- separated connected if it cannot be written as a union of two $I\alpha$- separated sets. An ideal space (X, τ, I) is called $\alpha \star_s$- connected if it is not the union of two $\alpha \star$- separated sets.

Definition 3.21 A nonempty subsets A, B of an ideal space (X, τ, I) are said to be $I\alpha$- separated sets if $I\alpha$- $cl(A) \cap B = \emptyset = A \cap \ cl_{\alpha}(B)$.

Definition 3.22 A subset A of an ideal space (X, τ, I) is called $I\alpha$- separated connected if it cannot be written as a union of two $I\alpha$- separated sets. An ideal space (X, τ, I) is called $I\alpha$- separated connected if it is not the union of two $I\alpha$- separated sets.

Remark 3.23

The following diagram shows the relationships between different types of separated sets

$$\begin{tikzcd}
 & I\alpha\text{- separated} \arrow[d] \\
\star\text{- separated} \arrow[r] & \alpha \star\text{- separated}
\end{tikzcd}$$

Remark 3.24

The following diagram shows the relationships between different types of separated connected sets

$$\begin{tikzcd}
 & I\alpha\text{- separated connected} \\
\star\text{- separated connected} & \alpha \star\text{- connected} \arrow[l] \arrow[u]
\end{tikzcd}$$

Proposition 3.25 Let (X, τ, I) be an ideal space. If A and B are $I\alpha$- separated subsets of X and $A \cup B \in \tau_{\alpha}$ then A and B are α- open and $I\alpha$- open respectively.

Remark 3.26 Let (X, τ, I) be an ideal space. If the union of two $I\alpha$- separated sets is an α-closed set, then one set is α-closed and the other is $I\alpha$-closed.

Proposition 3.27 Let (X, τ, I) be an ideal space. If A and B are nonempty disjoint subsets of X, such that A is α- open and B is $I\alpha$- open, then A and B are $I\alpha$- separated.

Proposition 3.28 Let (X, τ, I) be an ideal space. If A is $I\alpha$- separated connected subset of X and H, G are $I\alpha$- separated subsets of X with $A \subseteq H \cup G$, then either $A \subset H$ or $A \subset G$.

Corollary 3.29 If A is an $I\alpha$- separated connected subset of an ideal topological space (X, τ, I) and $A \subset B \subset I\alpha$-$cl(A)$, then B is $I\alpha$- separated connected.

Remark 3.30 Let A and B are two $I\alpha$- separated sets in an ideal topological space (X, τ, I). If C and D are nonempty subsets such that $C \subset A$ and $D \subset B$, then C and D are also $I\alpha$- separated..

4. CONCLUSION

General topology has many branches whether it is old or new. In this paper, we implement to a new part in this wide science by using $I\alpha$- open set in ideal topological space which introduced by M.E. Abd El-Monsef [1] in 2013, by bring down the concepts of general topology such as interior, exterior, boundary, closure and connected space on this new set and introduce a new form to this concepts. In future, anyone can study the rest of the concepts of general topology like separation axioms and compacted in the $I\alpha$- open set.

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