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Existence of positive solutions for the boundary valueproblem of a nonlinear fractional differential equat

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ABSTRACT

In this paper, we deal with the following nonlinear fractional boundary value problem

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<t<1, 4<\alpha\le 5,$$

$$u(0)=u(1)=u'(0)=u'(1)=u''(1)=0$$

where D_{0+}^{α} is the standard Riemann-Liouville differential operator of order α. We give some properties of Green's function for the problem. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions are obtained. Moreover, some concrete examples are given respectively.

KEYWORDS: Fractional differential equation; Boundary value problem; Positive solution; Green's function; Fixed-point theorems.

SUBJECT CLASSIFICATION: 26A33; 34B18; 34B27.

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1. INTRODUCTION

Recently, many books and papers on fractional differential equations have been studied extensively. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences, such as physics, engineering, economics, viscoelasticity and many other fields.

For the history, theory and applications of fractional calculus, we refer the readers to the books by Kilbas et al[11], Miller et al [17], and Podlubny [18]. Some basic theory for the initial value problems of fractional differential equations has been discussed by many authors, see for instance [3,7,13,14,21] and the references therein.

Moreover, there are some papers involving the existence and multiplicity of solutions for nonlinear fractional differential equations' boundary value problems. In [22] Zhang used cone theory and the theory of upper and lower solutions to show the existence of at least one positive solution of fractional order differential equation

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<t<1, 0<\alpha\le 1.$$

In [2], Bai and lü studied the existence of positive solutions of the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<\text{t}<1, 1<\alpha\le 2. \\ u(0)=u(1)=0, \end{cases}$$

El-Shahed [8] studied the following nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda a(t)f(t,u(t))=0, 0<\text{t}<1, 2<\alpha\le 3, \\ u(0)=u'(0)=u'(1)=0, \end{cases}$$

Recently, Liang and Zhang [16] considered the following nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<\text{t}<1, 3<\alpha\le 4, \\ u(0)=u'(0)=u''(0)=u'''(1)=0, \end{cases}$$

Also, in [20], Xu et al studied the following nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,u(t)), 0<\text{t}<1, 3<\alpha\le 4, \\ u(0)=u(1)=u'(0)=u'(1)=0, \end{cases}$$

In [16,20], the authors gave the existence of positive solutions for the above boundary value problems respectively. Similarly, it also should be noted that the papers [1,4,5,6,9,10,19,23] and the references therein.

Motivated by all the works above, in this paper, we discuss the following nonlinear fractional boundary value problem

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<t<1, 4<\alpha\le 5, \quad (1.1)$$

$$u(0)=u(1)=u'(0)=u'(1)=u''(1)=0 \quad (1.2)$$

where $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and D_{0+}^{α} is the standard Riemann-Liouvill

e fractional derivative.

In this paper, we firstly derive the corresponding Green's function known as fractional Green's functions. Then, some properties of the Green's function are given, which plays an important role in this paper. Consequently problem (1) is reduced to an equivalent Fredholm integral equation of the second kind. Finally, the existence and multiplicity of positive solutions are obtained in Theorem 3.1 and Theorem 3.2 by means of some fixed-point theorems.

2. PRELIMINARIES and LEMMAS

For completeness, in this section, we present here the necessary definitions and some basic results from fractional calculus theory. These can be found in the recent literatures such as [2,18,20].

Definition 1. The Riemann-Liouville fractional integral of order α of a function

$y:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds$$

provided the right side is pointwise defined on $(0,\infty)$ and $\Gamma(\alpha)$ is the Euler gamma function defined by

$$\Gamma(\alpha)=\int_0^{+\infty} x^{\alpha-1}e^{-x}ds, \alpha>0$$

Definition 2. The Riemann-Liouville fractional derivative of order α of a function $y:(0,\infty)\to R$ is given by

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}ds$$

where $n=[\alpha]+1$ provided that the right side is pointwise defined on $(0,\infty)$

From the definition of the Riemann-Liouville derivative, we can obtain the statements.

Lemma 1. Let $\alpha>0.$ If we assume $u\in C(0,1)\cap L(0,1),$ then the fractional differential equatio has

$$u(t)=C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_Nt^{\alpha-N},\ C_i\in R, i=1,2,\cdots,N$$

as unique solution, where N is the smallest integer greater than or equal to $\alpha.$

Lemma 2. Assume that $u\in C(0,1)\cap L(0,1)$ with a fractional derivative of order α that belongs to $C(0,1)\cap L(0,1)$, Then $D_{0+}^{\alpha}u(t)=0$

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_Nt^{\alpha-N}$$

for some $C_i\in R, i=1,2,\cdots,N,$ N is the smallest integer greater than or equal to $\alpha.$

In the following, we present the Green's function of fractional differential equation boundary value problem, which plays the major role in our next analysis.

Lemma 3. Given $h\in C[0,1]$ and $4<\alpha\le 5$ then the unique solution of

$$D_{0+}^{\alpha}u(t)=h(t), 0<t<1, 4<\alpha\le 5, \quad (2.1)$$

$$u(0)=u(1)=u'(0)=u'(1)=u''(1)=0 \quad (2.2)$$

is $u(t)=\int_0^1 G(t,s)h(s)ds$

where

$$G(t,s)=\begin{cases}\dfrac{g(t,s)-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\[2mm] \dfrac{g(t,s)}{\Gamma(\alpha)}, & 0\le t\le s\le 1,\end{cases} \quad (2.3)$$

with

$$g(t,s)=\frac{1}{2}(\alpha-2)(\alpha-3)t^{\alpha-1}(1-s)^{\alpha-1}-(\alpha-1)(\alpha-3)t^{\alpha-2}(1-s)^{\alpha-2}(t-s)$$
$$+\frac{1}{2}(\alpha-1)(\alpha-2)t^{\alpha-3}(1-s)^{\alpha-3}(t-s)^{2}$$

Here $G(t,s)$ is called Green's function of boundary value problem (2.1)-(2.2).

Proof. We apply Lemma 2.2 and Definition 2.1 to reduce (2.1) to an equivalent integral equation

$$u(t)=C_1t^{\alpha-1}+C_2t^{\alpha-2}+C_3t^{\alpha-3}+C_4t^{\alpha-4}+C_5t^{\alpha-5}-I_{0+}^{\alpha}h(t)$$
$$=C_1t^{\alpha-1}+C_2t^{\alpha-2}+C_3t^{\alpha-3}+C_4t^{\alpha-4}+C_5t^{\alpha-5}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds$$

for some $C_i\in R, i=1,2,\cdots,5.$

From (2.2), we obtain that $C_4=C_5=0$ and

$$C_1=\frac{1}{\Gamma(\alpha)}\int_0^1\left[\frac{1}{2}(\alpha-2)(\alpha-3)s^2+(\alpha-3)s+1\right](1-s)^{\alpha-3}h(s)ds$$
$$C_2=\frac{\alpha-1}{\Gamma(\alpha)}\int_0^1(1+\alpha s-3s)(1-s)^{\alpha-3}sh(s)ds$$
$$C_3=\frac{1}{\Gamma(\alpha)}\int_0^1\frac{1}{2}(\alpha-1)(\alpha-2)s^2(1-s)^{\alpha-3}h(s)ds.$$

Hence, the unique solution of problem (2.1)-(2.2) is

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^1\left[\frac{1}{2}(\alpha-2)(\alpha-3)s^2+(\alpha-3)s+1\right]t^{\alpha-1}(1-s)^{\alpha-3}h(s)ds$$
$$-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^1(1+\alpha s-3s)t^{\alpha-2}(1-s)^{\alpha-3}sh(s)ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^1\frac{1}{2}(\alpha-1)(\alpha-2)t^{\alpha-3}s^2(1-s)^{\alpha-3}h(s)ds$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^t\{\left[\frac{1}{2}(\alpha-2)(\alpha-3)s^2+(\alpha-3)s+1\right]t^{\alpha-1}(1-s)^{\alpha-3}$$
$$-(1+\alpha s-3s)st^{\alpha-2}(1-s)^{\alpha-3}+\frac{1}{2}(\alpha-1)(\alpha-2)t^{\alpha-3}s^2(1-s)^{\alpha-3}-(t-s)^{\alpha-1}\}h(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_t^1\{\left[\frac{1}{2}(\alpha-2)(\alpha-3)s^2+(\alpha-3)s+1\right]t^{\alpha-1}(1-s)^{\alpha-3}-(\alpha-1)(1+\alpha s-3s)st$$
$$+\frac{1}{2}(\alpha-1)(\alpha-2)s^2\}t^{\alpha-3}(1-s)^{\alpha-3}h(s)ds$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^t\{[\frac{1}{2}(\alpha-2)(\alpha-3)t^2-(\alpha-1)(\alpha-3)t(1-s)(t-s)$$
$$+\frac{1}{2}(\alpha-1)(\alpha-2)(t-s)^2]t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}\}h(s)ds$$

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$$+\frac{1}{\Gamma(\alpha)}\int_t^1\{[\frac{1}{2}(\alpha-2)(\alpha-3)t^2-(\alpha-1)(\alpha-3)t(1-s)(t-s)$$

$$+\frac{1}{2}(\alpha-1)(\alpha-2)(t-s)^2]t^{\alpha-1}(1-s)^{\alpha-3}\}h(s)ds$$

$$=\int_0^1 G(t,s)h(s)ds$$

The proof is completed. □

The following properties of Green's function form the basis of our main work in this paper.

Lemma 4. When $1\le t\le s\le 1,$ the function $g(t,s)$ satisfies the following properties:

(1) $$\int_t^1(1-s)^{\alpha-2}(s-t)\,ds=(1-t)^{\alpha-1}B(\alpha-1,2)$$

(2) $$\int_t^1(1-s)^{\alpha-3}(s-t)^2\,ds=(1-t)^{\alpha-1}B(\alpha-2,3)$$

(3) $$\int_0^t s^3(t-s)^{\alpha-4}\,ds=t^{\alpha}\beta(\alpha-3,4)$$

(4) $$\int_t^1 g(t,s)ds=\frac{1}{2\alpha}t^{\alpha-1}(1-t)^{\alpha}(\alpha-2)(\alpha-3)+\frac{1}{\alpha}(\alpha-3)t^{\alpha-2}(1-t)^{\alpha}+\frac{1}{\alpha}t^{\alpha-3}(1-t),$$

where $B(p,q)$ is Bata function $B(\mathrm{p},q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,\ p>0,q>0$

Proof. (1) Set $\tau=\frac{s-t}{1-t}$ then $s=\tau(1-t)+t,1-s=(1-t)(1-\tau)$

$$\int_t^1(1-s)^x(s-t)^y\,ds==(1-t)^{x+y+1}B(x+1,y+1)$$

Letting $x=\alpha-2,\ y=1$ then we obtain (1), The proof of (2), (3), (4) is similar. □

Lemma 5. The function $G(t,s)$ defined by (2.3) satisfies the following conditions:

(1) $$G(t,s)=G(1-s,1-t),\ t,s\in(0,1)$$

(2) $$\frac{a}{3}s^3(1-t)^3(t-s)^{\alpha-4}\le\Gamma(\alpha)G(t,s)\le\frac{a}{3}s^3(1-t)^3t^{\alpha-4}(1-s)^{\alpha-4}\le\frac{a}{3}s^3(1-s)^{\alpha-4}$$

with $a=\frac{1}{2}(\alpha-1)(\alpha-2)(\alpha-3)$ for all $0\le s\le t\le 1$.

(3) $$\frac{(1-t)^3t^{\alpha}}{\alpha}\le\Gamma(\alpha)\int_0^t G(t,s)ds\le\frac{1}{\alpha}(1-t)^3t^{\alpha-4}$$

(4) $$\frac{1}{\alpha}(1-t)^3t^{\alpha}+\frac{1}{2\alpha}(\alpha-2)(\alpha-3)t^{\alpha-1}(1-t)^{\alpha}+\frac{\alpha-3}{\alpha}t^{\alpha-2}(1-t)^{\alpha}+\frac{1}{\alpha}t^{\alpha-3}(1-t)^{\alpha}$$

$$\le\Gamma(\alpha)\int_0^1 G(t,s)ds\le\frac{\alpha^2-3\alpha+4}{2\alpha}$$

Proof. (1) Noticing the expressing of $G(t,s)$, it is clear that $G(t,s)=G(1-s,1-t)$ for $t,s\in(0,1)$.

(2) When $0\le s\le t\le 1,$ we have

$$\Gamma(\alpha)G(t,s)=g(t,s)-(t-s)^{\alpha-1}$$

$$=\frac{1}{2}(\alpha-2)(\alpha-3)[(t-ts)^{\alpha-1}-(t-s)^{\alpha-1}]$$

$$-(\alpha-1)(\alpha-3)\left[(t-ts)^{\alpha-2}-(t-s)^{\alpha-2}\right](t-s)$$

$$+\frac{1}{2}(\alpha-1)(\alpha-2)\left[(t-ts)^{\alpha-3}-(t-s)^{\alpha-3}\right](t-s)^{2}$$

$$=\frac{1}{2}(\alpha-1)(\alpha-2)(\alpha-3)\int_{t-s}^{t-ts}\zeta^{\alpha-4}\left[\zeta^{2}-2\zeta(t-s)+(t-s)^{2}\right]d\zeta$$

$$=\frac{1}{2}(\alpha-1)(\alpha-2)(\alpha-3)\int_{t-s}^{t-ts}\zeta^{\alpha-4}\left[\zeta-(t-s)\right]^{2}d\zeta.$$

Noting that

$$(t-s)^{\alpha-4}\le\zeta^{\alpha-4}\le(t-ts)^{\alpha-4},\int_{t-s}^{t-ts}\left[\zeta-(t-s)\right]^{2}d\zeta=\frac{1}{3}s^{3}(1-t)^{3},$$

therefore, we have

$$\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{6}s^{3}(1-t)^{3}(t-s)^{\alpha-4}\le\Gamma(\alpha)G(t,s)$$

$$\le\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{6}s^{3}(1-t)^{3}t^{\alpha-4}(1-s)^{\alpha-4}$$

where $\alpha>4$ is used. This completes the proof.

(3) It is a direct consequence of (2). We omit the proof.

(4)Combining (3) in Lemma 2.5 and (4) in Lemma 2.4, we deduce that

$$\Gamma(\alpha)\int_{0}^{1}G(t,s)ds=\int_{0}^{t}\left[\Gamma(\alpha)G(t,s)\right]ds+\int_{t}^{1}g(t,s)ds$$

$$\le\frac{1}{\alpha}(1-t)^{3}t^{\alpha-4}+\frac{1}{2\alpha}t^{\alpha-1}(1-t)^{\alpha}(\alpha-2)(\alpha-3)$$

$$+\frac{1}{\alpha}(\alpha-3)t^{\alpha-2}(1-t)^{\alpha}+\frac{1}{\alpha}t^{\alpha-3}(1-t)^{\alpha}$$

$$\le\frac{1}{\alpha}+\frac{(\alpha-2)(\alpha-3)}{2\alpha}+\frac{\alpha-3}{\alpha}+\frac{1}{\alpha}=\frac{\alpha^{2}-3\alpha+4}{2\alpha}.$$

On the other hand,

$$\Gamma(\alpha)\int_{0}^{1}G(t,s)ds=\int_{0}^{t}\left[g(t-s)-(t-s)^{\alpha-1}\right]ds+\int_{t}^{1}g(t,s)ds$$

$$\ge\frac{1}{\alpha}(1-t)^{3}t^{\alpha}+\frac{1}{2\alpha}t^{\alpha-1}(1-t)^{\alpha}(\alpha-2)(\alpha-3)+\frac{\alpha\text{-}3}{\alpha}t^{\alpha-2}(1-t)^{\alpha}+\frac{1}{\alpha}t^{\alpha-3}(1-t)^{\alpha}.$$

The proof is complete. □

In the proof of our main results, we will use the following Lemma, the proof of which is simple.

Lemma 6. Supposing $h_{\beta,\gamma}(t)=t^{\beta}(1-t)^{\gamma}$, with $(\beta,\gamma)=(\alpha,3),(\alpha-1,\alpha),(\alpha-2,\alpha),$ $(\alpha-3\alpha$ respectively, we have the following properties

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} h_{\alpha,3}(t)=h_{\alpha,3}(\frac{1}{4})=\frac{3^3}{4^{\alpha+3}},\ \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} h_{\alpha-1,\alpha}(t)=h_{\alpha-1,\alpha}\left(\frac{3}{4}\right)=\frac{3^{\alpha-1}}{4^{2\alpha-1}}$$

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} h_{\alpha-2,\alpha}(t)=h_{\alpha-2,\alpha}\left(\frac{3}{4}\right)=\frac{3^{\alpha-2}}{4^{2\alpha-2}},\ \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} h_{\alpha-3,\alpha}(t)=h_{\alpha-3,\alpha}\left(\frac{3}{4}\right)=\frac{3^{\alpha-3}}{4^{2\alpha-3}}$$

The following fixed-point theorems are fundamental in the proof of our main results.

Lemma 7. *[12]* Let E be a Banach space $P\subseteq E$ a cone, and $\Omega_1,\ \Omega_2$ two bounded open balls of E centered the origin with $\overline{\Omega_1}\subset\Omega_2$. Suppose that $A:P\cap(\overline{\Omega_2}\setminus\Omega_1)\to P$ is a completely continuous operator such that either

(i) $\|Ax\|\le x, x\in P\bigcap\partial\Omega_1$ and $\|Ax\|\ge\|x\|, x\in P\bigcap\partial\Omega_2$ or

(ii) $\|Ax\|\ge\|x\|, x\in P\bigcap\partial\Omega_1$ and $\|Ax\|\le\|x\|, x\in P\bigcap\partial\Omega_2$

holds, Then A has a fixed point in $P\cap(\overline{\Omega_2}\setminus\Omega_1)$.

Lemma 8. *[15]* Let P be a cone in a real Banach space E, $P_c=\left\{x\in P \middle| \|x\|\le c\right\}$, θ a nonnegative continuous concave function on P such that $\theta(x)\le\|x\|$, for all $x\in\overline{P}_c$ and $P(\theta,\mathrm{b},\mathrm{d})=\left\{x\in P\middle| b\le\theta(x),\|x\|<d\right\}$, suppose $A:\overline{\mathrm{P}}_c\to\overline{\mathrm{P}}_c$ is completely continuous and there exist constants $0<a<b<d\le c$ such that

(C1) $\left\{x\in P(\theta,\mathrm{b},\mathrm{d})\middle|\theta(x)>b\right\}\ne\varnothing$ and $\theta(AX)>b, x\in P(\theta,\mathrm{b},\mathrm{d});$

(C2) $\|Ax\|<a, x\le a;$

(C3) $\theta(Ax)>b$ for $x\in P(\theta,b,c)$ with $\|Ax\|>d$.

Then A at least three fixed points x_1,x_2,x_3 with

$$\|x_1\|<a, b<\theta(x_2), a<\|x_3\|, \theta(x_3)<b.$$

Remark 1. If there holds $d=c$, then the condition (C1) of Lemma 2.8 implies condition (C3) of Lemma 2.8.

3. MAIN RESULTS

In this section, we will apply Lemma 2.7 and lemma 2.8 to establish some results of existence and multiplicity of positive solutions for problem (1)-(2).

Let $E=C([0,1],\|\cdot\|)$ be endowed with the maximum norm, $\|u\|=\max\limits_{0\le t\le1}|u(t)|$, then E is a Banach space. Define the cone $P\subset E$ by

$$P=\left\{u\in E\,|\,u(t)>0, t\in[0,1]\right\},$$

and the operator $A:P\to E$ by

$$Au(t)=\int_0^1 G(t,s)f(s,u(s))ds.$$

Notice that the fixed points of A are solutions of (1.1)-(1.2). In order to apply Lemma 2.7 and Lemma 2.8, we must show that $A:P\to P$ is completely continuous.

Lemma 1. Let $f:[0,1]\times[0,\infty)\to[0,\infty)$ be continuous, then the operator $A:P\to P$ is completely continuous.

Proof. The operator $A:P\to P$ is continuous in view of nonnegativeness and continuity of $G(t,s)$ and $f(t,s)$ as well as Lemma 2.5.

Let $\Omega \subset P$ be bounded, i.e. there exists a positive constant $L > 0$ such that $\|u\| \le L$ for all $u \in \Omega$. Let $K = \max_{0\le t\le 1, 0\le u\le L} f(t,u(t))$ then from (4) of Lemma 2.5, we have

$$Au(t) = \int_0^1 G(t,s)\, f(s,u(s))ds \le K\int_0^1 G(t,s)\,ds \le K\frac{\alpha^2-3\alpha+4}{2\alpha\Gamma(\alpha)}.$$

Therefore, $\|Au\| \le K\frac{\alpha^2-3\alpha+4}{2\alpha\Gamma(\alpha)}$, and so $A(\Omega)$ is uniformly bounded.

On the other hand, for any given $\varepsilon > 0$, taking $\delta = \frac{\varepsilon}{C}$ with some positive constant C to be chosen later, then for each $u \in \Omega, t_1, t_2 \in [0,1]$, and $0 < t_2 - t_1 < \delta$, we have

$$\left|Au(t_2) - Au(t_1)\right| < \varepsilon.$$

That is to say, $A(\Omega)$ is equicontinuity.

Indeed,

$$\left|Au(t_2) - Au(t_1)\right| = \left|\int_0^t (G(t_2,s) - G(t_1,s)) f(s,u(s))ds\right|$$

$$= \int_0^{t_1} \left|G(t_2,s) - G(t_1,s)\right| f(s,u(s))ds + \int_{t_1}^{t_2} \left|G(t_2,s) - G(t_1,s)\right| f(s,u(s))ds$$

$$+\int_{t_2}^{1} \left|G(t_2,s) - G(t_1,s)\right| f(s,u(s))ds$$

$$\le K\int_0^{t_1} \frac{1}{2}(\alpha-2)(\alpha-3)(1-s)^{\alpha-1}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)ds$$

$$+K\int_0^{t_1}(\alpha-1)(\alpha-3)(1-s)^{\alpha-2}\left[t_2^{\alpha-1} - t_1^{\alpha-1} + s\left(t_1^{\alpha-2} - t_2^{\alpha-2}\right)\right]ds$$

$$+K\int_0^{t_1}\frac{1}{2}(\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\left[t_2^{\alpha-3}(t_2-s)^2 - t_1^{\alpha-3}(t_1-s)^2\right]ds$$

$$+K\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]ds + \int_{t_1}^{t_2}\left|G(t_2,s) - G(t_1,s)\right| f(s,u(s))ds$$

$$+\int_{t_2}^{1}\left|G(t_2,s) - G(t_1,s)\right| f(s,u(s))ds$$

Noticing that for each $k = 1,2,3$, there exists $\zeta_k \in (t_2,t_1) \subset [0,1]$ such that

$$\left|t_2^{\alpha-k} - t_1^{\alpha-k}\right| = (\alpha-k)\zeta_k^{\alpha-K-1}\left|t_2 - t_1\right|,$$

hence, we obtain

$$\left|Au(t_2) - Au(t_1)\right| \le K\frac{1}{2}(\alpha-1)(\alpha-2)(\alpha-3)\int_0^{t_1}(1-s)^{\alpha-1}ds\left|t_2 - t_1\right|$$

$$+K(\alpha-1)(\alpha-3)(\alpha-3)\int_0^{t_1}(1-s)^{\alpha-2}ds\left|t_2 - t_1\right|$$

$$+K\frac{1}{2}(\alpha-1)(\alpha-2)\int_0^{t_1}(1-s)^{\alpha-3}\left[(\alpha-1) + 2s(\alpha-2) + s^2(\alpha-3)\right]ds\left|t_2 - t_1\right|$$

$$+K(\alpha-1)\int_0^{t_1}(1-s)^{\alpha-2}ds|t_2-t_1|+\int_{t_1}^{t_2}|G(t_2,s)-G(t_1,s)|f(s,u(s))ds$$
$$+\int_{t_2}^{1}|G(t_2,s)-G(t_1,s)|f(s,u(s))ds$$
$$=c_1|t_2-t_1|+c_2|t_2-t_1|+c_3|t_2-t_1|+c_4|t_2-t_1|+\int_{t_1}^{t_2}|G(t_2,s)-G(t_1,s)|f(s,u(s))ds$$
$$+\int_{t_2}^{1}|G(t_2,s)-G(t_1,s)|f(s,u(s))ds$$

Similarly, we can obtain

$$K\int_{t_2}^{1}|G(t_2,s)-G(t_1,s)|ds\le c_5|t_2-t_1|.$$

From the inequality (4) of Lemma 2.5, we obtain

$$K\int_{t_1}^{t_2}|G(t_2,s)-G(t_1,s)|ds\le K\frac{\alpha^2-3\alpha+4}{2\Gamma(\alpha)\alpha}|t_2-t_1|=c_6|t_2-t_1|.$$

Therefore, we deduce that

$$|Au(t_2)-Au(t_1)|\le\sum_{i=1}^{6}c_i|t_2-t_1|=C|t_2-t_1|\le C\delta=\varepsilon.$$

That is A is equicontinuous on Ω. Thanks to the Arzela-Ascoli Theorem, We get that A is completely continuous. This completes the proof. □

In our first result, we show the existence of at least one positive solution of (1.1)-(1.2).

Theorem 1. Let $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$. Assume that there exist two positive constants $r_2>r_1>0$ such that the following conditions hold

(H1) $f(t,u)\le Mr_2$ for $(t,u)\in[0,1]\times[0,r_2]$;

(H2) $f(t,u)\ge Nr_1$ for $(t,u)\in[0,1]\times[0,r_1]$,

where

$$M=\frac{2\Gamma(\alpha+1)}{\alpha^2\text{-}3\alpha+4},$$

$$N=2\Gamma(\alpha+1)\left[\frac{1}{2^{\alpha+2}}+\frac{(\alpha-2)(\alpha-3)}{2^{2\alpha-1}}+\frac{\alpha-3}{2^{2\alpha-3}}+\frac{1}{2^{2\alpha-4}}\right]^{-1}.$$

Then the problem (1.1)-(1.2) has at least one positive solution such that $r_1\le\|u\|\le r_2$.

Proof. From Lemma 2.3 and Lemma 3.1, we know that $A:P\to P$ is completely continuous and problem (1)-(2) has a solutions $u=u(t)$ if and only if $u=Au$. Now, we are in the position to show that the condition (ii) of Lemma 2.7 is satisfied.

Define $\Omega_2=\{u\in P\,|\,\|u\|\le r_2\}$. For $u\in P\cap\partial\Omega_2$, it follows from (H1) and (4) of Lemma 2.5 that for $t\in[0,1]$

$$Au(t)=\int_0^1G(t,s)f(s,u(s))ds\le Mr_2\int_0^1G(t,s)ds\le Mr_2\frac{\alpha^2-3\alpha+4}{2\alpha\Gamma(\alpha)}=r_2,$$

$$Au(t)\ge\frac{Nr_1}{2\alpha\Gamma(\alpha)}\left[2t^{\alpha}(1-t)^3+(1-t)^{\alpha}t^{\alpha-1}(\alpha-2)(\alpha-3)+2(\alpha-3)(1-t)^{\alpha}t^{\alpha-2}+2(1-t)^{\alpha}t^{\alpha-3}\right].$$ which

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implies that $\|Au\| \le \|u\|$ for $u \in P \bigcap \partial\Omega_2$.

On the other hand, define $\Omega_1 = \{u \in P, \|u\| \le r_1\}$. For $u \in P \cap \partial\Omega_1$, it follows from (H2) and (4) of Lemma 2.5 that for $t \in [0,1]$

$$Au(t) \ge \frac{Nr_1}{2\alpha\Gamma(\alpha)}\left[2t^\alpha(1-t)^3 + (1-t)^\alpha t^{\alpha-1}(\alpha-2)(\alpha-3) + 2(\alpha-3)(1-t)^\alpha t^{\alpha-2} + 2(1-t)^\alpha t^{\alpha-3}\right].$$

Setting $t = \frac{1}{2}$, from the definition of N, then the last inequality implies that

$$Au\left(\frac{1}{2}\right) \ge \frac{Nr_1}{2\alpha\Gamma(\alpha)}\left[\frac{1}{2^{\alpha+2}} + \frac{(\alpha-2)(\alpha-3)}{2^{2\alpha-1}} + \frac{\alpha-3}{2^{2\alpha-3}} + \frac{1}{2^{2\alpha-4}}\right] = r_1 = \|u\|.$$

So, $\|Au\| \ge \|u\|$ for $u \in P \cap \partial\Omega_1$. Therefore, we complete the proof by (ii) of Lemma 2.7. □

Example 3.1. Consider the boundary value problem

$$D^{4.5}u(t) + u^2 + \frac{\sin t}{20} + 8 = 0, 0 < t \le 1, \tag{3.1}$$

$$u(0) = u(1) = u'(0) = u'(1) = u''(1) = 0 \tag{3.2}$$

that is $f(t,u) = u^2 + \frac{\sin t}{20} + 8$ and $\alpha = 4.5$.

A simple computation shows that $M \approx 9.7357$ and $N \approx 1302.2954$. Choosing $r_2 = 1$ and $r_1 = 0.006$, we deduce that

$$f(t,u) = u^2 + \frac{\sin t}{20} + 8 \ge 8 \ge 7.8137724 = Nr_1, (t,u) \in [0,1] \times [0, 0.006],$$

$$f(t,u) = u^2 + \frac{\sin t}{20} + 8 \le 1 + \frac{1}{20} + 8 \le 9.7357 = Mr_1, (t,u) \in [0,1] \times [0,1].$$

Thus, by Theorem 3.1 we know that the boundary value problem (3.1)-(3.2) has at least a positive solution $u(t)$ such that $0.006 \le \|u\| \le 1$.

In the next result, we show the existence of at least three positive solutions of (1.1)-(1.2).

We define the nonnegative continuous concave functional θ on the cone P by

$$\theta(u) = \min_{\frac{1}{4} \le t \le \frac{3}{4}} |u(t)|.$$

Theorem 2. Let $f(t,u)$ is continuous on $[0,1] \times [0, +\infty)$. Assume that there exist three positive constants $0 < a < b < c$, such that the following assumptions hold

(A1) $f(t,u) < Ma, (t,u) \in [0,1] \times [0,a]$;

(A2) $f(t,u) \ge Nb, (t,u) \in \left[\frac{1}{2}, \frac{3}{4}\right] \times [\mathrm{b}, \mathrm{c}]$;

(A3) $f(t,u) \le Mc, (t,u) \in [0,1] \times [0, \mathrm{c}]$;

Where

$$M=\frac{2\Gamma(\alpha+1)}{\alpha^2\text{-}3\alpha+4},$$

$$\overline{N}=\Gamma(\alpha+1)\left[\frac{27}{4^{\alpha+3}}+\frac{(\alpha-2)(\alpha-3)3^{\alpha-1}}{2\times4^{2\alpha-1}}+\frac{(\alpha-3)3^{\alpha-3}}{4^{2\alpha-2}}+\frac{3^{\alpha-3}}{4^{2\alpha-3}}\right]^{-1}.$$

Then the problem (1.1)-(1.2) has at least three positive solutions u_1,u_2,u_3 such that

$$\max_{0\le t\le1}|u_1(t)|<a,\ b<\min_{\frac14\le t\le\frac34}|u_2(t)|<\max_{0\le t\le1}|u_2(t)|\le c,$$

$$\text{a}<\max_{0\le t\le1}|u_3(t)|\le c,\ \min_{\frac14\le t\le\frac34}|u_3(t)|<b.$$

Proof. We show all the conditions of Lemma 2.8 are satisfied.

Let $u\in\overline{P}_c$, that is $\|u\|\le c.$ By (A3) and (4) of Lemma 2.5, we have

$$\|Au\|=\max_{0\le t\le1}|\int_0^1G(1,s)f(s,u(s))ds|\le\frac{2\Gamma(\alpha+1)}{\alpha^2\text{-}3\alpha+4}Mc=c.$$

Thus, $A:\overline{P}_c\to\overline{P}_c$, by Lemma 3.1 A is completely continuous. Using an analogous argument, it follow from (A1) that if $u\in\overline{P}_a$, then $\|Tu\|\le\|u\|$. Condition (C2) of Lemma 2.8 is satisfied.

To check condition (C1) of Lemma 2.8 holds, we choose $u(t)=\frac{(b+c)}{2},0\le t\le1.$ It is easy to check that

$$u(t)=\frac{(b+c)}{2}\in P(\theta,\text{b},\text{d}),\theta(u)=\theta\left(\frac{(b+c)}{2}\right)>b,$$

which implies that $\{u\in P(\theta,\text{b},\text{d})|\theta(u)>b\}$ is nonempty. Hence,if $u\in P(\theta,b,c)$, then

$b\le u(t)\le c$, for $\frac12\le t\le\frac34$, By assumption (A2), we deduce

$$\theta(Au)=\min_{\frac14\le t\le\frac34}|(Au)(t)|=\min_{\frac14\le t\le\frac34}\int_0^1G(t,s)f(s,\text{u}(s))ds\ge\min_{\frac14\le t\le\frac34}\int_0^1G(t,s)dsN\overline{b}.$$

By Lemma 2.5 and Lemma 2.6, we obtain

$$\theta(Au)\ge\frac{\overline{N}b}{\Gamma(\alpha)}\min_{\frac14\le t\le\frac34}[\frac{1}{\alpha}(1-t)^3t^\alpha+\frac{1}{2\alpha}(\alpha-2)(\alpha-3)t^{\alpha-1}(1-t)^\alpha$$

$$+\frac{\alpha-3}{\alpha}t^{\alpha-2}(1-t)^\alpha+\frac{1}{\alpha}t^{\alpha-3}(1-t)^\alpha]$$

$$\ge\frac{\overline{N}b}{\Gamma(\alpha)}[\min_{\frac14\le t\le\frac34}\frac{1}{\alpha}(1-t)^3t^\alpha+\min_{\frac14\le t\le\frac34}\frac{1}{2\alpha}(\alpha-2)(\alpha-3)t^{\alpha-1}(1-t)^\alpha$$

$$+\min_{\frac14\le t\le\frac34}\frac{\alpha-3}{\alpha}t^{\alpha-2}(1-t)^\alpha+\min_{\frac14\le t\le\frac34}\frac{1}{\alpha}t^{\alpha-3}(1-t)^\alpha]$$

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$$\geq \frac{\overline{N}b}{\Gamma(\alpha)}[\frac{27}{4^{\alpha+3}}+\frac{(\alpha-2)(\alpha-3)3^{\alpha-1}}{2\times 4^{2\alpha-1}}+\frac{(\alpha-3)3^{\alpha-3}}{4^{2\alpha-2}}+\frac{3^{\alpha-3}}{4^{2\alpha-3}}]$$
$$=b,$$

that is $\theta(Au)>b$ for all $u\in P(\theta,b,c)$. Condition (C1) of Lemma 2.8 holds.

Finally, if $u\in P(\theta,b,c)$, with $\|Au\|\geq d$, then $\|u\|\leq c$ and $\min_{1/4\leq t\leq 3/4} u(t)\geq b$. From assumption (A2) and Remark 2.1. we can also get $\|Au\|\geq b$. Condition (C3) of Lemma 2.8 holds.

As a consequence of Lemma 2.8, A has at least three fixed points u_1,u_2,u_3 such that $\|u_1\|<a, b<\theta(u_2), a<\|u_3\|$, and $\theta(u_2)<b$. These fixed points are solutions of (1.1) -(1.2). The proof is complete. □

Example 3.2. Consider the boundary value problem

$$D_{0+}^{4.5}u(t)+f(t,u)=0, 0<t\leq 1, \quad (3.3)$$

$$u(0)=u(1)=u'(0)=u'(1)=u''(1)=0 \quad (3.4)$$

that is $\alpha=4.5$, where

$$f(t,u)=\begin{cases}\frac{3t}{2}+10076u^7, u\leq 1\\ \frac{3t}{2}+u+10075, u>1.\end{cases}$$

Then,we have that $M\approx 9.7357$ and $\overline{N}\approx 10775.2818$. Choosing $a=0.25, b=1, c=1234$,

We obtain

$$f(t,u)=\frac{3t}{2}+10076u^7\leq 2.1578\leq Ma\approx 2.4339, (t,u)\in[0,1]\times[0,0.25],$$

$$f(t,u)=\frac{3t}{2}+u+10075\geq 10776.625\geq \overline{N}b\approx 10775.2128, (t,u)\in[\frac{1}{4},\frac{3}{4}]\times[1,1234],$$

and

$$f(t,u)=\frac{3t}{2}+10076u^7\leq 10777.5, (t,u)\in[0,1]\times[0,1],$$

$$f(t,u)=\frac{3t}{2}+u+10075\leq 12010.5, (t,u)\in[0,1]\times[1234],$$

I.e. $f(t,u)\leq Mc\approx 12013.8538$, for $(t,u)\in[0,1]\times[0,1234]$.

By Theorem 3.2, the problem (3.3)-(3.4) has at least three positive solutions u_1,u_2,u_3 such that

$$\max_{0\leq t\leq 1}|u_1(t)|<1, 1<\min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|u_2(t)|<\max_{0\leq t\leq 1}|u_2(t)|\leq 1234,$$

$$0.25<\max_{0\leq t\leq 1}|u_3(t)|\leq 1234, \min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|u_3(t)|<1.$$

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