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Some Properties of a Subclass of Univalent Functions

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Abstract: In this paper, we introduce a certain subclass of univalent functions $\tilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. We obtain some results, like, coefficient inequality, distortion theorem, extreme points, radii of close to convex and convexity for this class and convolution operator, integral representation, inclusive properties and weighted mean.

Keywords: Univalent function, Distortion theorem, Radius of convexity, Extreme points, Convolution operator, Integral representation, Weighted mean.

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1. INTRODUCTION:

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \quad n \in \mathbb{N}, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Consider a subclass $\tilde{I}$ of the class $\mathcal{A}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \quad (a_n \geq 0; n \in \mathbb{N}). \tag{1.2}$$

The Hadamard product of two functions, f is given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad , \quad n \in \mathbb{N}, \tag{1.3}$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Definition (1.1): Let $f \in \mathcal{A}$ given by (1.1). Then f be in the class $\mathcal{H}(f, g, V, M, \alpha, \beta, \gamma)$ if it satisfies the following condition:

$$\left| \frac{\frac{z(f*g)''(z)}{(f*g)'(z)} - \alpha\gamma \left|\frac{z(f*g)''(z)}{(f*g)'(z)}\right|}{M\left[\frac{z(f*g)''(z)}{(f*g)'(z)} - \alpha\gamma \left|\frac{z(f*g)''(z)}{(f*g)'(z)}\right|\right] - (V - M)} \right| < \beta, \tag{1.4}$$

where $\alpha \geq 0, 0 < \beta \leq 1, -1 \leq M < V \leq 1, -1 \leq M < 0, \gamma \geq 0$.

We define the subclass $\widetilde{\mathcal{H}}(f, g, V, M, \alpha, \beta, \gamma) = \tilde{I} \cap \mathcal{H}(f, g, V, M, \alpha, \beta, \gamma)$.

Such type of study was carried out by several different authors for another class, like, Schild and Silverman [8], Gupta and Jain [6] and Goodman [5].

2.Coefficient inequality

The first theorem gives a necessary and sufficient condition for a function f to be in the class $\mathcal{H}(f, g, V, M, \alpha, \beta, \gamma)$.

Theorem(2.1): Let the function $f(z)$ defined (1.1). If

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma) + \beta(V-M)]|a_n||b_n| \leq \beta(V-M), \tag{2.1}$$

where $(\alpha \geq 0, 0 < \beta \leq 1, -1 \leq M < V \leq 1, -1 \leq M < 0, \gamma \geq 0)$,then $f \in \mathcal{H}(f, g, V, M, \alpha, \beta, \gamma)$.

Proof: Let the condition (2.1) holds true and let $|z| = 1$.Then we have

$$\left|z(f*g)''(z) - \alpha\gamma e^{i\theta}\left|z(f*g)''(z)\right|\right| - \beta\left|(V-M)(f*g)'(z) - M\left[z(f*g)''(z) - \alpha\gamma e^{i\theta}\left|z(f*g)''(z)\right|\right]\right|$$

$$= \left|\sum_{n=2}^{\infty} n(n-1)a_n b_n z^{n-1} - \alpha\gamma e^{i\theta}\left|\sum_{n=2}^{\infty} n(n-1)a_n b_n z^{n-1}\right|\right| -$$

$$\beta\left|(V-M) + (V-M)\sum_{n=2}^{\infty} n a_n b_n z^{n-1} - M\left[\sum_{n=2}^{\infty} n(n-1)a_n b_n z^{n-1} - \alpha\gamma e^{i\theta}\left|\sum_{n=2}^{\infty} n(n-1)a_n b_n z^{n-1}\right|\right]\right|$$

$$\leq (1+\alpha\gamma)\sum_{n=2}^{\infty} n(n-1)|a_n||b_n||z|^{n-1} - \beta(V-M) + \beta(V-M)\sum_{n=2}^{\infty} n|a_n||b_n||z|^{n-1} + \beta|M|(1+\alpha\gamma)\sum_{n=2}^{\infty} n(n-1)|a_n||b_n||z|^{n-1}$$

$$\leq \sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma) + \beta(V-M)]|a_n||b_n| - \beta(V-M) \leq 0, \text{by hypothesis.}$$

Hence, by the principle of maximum modulus, $f \in \mathcal{H}(f, g, V, M, \alpha, \beta, \gamma)$.

Theorem(2.2): Let the function $f(z)$ defined by (1.2) be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]a_n b_n \leq \beta(V-M). \qquad (2.2)$$

$$(\alpha \geq 0, 0<\beta \leq 1, -1 \leq M < V \leq 1, -1 \leq M < 0, \gamma \geq 0).$$

Proof: we only need to prove the "only if " part of Theorem (2.1). For functions $f(z) \in \tilde{I}$, we can write

$$\left|\frac{\frac{z(f*g)''(z)}{(f*g)'(z)}-\alpha\gamma\left|\frac{z(f*g)''(z)}{(f*g)'(z)}\right|}{M\left[\frac{z(f*g)''(z)}{(f*g)'(z)}-\alpha\gamma\left|\frac{z(f*g)''(z)}{(f*g)'(z)}\right|\right]-(V-M)}\right|$$

$$=\left|\frac{z(f*g)''(z)-\alpha\gamma e^{i\theta}\left|z(f*g)''(z)\right|}{M[z(f*g)''(z)-\alpha\gamma e^{i\theta}|z(f*g)''(z)|-(V-M)(f*g)'(z)]}\right|$$

$$=\left|\frac{\left(1+\alpha\gamma e^{i\theta}\right)\sum_{n=2}^{\infty} n(n-1)a_n b_n|z|^{n-1}}{(V-M)-(V-M)\sum_{n=2}^{\infty} n a_n b_n z^{n-1}+M(1+\alpha\gamma e^{i\theta})\sum_{n=2}^{\infty} n(n-1)a_n b_n|z|^{n-1}}\right|<\beta.$$

Since $Re(z) \leq |z|, (z \in U)$, we thus find that

$$Re\left(\frac{\left(1+\alpha\gamma e^{i\theta}\right)\sum_{n=2}^{\infty} n(n-1)a_n b_n|z|^{n-1}}{(V-M)-(V-M)\sum_{n=2}^{\infty} n a_n b_n z^{n-1}+M(1+\alpha\gamma e^{i\theta})\sum_{n=2}^{\infty} n(n-1)a_n b_n|z|^{n-1}}\right)<\beta.$$

If we now choose$z$ to be real and let $z \to 1^{-}$, we get

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]a_n b_n \leq \beta(V-M),$$

which is equivalent to (2.2) .■

Corollary (2.1): Let the function $f(z)$ defined by (2.1) be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then

$$a_n \leq \frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}.$$

The result is sharp for the function

$$f(z)=z-\frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}z^n. \qquad (2.3)$$

3.Distortion and Growth Theorem

Next, we obtain the distortion and growth theorems for a function f to be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Theorem (3.1): Let the function $f(z)$ defined by (1.2) be in the class$\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.Then for $z \in U$, we have

$$|z|-\frac{\beta(V-M)}{2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z|^2 \leq |f(z)| \leq |z|+\frac{\beta(V-M)}{2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z|^2, |z|<1. \qquad (3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z-\frac{\beta(V-M)}{2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}z^2. \qquad (3.2)$$

Proof: It is easy to see from Theorem (2.2) that

$$2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]a_n b_n \leq \beta(V-M).$$

Then

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta(V-M)}{2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}. \qquad (3.3)$$

Making use of (3.3), we have

$$|f(z)| \geq |z|-|z|^2\sum_{n=2}^{\infty} a_n$$

$$|f(z)| \geq |z|-\frac{\beta(V-M)}{2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z|^2,$$

and

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n$$

$$|f(z)| \leq |z| + \frac{\beta(V-M)}{2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z|^2. \blacksquare$$

Theorem (2.3): Let the function $f(z)$ defined by (1.2) be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then for $z \in U$, we have

$$1 - \frac{\beta(V-M)}{[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z| \leq |f'(z)| \leq 1 + \frac{\beta(V-M)}{[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z|, |z| < 1, \quad (3.4)$$

with equality for

$$f(z) = z - \frac{\beta(V-M)}{2[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}z^2.$$

Proof: From (3.3) and Theorem (2.2) that

$$\sum_{n=2}^{\infty} na_n \leq \frac{\beta(V-M)}{[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}.$$

Consequently, we have

$$|f'(z)| \geq 1 - |z| \sum_{n=2}^{\infty} na_n$$

$$\geq 1 - \frac{\beta(V-M)}{[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z|,$$

and

$$|f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} na_n$$

$$\leq 1 + \frac{\beta(V-M)}{[(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_2}|z|. \blacksquare$$

4. Closure Theorems

We will consider the functions $f_j(z)$defined, for $j = 1,2,3,\dots,l$ by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \geq 0). \qquad (4.1)$$

In the following, we prove closure theorem.

Theorem (4.1): Let the functions $f_j(z)$ defined by (4.1) be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{l} c_j f_j(z) \text{ and } \sum_{j=1}^{l} c_j = 1, c_j \geq 0$$

is in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Proof: By definition of h, we have

$$h(z) = \left[\sum_{j=1}^{l} c_j\right] z - \sum_{n=2}^{\infty}\left[\sum_{j=1}^{l} c_j a_{n,j}\right] z^n,$$

further. Since f_j are in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$, for every $j = 1,2,3,\dots,l$, we get

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]a_{n,j} \leq \beta(V-M),$$

for every $j = 1,2,3,\dots,l$. Hence, we can see that

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]\left[\sum_{j=1}^{l} c_j\, a_{n,j}\right]$$
$$=\sum_{j=1}^{l} c_j\left[\sum_{n=2}^{\infty} n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]a_{n,j}\right]$$
$$\leq \sum_{j=1}^{l} c_j\ \beta(V-M)=\beta(V-M),$$

which implies that $h(z) \in \widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.■

Theorem (4.2): Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n} z^n.$$

Then $f(z)$ is in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} d_n\, f_n(z), \tag{4.2}$$

where $d_n \geq 0$ and $\sum_{n=1}^{\infty} d_n = 1$.

Proof: Assume that

$$f(z) = \sum_{n=1}^{\infty} d_n\, f_n(z) = z - \frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n} d_n z^n.$$

Then it follows that

$$\sum_{n=1}^{\infty} \frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)} \frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n} d_n$$
$$=\sum_{n=2}^{\infty} d_n = 1 - d_1 \leq 1,$$

which implies that the function $f(z) \in \widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Conversely, assume that the function $f(z)$ defined by (1.2) be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then

$$a_n \leq \frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}.$$

Setting

$$d_n = \frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)} a_n,$$

where $d_1 = 1 - \sum_{n=2}^{\infty} d_n$, we can see that the function $f(z)$can be expressed in the form (4.2).■

Corollary (4.1): The extreme points of the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$ are the functions $f_1(z) = z$ and

$$f_n(z) = z - \frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n} z^n.$$

5. Radii of Close-to-convex and Convexity

Next, we discuss the radii of close-to-convexity and convexity.

Theorem (5.1): Let the function $f(z)$ defined by (1.2) be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then $f(z)$ is close-to-convex of order $(0 \leq \rho < 1)$ in $|z| < r_1$, where

$$r_1 = inf_{n\geq 2}\left\{\frac{(1-\rho)[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)}\right\}^{\frac{1}{n-1}}. \tag{5.1}$$

The result is sharp, the external function given by (2.3).

Proof: We must show that

$$\left|f(z)' - 1\right| \le 1-\rho \text{ for } |z| \le r_1, \tag{5.2}$$

where r_1 is given by (5.1). Indeed we find from (1.2) that

$$\left|f(z)' - 1\right| \le \sum_{n=2}^{\infty} n\, a_n |z|^{n-1}.$$

Thus

$$\left|f(z)' - 1\right| \le 1-\rho \quad \text{if}$$

$$\sum_{n=2}^{\infty} \frac{n}{(1-\rho)} a_n |z|^{n-1} \le 1, \tag{5.3}$$

but by using Theorem (2.2), (5.3) will be true if

$$\frac{n}{(1-\rho)}|z|^{n-1} \le \frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)},$$

then

$$|z| \le \left\{\frac{(1-\rho)[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)}\right\}^{\frac{1}{n-1}}. \tag{5.4}$$

The result follows easily from (5.4).■

Theorem (5.2): Let the function $f(z)$ defined by (1.2)be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then $f(z)$ is convex of order $\rho\ (0 \le \rho < 1)$ in $|z| < r_2$,

where

$$r_2 = inf_{n\ge 2}\left\{\frac{(1-\rho)[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(n-\rho)(V-M)}\right\}^{\frac{1}{n-1}}. \tag{5.5}$$

The result is sharp with the external function given by (2.3).

Proof: We must show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\rho, \qquad \text{for } |z| \le r_2,$$

where r_2 is given by (5.5). Indeed we find from (1.2) that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\rho,$$

if

$$\sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right)a_n|z|^{n-1} \le 1. \tag{5.6}$$

But by using Theorem (2.2), (5.6) will be true if

$$\left(\frac{n(n-\rho)}{1-\rho}\right)|z|^{n-1} \le \frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)},$$

then

$$|z| \le \left\{\frac{(1-\rho)[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(n-\rho)(V-M)}\right\}^{\frac{1}{n-1}}. \tag{5.7}$$

The result follows easily from (5.7).■

6. Convolution Operator

Definition (6.1)[7]: The Gaussian hypergeometric function denoted by

$$ {}_2F_1\ (a,b;c;z)=\sum_{\mathrm{n=o}}^{\infty}\frac{(a)_{\mathrm{n}}(b)_{\mathrm{n}}}{(c)_{\mathrm{n}}\mathrm{n!}}z^{\mathrm{n}},\quad |z|<1, $$

where $c>b>0, c>a+b$ and

$$ (x)_n\begin{cases} x(x+1)(x+2)\dots(x+n-1), & \text{for } n=1,2,3,\dots \\ 1 & n=0 \end{cases} $$

Definition (6.2)[7]: For every $f\in\mathcal{A}$,the convolution operator is defined by

$$ \mathcal{W}_{a,b,c}(f)(z)=\ {}_2F_1\ (a,b;c;z)*f(z)=z-\sum_{\mathrm{n=2}}^{\infty}\frac{(a)_{\mathrm{n}}(b)_{\mathrm{n}}}{(c)_{\mathrm{n}}\mathrm{n!}}z^{\mathrm{n}}, $$

where ${}_2F_1\ (a,b,c;z)$is Gaussian hypergeometric function (see [1] and [7]) introduced in Definition (6.1).

Theorem (6.1): Let f is given by (1.2) be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then the convolution operator $\mathcal{W}_{a,b,c}(f)(z)$ is in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\theta,\gamma)$ for $|z|\le r(\beta,\theta)$, where

$$ r(\beta,\theta)=inf_n\left\{\frac{\theta[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]}{\beta[(n-1)(1-\theta M)(1+\alpha\gamma)+\theta(V-M)]\frac{(a)_{\mathrm{n}}(b)_{\mathrm{n}}}{(c)_{\mathrm{n}}\mathrm{n!}}}\right\}^{\frac{1}{n-1}}. $$

The result is sharp for the function

$$ f_n(z)=z-\frac{\beta(V-M)}{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}z^n, n=2,3,\dots\quad . $$

Proof: Since $f\in\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$, we have

$$ \sum_{n=2}^{\infty}\frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)}a_n\le 1. $$

It is sufficient to show that

$$ \sum_{n=2}^{\infty}\frac{n[(n-1)(1-\theta M)(1+\alpha\gamma)+\theta(V-M)]b_n\frac{(a)_{\mathrm{n}}(b)_{\mathrm{n}}}{(c)_{\mathrm{n}}\mathrm{n!}}}{\theta(V-M)}a_n|z|^{n-1}\le 1. \qquad (6.1) $$

Note that (6.1) is satisfied if

$$ \frac{n[(n-1)(1-\theta M)(1+\alpha\gamma)+\theta(V-M)]b_n\frac{(a)_{\mathrm{n}}(b)_{\mathrm{n}}}{(c)_{\mathrm{n}}\mathrm{n!}}}{\theta(V-M)}a_n|z|^{n-1}\le\frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)}a_n, $$

solving for $|z|$, we get the result.

7. Integral Representation

In the following theorem, we obtain integral representation for the function $f\in\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Theorem(7.1): Let f and $g\in\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then

$$ (f*g)(z)=\int_0^z e^{\int_0^z\frac{(M-V)\beta\phi(t)}{t[(1-\alpha\gamma)(1-\beta M\phi(t))]}dt}dt. $$

Proof: By putting $\mathrm{N}(z)=\frac{z(f*g)''(z)}{(f*g)'(z)}$ in (1.4), we have

$$ \left|\frac{\mathrm{N}(z)-\alpha\gamma|\mathrm{N}(z)|}{M[\mathrm{N}(z)-\alpha\gamma|\mathrm{N}(z)|]-(V-M)}\right|<\beta, $$

or equivalently

$$\frac{\mathrm{N}(z)-\alpha\gamma\,\mathrm{N}(z)}{M[\mathrm{N}(z)-\alpha\gamma\,\mathrm{N}(z)]-(V-M)}=\beta\phi(z),$$

where $|\phi(z)|<1$, $z\in U$.

So

$$\frac{(f*g)''(z)}{(f*g)'(z)}=\frac{(M-V)\beta\phi(z)}{z[(1-\alpha\gamma)(1-\beta M\phi(z))]},$$

after integration,we obtain

$$\log\left((f*g)'(z)\right)=\int_0^z\frac{(M-V)\beta\phi(t)}{t[(1-\alpha\gamma)(1-\beta M\phi(t))]}dt.$$

Thus

$$(f*g)'(z)=e^{\int_0^z\frac{(M-V)\beta\phi(t)}{t[(1-\alpha\gamma)(1-\beta M\phi(t))]}dt}.$$

After integration, we have

$$(f*g)(z)=\int_0^z e^{\int_0^z\frac{(M-V)\beta\phi(t)}{t[(1-\alpha\gamma)(1-\beta M\phi(t))]}dt}\,dt$$

and this gives the result.

8. Inclusive Properties

Now, we obtain the inclusive properties of the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Theorem(8.1): Let $\alpha\geq 0{,}0<\beta\leq 1,-1\leq M<V\leq 1,-1\leq M<0,\gamma\geq 0$.

Then $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)\subset\widetilde{\mathcal{H}}(f,g,V,0,\alpha,\tau,\gamma)$, where

$$\tau\leq\frac{(n-1)(1+\alpha\gamma)(V-M)\beta}{(n-1)(1-\beta M)(1+\alpha\gamma)+(V-M)(1-V)\beta}.$$

Proof: Let the function f given by (1.2) belongs to the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Then in view of theorem (2.2), we have

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)}a_n\leq 1. \tag{8.1}$$

We want to find the value τ such that

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\alpha\gamma)+V\tau]b_n}{V\tau}a_n\leq 1. \tag{8.2}$$

The inequality (8.1) would obviously imply (8.2) if

$$\frac{n[(n-1)(1+\alpha\gamma)+V\tau]b_n}{V\tau}\leq\frac{n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]b_n}{\beta(V-M)}.$$

Rewriting the inequality, we have

$$\tau\leq\frac{(n-1)(1+\alpha\gamma)(V-M)\beta}{(n-1)(1-\beta M)(1+\alpha\gamma)+(V-M)(1-V)\beta}.$$

This completes the proof.

9. Weighted Mean

Definition(9.1): Let $f*g$ and $h*k$ be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then, the weighted mean E_q of $f*g$ and $h*k$ given by

$$E_q(z)=\frac{1}{2}[(1-q)(f*g)(z)+(1+q)(h*k)(z)],0<q<1.$$

Theorem(9.1): Let $f*g$ and $h*k$ be in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$. Then, the weighted mean of $f*g$ and $h*k$ is also in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

Proof: By Definition (9.1), we have

$$E_q(z)=\frac{1}{2}[(1-q)(f*g)(z)+(1+q)(h*k)(z)]$$

$$=\frac{1}{2}\left[(1-q)\left(z-\sum_{n=2}^{\infty}a_nb_nz^n\right)+(1+q)\left(z-\sum_{n=2}^{\infty}c_nd_nz^n\right)\right]$$

$$=z-\sum_{n=2}^{\infty}\frac{1}{2}\big((1-q)a_nb_n+(1+q)c_nd_n\big)z^n.$$

Since $f*g$ and $h*k$ are in the class $\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$ so by Theorem (2.2), we get

$$\sum_{n=2}^{\infty}n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]a_nb_n\leq\beta(V-M)$$

and

$$\sum_{n=2}^{\infty}n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]c_nd_n\leq\beta(V-M).$$

Hence

$$\sum_{n=2}^{\infty}n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]\left(\frac{1}{2}(1-q)a_nb_n+\frac{1}{2}(1+q)c_nd_n\right)$$

$$=\frac{1}{2}(1-q)\sum_{n=2}^{\infty}n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]a_nb_n$$

$$+\frac{1}{2}(1+q)\sum_{n=2}^{\infty}n[(n-1)(1-\beta M)(1+\alpha\gamma)+\beta(V-M)]c_nd_n$$

$$\leq\frac{1}{2}(1-q)\beta(V-M)+\frac{1}{2}(1+q)\beta(V-M)=\beta(V-M).$$

This shows $E_q\in\widetilde{\mathcal{H}}(f,g,V,M,\alpha,\beta,\gamma)$.

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