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THE MOMENT PROBLEM IN HYPERCOMPLEX SYSTEMS

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Abstract.

This paper is devoted to give the necessary and sufficient conditions guarantees that the product of two generalized moment functions defined in a hypercomplex system $L_1(Q,m)$ is also generalized moment function in $L_1(Q,m)$. Also, we prove that a bounded continuous function ϕ in a hypercomplex system $L_1(Q,m)$ is conditionally positive definite if and only if ϕ is generalized moment functions defined in $L_1(Q,m)$. Moreover, we will give the integral representations of a generalized moment functions defined in $L_1(Q,m)$.

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1 INTRODUCTION

Harmonic analysis in hypercomplex system dates back to J. Delsarte's and B. M. Levitan's work during the 1930s and 1940s, but the substantial development had to wait till the 1990s when Berezansky and Kondratiev [4] put hypercomplex system in the right setting for harmonic analysis. A central idea in harmonic analysis in various settings has been the existence of a product, usually called convolution, for functions and measures. In some cases, an investigation begins with a convolution algebra of measures as the primitive object upon which to build a theory; this is the case of the analysis of the objects called generalized hypergroups"hypercomplex system" which are the generalizations of the convolution algebra of Borel measures on a group. A hypercomplex system with a locally compact basis Q (see [2, 3, 12]) is a set $L_1(Q,m)$ with generalized convolution, which can be defined in terms of a structure measure $c(A,B,r), \quad A,B \subset Q, r \in Q$. One important reason that explain why the harmonic analysts did not attracted to study Fourier algebra over hypercomplex system is that, the product of two continuous positive definite functions in a hypercomplex system is not necessarily positive definite in general. Consider the space $L_1(Q,m) = L_1$ of functions on Q integrable with respect to the multiplicative measure m i.e., a regular Borel measure m positive on open sets such that

$$\int c(A,B,r)dm(r) = m(A)m(B) \quad (A,B \in B_0(Q))$$

It is some-times convenient to consider, together with the measure $c(A,B,r)$, its extension to the sets form $Q \times Q$. For this purpose, we fix r and, for any $(A,B \in (Q))$, put $m_r(A \times B) = c(A,B,r)$. The space $L_1(Q,m)$ with the convolution

$$(f * g)(r) = \int\int f(p)g(q)dm_r(p,q) \tag{1.1}$$

is called a hypercomplex system with basis Q. The rule played by the generalized translation R_p acting upon functions of a point $q \in Q$ and satisfying

$$(R_p \chi(.,\lambda))(q) = \chi(p,\lambda)\chi(q,\lambda) \tag{1.2}$$

A continuous bounded function $\phi(r) \quad (r \in Q)$ is called positive definite if

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda}_j (R_{r_i} * \phi)(r_i) \geq 0 \tag{1.3}$$

for all $r_1, r_2, ..., r_n \in Q, \lambda_1, \lambda_2, ..., \lambda_n \in C$ and $n \in N$. As pointed in [1], every continuous positive definite function $\phi \in P(Q)$ admits a unique representation in the form of an integral

$$\phi(r) = \int_{X_h} \chi(r)d\mu(\chi) \quad (r \in Q) \tag{4}$$

where μ is a nonnegative finite regular measure on the space of continuous bounded characters X_h.

The main task in our previous works [7, 8] was to give "the necessary and sufficient conditions guarantees that the product of two positive definite functions defined on hypergroup H is also positive definite on H ". In this paper, we will prove that the product of two generalized moment functions defined in a hypercomplex system $L_1(Q,m)$ is also generalized moment function in $L_1(Q,m)$. Also, we prove that a bounded continuous function ϕ in a hypercomplex system $L_1(Q,m)$ is conditionally positive definite if and only if ϕ is generalized moment functions defined in $L_1(Q,m)$. Moreover, we will give the integral representations of a generalized moment functions defined in $L_1(Q,m)$.

2 Generalized moment functions

The classical moment problem stated by Stieljes in the following form: For any sequence of real numbers $s_0, s_1, \ldots$. Find necessary and sufficient conditions for the existence of a measure μ on $[0,\infty[$ such that

$$s_n = \int_0^\infty x^n \quad d\mu(x)$$

holds for $n = 0,1,\ldots$. The F-moment problem:

$$s_n = \int_F x^n \quad d\mu(x)$$

carries the name of Stieltjes (for $F = R_+$), Hamburger (for $F = R$), Hausdorff (for $F = [0,1]$) and Toeplitz (for $F = T$) [5, 6, 11]. A hypercomplex system $L_1(Q,m)$ is commutative if its structure measure is commutative. An essentially bounded measurable complex-valued function $\chi(r)(r \in Q)$ not identically equal to zero on a set of positive measure is called a character of a hypercomplex system if

$$\int c(A,B,r)\chi(r)dx = \chi(A)\chi(B)$$

holds for any $A,B \in B_0(Q)$. For any nonnegative integer n the complex valued continuous function ϕ in the hypercomplex system $L_1(Q,m)$ is called a generalized moment function of order n, if there exist complex valued continuous functions $\phi_k : Q \to C$ for $k = 0,1,\ldots,n$ such that $\phi_0 \neq 0$, $\phi_n = \phi$ and

$$\phi_k(x*y) = \sum_{i=0}^{k} \binom{k}{i} \phi_i(x)\phi_{k-i}(y) \tag{2.1}$$

holds for $k = 0,1,\ldots,n$ and for all $x, y \in Q$.

Let $L_1(Q,m)$ be a commutative hypercomplex system, $n1$ an integer and $(\phi_k)_{k=0}^{n}$ a sequence of generalized moment functions with $\phi_1 \neq 0$. Then the functions $\phi_0,\phi_1,\ldots,\phi_n$ are linearly independent. In particular, non of them identically equal zero.

Let $n = 1$ and suppose that $\phi_1 = \lambda\phi_0$ with some non zero complex λ. Then by (2.1), it follows

$$\phi_0(x)\phi_0(y) = \phi_0(x*y) = \frac{1}{\lambda}\phi_1(x*y) =$$

$$\frac{1}{\lambda}\phi_1(x)\phi_0(y) + \frac{1}{\lambda}\phi_0(x)\phi_1(y) = 2\phi_0(x)\phi_0(y)$$

a contradiction.

Now, let $n2$ be any integer and suppose that we have proved our statement for all integers not greater than n. For $n+1$ we suppose the contrary that is, there are complex numbers $c_i(i = 0,1,\ldots,n)$ such that

$$\phi_{n+1}(x*y) = \sum_{i=0}^{n} c_i\phi_i(x*y) \tag{2.2}$$

holds for each $x, y \in Q$. By (2.1) we have

$$\sum_{j=0}^{n+1} \binom{n+1}{j} \phi_j(x)\phi_{n+1-j}(y) = \sum_{i=0}^{n}\sum_{j=0}^{i} c_i \binom{i}{j} \phi_i(x)\phi_{i-j}(y) \tag{2.3}$$

for each $x, y \in Q$. Using (2.1),(2.2) and reordering the sum on the right hand side after simplification we get

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$$\sum_{j=0}^{n}[\binom{n+1}{j}\phi_{n+1-j}(y)+c_j\phi_0(y)-\sum_{i=j}^{n}c_i\binom{i}{j}\phi_{i-j}(y)]\phi_j(x)=0 \tag{2.4}$$

By our assumption, the coefficient of ϕ_n must be zero for each $y\in Q$, so

$$(n+1)\phi_1(y)=0$$

which contradict our assumption.

Suppose $H=L_1(Q,m)$ be a commutative hypercomplex system, $n1$ an integer and $(\phi_k)_{k=0}^{n}$, $(\psi_k)_{k=0}^{n}$ are two sequences of generalized moment functions, with $\phi_1\neq 0,\psi_1\neq 0$, orthogonal to each other in the sense that $\phi_i(x).\psi_j(y)=0$ for all $i,j=1,...,n$ and all $x,y\in Q$. The product of this two generalized moment functions is also generalized moment function in H.

We will prove this theorem by using mathematical induction. Let $x,y\in Q$ and for $n=1$ we have:

$$\phi_1(x*y)=\phi_1(x)\phi_0(y)+\phi_0(x)\phi_1(y)$$

$$\psi_1(x*y)=\psi_1(x)\psi_0(y)+\psi_0(x)\psi_1(y)$$

Regarding the orthogonality property, we have

$$\Gamma_1(x*y):=(\phi_1.\psi_1)(x*y)=\phi_1(x)\psi_1(x)\psi_0(y)\phi_0(y)+\phi_0(x)\psi_0(x)\psi_1(y)\phi_1(y)$$

So

$$\Gamma_1(x*y)=\Gamma_1(x)\Gamma_0(y)+\Gamma_0(x)\Gamma_1(y) \tag{2.5}$$

where

$$\Gamma_i(x):=(\phi_i.\psi_i)(x) \quad for \quad all \quad x\in Q, i=0,...,n$$

Suppose the relation is true for $n=m-1$, i.e,

$$\Gamma_{m-1}(x*y)=\sum_{i=0}^{m-1}\binom{m-1}{i}\Gamma_i(x)\Gamma_{m-i-1}(y) \tag{2.6}$$

Since

$$\Gamma_m(x*y)=(\phi_m.\psi_m)(x*y)=(\phi_m)(x*y).(\psi_m)(x*y)$$

expanding the right hand side we get

$$\Gamma_m(x*y)=\sum_{i=0}^{m}\binom{m}{i}\phi_i(x)\phi_{m-i}(y).\sum_{j=0}^{m}\binom{m}{i}\phi_j(x)\phi_{m-j}(y)$$

$$=[\phi_m(x)\phi_0(y)+\phi_0(x)\phi_m(y)+\sum_{i=0}^{m-1}\binom{m-1}{i}\phi_i(x)\phi_{m-i-1}(y)]$$

$$.[\psi_m(x)\psi_0(y)+\psi_0(x)\psi_m(y)+\sum_{j=0}^{m-1}\binom{m-1}{i}\phi_j(x)\phi_{m-j-1}(y)]$$

Therefore, by virtue of equation (2.6) and applying the orthogonality property for the right hand side, the reader can easily obtain

$$\Gamma_m(x*y)=\sum_{i=0}^{m}\binom{m}{i}\phi_i(x)\psi_i(x)\phi_{m-i}(y)\psi_{m-i}(y)$$

hence the require is completely proved.

3 $SU(2)$-hypercomplex system

Let G denote the $SU(2)$-hypercomplex system, i.e, the set of all continuous unitary irreducible representations of the group $G=SU(2)$, the special linear group in two dimensions . The dual $\hat{G}$ consists of equivalence classes of continuous irreducible representations of G. For any two classes U,V of this this type their tensor product can be decomposed into its irreducible components $U_1,U_1,...,U_n$ with the respective multiplicities $m_1,m_2,...,m_n$. We define convolution on $\hat{G}$ by

$$\delta_U*\delta_V=\sum_{i=1}^{n}\frac{m_i d(U_i)}{d(U)d(V)}\delta_{U_i} \tag{3.1}$$

where $d(U)$ denotes the dimension of U and δ_U is the dirac measure concentrated at U. Then $\hat{G}$ with this convolution and with respect to the discrete topology form a commutative hypercomplex system. A function $E:N\to C$ is called exponential if it satisfies

$$E(m)E(n)=E(m*n)=\sum_{k=|m-n|}^{m+n}\frac{k+1}{(m+1)(n+1)}E(k) \tag{3.2}$$

for all natural numbers $m,n.$

The function $E:N\to C$ is an exponential in the $SU(2)$-hypercomplex system if and only if there exists a complex number λ such that

$$E(n)=\frac{sinh[(n+1)\lambda]}{(n+1)sinh\lambda} \tag{3.3}$$

holds for all natural number λ.

It is easy to prove that any function E of the given form is an exponential on the $SU(2)$-hypercomplex system. Conversely, let $E:N\to C$ be a solution of (3.2) and let

$$f(n)=(n+1)E(n) \tag{3.4}$$

for each $n\in N$. Then we have

$$f(m)f(n)=\sum_{k=|m-n|}^{m+n}f(k)$$

for each $m,n\in N$. With $m=1$ it follows that f satisfies the following second order homogenous linear difference equation

$$f(n+2)-f(1)f(n+1)+f(n)=0 \tag{3.5}$$

for each $n\in N$ with $f(0)=1$.

Suppose that $f(1)=2$. Then from (3.5) we infer that $f(n)=n+1$ and $E=1$ which corresponds to the case $\lambda=1$ in (3.3). Otherwise $f(1)\neq 2$ and let $\lambda\neq 0$ be a complex number with $f(1)=2cosh\lambda$. Then we have that

$$f(n) = \alpha exp(n\lambda) + \beta exp(-n\lambda) \tag{3.6}$$

holds for any $n \in N$ with some complex numbers α, β satisfying $\alpha + \beta = 1$. This implies

$$f(n) = \frac{sinh[(n+1)\lambda]}{sinh\lambda}$$

holds for each $n \in N$. So

$$E(n) = \frac{sinh[(n+1)\lambda]}{(n+1)sinh\lambda}$$

4 Conditionally exponential convex functions

A hypercomplex system $L_1(Q,m)$ is commutative if its structure measure is commutative. An essentially bounded measurable complex-valued function $\chi(r)(r \in Q)$ not identically equal to zero on a set of positive measure is called a character of a hypercomplex system if

$$\int c(A,B,r)\chi(r)dx = \chi(A)\chi(B)$$

holds for any $A, B \in B_0(Q)$. In this section we will give some properties of the class of conditionally exponential convex functions defined in hypercomplex systems. The study of continuous conditionally exponential convex functions leads to the characterisation of convolution semigroups discussed in [9], which represents the essential datum for the potential theory. The function $\psi : L_c^\infty(Q) \to C$ is said to be conditionally exponential convex if for all $n \in N$ and any $y_1, y_2, ..., y_n \in Q^*$ and $c_1, c_2, ..., c_n \in C$ such that $\sum_{k=0}^{n} c_k = 0$ we have:

$$\sum_{i,j=1}^{n} [\psi(y_i) + \overline{\psi(y_j)} - R_{y_j^*}\psi(y_i)]c_i\overline{c_j} \geq 0$$

The following two lemmas are in fact, an adaption of whatever done for semigroups in [1, Berg et al]. We will not repeat the proof, wherever the proof for semigroups can be applied to the hypercomplex systems with necessary modification.

(i) The sum and the point-wise limit of conditionally exponential convex functions in hypercomplex systems are also positive definite.

(ii) Let ϕ be a continuous conditionally exponential convex function on Q and define $L_1(Q,m) \to C$ by $\Phi(s) := \int \phi(s)dm(s)$. Then Φ is conditionally exponential convex in $L_1(Q,m)$.

A bounded measurable function $\phi \in C_c(Q)$ is conditionally exponential convex if and only if there exists a ψ in $L_2(Q,m)$ such that $\phi = \psi \bullet \check{\psi}$, where

$$f \bullet \check{g}(r) = \int_Q f(r * s)\overline{g(s)}dm(s).$$

for all $f, g \in C_c(Q)$

The proof is as in [10].

Let ϕ_1 and ϕ_2 belongs to $C_c(Q)$ then the product $\phi_1.\phi_2$ is conditionally exponential convex on Q if and only if ϕ_1

and ϕ_2 are conditionally exponential convex on Q.

From the above lemma there exists $\psi_1, \psi_2 \in L_2(Q,m)$ such that $\phi_1 = \psi_1 \bullet \tilde{\psi}_1$ and $\phi_2 = \psi_2 \bullet \tilde{\psi}_2$, so

$$\phi_1.\phi_2(r) = (\psi_1 \bullet \tilde{\psi}_1(r)).(\psi_2 \bullet \tilde{\psi}_2(r))$$

$$= \int_Q \psi_1(r*s)\overline{\psi_1(s)}dm(s) \int_Q \psi_2(r*t)\overline{\psi_2(t)}dm(t)$$

$$= \int_Q \int_Q \psi_1(r*s)\psi_2(r*t)\overline{\psi_1(s)}\overline{\psi_2(t)}dm(s)dm(t)$$

$$= \int_Q \int_Q \psi_1.\psi_2(r*s, r*t)\overline{\psi_1.\psi_2(r,t)}dm(s)dm(t)$$

$$= \int_Q \int_Q \psi_1.\psi_2(r*(s,t))\overline{\psi_1.\psi_2(s,t)}dm(s)dm(t)$$

Applying Fubini's theorem to the right hand side we get

$$\phi_1.\phi_2(r) = \int_{Q\times Q} \psi_1.\psi_2(r*(s,t))\overline{\psi_1.\psi_2(s,t)}dn(s,t)$$

This implies $\phi_1.\phi_2(r) = \psi_1.\psi_2 \bullet \tilde{\psi_1.\psi_2}(r).$

Theorem 4.4. A continuous function $\psi \in L_c^\infty(Q)$ such that $\psi : Q \to C$ is conditionally exponential convex if the following conditions are satisfied: (i) $\psi(0)0$ (ii) The function $\Psi_t(y) = exp[-t\psi(y)]$ is continuous and conditionally exponential convex for all t.

Proof. suppose that ψ is a continuous conditionally exponential convex function, then (i) is easily satisfied. To establish (ii) we have:

$$\sum_{i,j=1}^{n} [\psi(y_i) + \overline{\psi(y_j)} - R_{y_j^*}\psi(y_i)]c_i\overline{c_j} \geq 0$$

which implies that:

$$\sum_{i,j=1}^{n} exp[\psi(y_i) + \overline{\psi(y_j)} - R_{y_j^*}\psi(y_i)]c_i\overline{c_j} \geq 0$$

So, we have for $t = 1$,

$$\sum_{i,j=1}^{n} R_{y_j^*}\Psi_1(y_i)c_i\overline{c_j} = \sum_{i,j=1}^{n} exp[-R_{y_j^*}\psi(y_i)]c_i\overline{c_j}$$

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$$= \sum_{i,j=1}^{n} exp[\psi(y_i) + \overline{\psi(y_j)} - R_{y_j^*}\psi(y_i)]c_i'\overline{c_j'}$$

where $c_k' = c_k exp[-\psi(y_k)]$. Hence, $\Psi_1(y)$ is conditionally exponential convex. Since $t\psi(t)$ is conditionally exponential convex, then its clear that $\Psi_t(y)$ is conditionally exponential convex for all $t > 0$. To prove the converse, let (i) and (ii) be satisfied. By (i) we have $exp[-t\psi(0)]1$ for all $t > 0$. So, $\Psi_t(y) = \frac{1}{t}[1 - exp(-t\psi(y))]$ is conditionally exponential convex for all $t > 0$. Using Fattou's lemma we can easily get that $\psi_t(y) = \lim \Psi_t(y)$ is conditionally exponential convex.

Corollary 4.5. Let $\psi : L_c^\infty(Q) \to C$ be a conditionally exponential convex and suppose that $\psi(0)0$ then $\frac{1}{\psi}$ is conditionally exponential convex.

Proof. Since ψ is a conditionally exponential convex function, then the function exp$[-t\psi(g)]$ is exponentially convex for all $t > 0$. The function $\frac{1}{\psi}$ can be written in the form:

$$\frac{1}{\psi(g)} = \int_0^\infty exp[-t\psi(g)]dt$$

Hence,

$$\sum_{i,j=1}^{n} \frac{1}{R_{g_j^*}\psi(g_i)} c_i\overline{c_j} = \sum_{i,j=1}^{n} c_i\overline{c_j} \int_0^\infty exp[-tR_{g_j^*}\psi(g_i)]dt$$

$$= \int_0^\infty \{\sum_{i,j=1}^{n} exp[-tR_{g_j^*}\psi(g_i)]c_i\overline{c_j}\}dt \geq 0.$$

Thus, $\frac{1}{\psi}$ is conditionally exponentially convex.

Remark. In [7], depending on the results given by Pederson[10, lemma 7.2.4], we were proved the stability of the set of continuous positive definite functions with compact support on hypergroups. In the same direction, the following Theorem ensure that, the product of two conditionally exponentially convex functions in L1(Q,m), also conditionally exponentially convex.

Theorem 4.6. There is a one to one correspondence between convolution semigroups $(\mu_t)_{t>0}$ in $L_1(Q,m)$ and the set of all continuous conditionally exponential convex functions defined in $L_c^\infty(Q)$. More precisely, if $(\mu_t)_{t>0}$ is a convolution semigroup in $L_1(Q,m)$, then there exists a uniquely determined continuous conditionally exponential convex function ψ in $L_c^\infty(Q)$ such that:

$$\hat{\mu}_t(g) = exp[-t\psi(g)] \qquad for \quad t > 0 \quad and \quad g \in L_c^\infty(Q)$$

Here, $\hat{\mu}_t$ denotes the Fourier transform of μ_t. Conversely, given a continuous conditionally exponential convex function defined in $L_c^\infty(Q)$, then the above representation determines a convolution semigroup $(\mu_t)_{t>0}$ in $L_1(Q,m)$.

Proof. Let $(\mu_t)_{t>0}$ be a convolution semigroup in $L_1(Q,m)$ and consider for fixed $g \in L_c^\infty(Q)$, the continuous function $\psi_g : (0,\infty) \to C$ defined by:

$$\psi_g(t) = \hat{\mu}_t(g) \qquad for \quad t > 0.$$

satisfies:

$$\psi_g(s+t) = \psi_g(s)\psi_g(t)$$

and there exists a uniquely determined complex number $\phi(g)$ such that:

(i) $\phi(0) \geq 0$

(ii) $\psi_g(t) = exp[-t\phi(g)] \quad for \quad t > 0$

(iii) $g \to exp[-t\phi(g)] = \hat{\mu}_t(g)$ is continuous and exponentially convex for $t > 0$.

Then by Theorem 5.1 it follows that ϕ is conditionally exponential convex. Moreover, the measure ρ defined by

$$< \rho, f >= \int_0^\infty exp(-t) < \mu_t, f > dt \quad f \in L_1(Q,m),$$

is positive and bounded with total mass less than one. So

$$\hat{\rho}(g) = \int_0^\infty exp(-t)\,\hat{\mu}_t(g)dt = \int_0^\infty exp[-t(1+\phi(g))] = \frac{1}{1+\phi(g)}; \quad g \in L_c^\infty(Q)$$

Since $\hat{\rho}$ is continuous, then ϕ is a continuous conditionally exponential convex function.

Conversely, let ϕ be a continuous conditionally exponential convex function. For $t > 0$, the function $g \to exp[-t\phi(g)]$ is continuous and exponentially convex. Consequently, there exists a positive bounded measure μ_t on $L_1(Q,m)$ such that

$$\hat{\mu}_t(g) = exp[-t\phi(g)] \quad for \quad g \in L_c^\infty(Q)$$

Now, let us prove that the family $(\mu_t)_{t>0}$ is a convolution semigroup. Since $\phi(0) \geq 0$, it is clear that

$$\mu_t(X) = \hat{\mu}_t(0) = exp(-t\phi(0)) \leq 1 \quad for \quad all \quad t \geq 0$$

Furthermore, for $t, s > 0$ and $g \in L_c^\infty(Q)$ we have:

$$\hat{\mu}_t(g).\hat{\mu}_s(g) = exp[-(t+s)\phi(g)] = \hat{\mu}_{t+s}(g)$$

So, $\mu_t * \mu_s = \mu_{t+s}$. Finally, since ϕ is continuous, it is bounded on compact sets and we have:

$$lim_{t\to 0} \quad \hat{\mu}_t(g) = lim_{t\to 0} \quad exp[-t\phi(g)] = 1,$$

uniformly, on compact subsets of $L_c^\infty(Q)$, and it follows

$$lim_{t \to 0} \quad \mu_t = \varepsilon_0$$

in the Bernoulli topology, hence the proof.

References

[1] C. Berg, J. P. R. Christensen and P. Ressel, Harmonic analysis on semigroups. Theory of positive definite and related functions, Graduated texts in Math. 100, Springer-Verlag, Berlin-Heidelberg-New-York, (1984).

[2] Y. M. Berezansky and A. A. Kalyuzhnyi, Harmonic analysis in hypercomplex systems, Kluwer Academic Publisher, Dordrecht-Boston-London, (1998).

[3] Y. M. Berezansky and A. A. Kalyuzhnyi, Nuclear function spaces on the base of hypercomplex systems, Inst. Mat. 35 (1982), 9-17.

[4] Y. M. Berezansky and Y. G. Kondratiev, Spectral Methods in Infinite-Dimensional Analysis, Kluwer Academic Publishers , 1995.

[5] F. Hausdorff, Summationmethoden und Momentfolgen. I, Math. Z. 9 (1921), 74-109.

[6] F. Hausdorff, Summationmethoden und Momentfolgen. II, Math. Z. 9 (1921), 280-299.

[7] A. S. Okb El Bab and H. A. Ghany, Harmonic analysis on hypergroups, International Journal of Pure and Applied Mathematics, 64 (2010), 22-31.

[8] A. S. Okb El Bab, H. A. Ghany and S. Ramadan, On strongly negative definite functions for the product of commutative hypergroups , International Journal of Pure and Applied Mathematics, 71 (4), (2011) 581-594.

[9] A. S. Okb El Bab and M. S. Shazly, Characterisation of convolution semigroups, Indian Journal of Theoretical Physics. 4, (1987) 300-311.

[10] G. K. Pederson, C^*-algebra and their automorphism groups, London Mathematical

Society Monograph, no.14, Academic Press, London- New york, (1979).

[11] T. J. Stieltjes, Recherches sur les fractions continues, Ann. Fac. Sci., 8 (4), (1894) 1-122.

[12] A. M. Zabel and B. A. Bin Dehaish, Levy khinchin formula on hypercomplex systems, Kyungpook Math. J, 48 (2008), 559-575.