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# Minimal Central Series of a Nilpotent Product of Abelian Lie Algebras

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**ABSTRACT**

We determine the structure of the lower central terms and the structure of the minimal central terms of the nilpotent product of free abelian Lie algebras of finite rank.

**Keywords.** Free abelian Lie algebra; Lower central series; Nilpotent product.

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## 1 INTRODUCTION

Let $F$ be a free Lie algebra of finite rank over a field $K$ of characteristic zero. By $\gamma_n(F)$ we denote the $n-th$ term of the lower central series of $F$.

The minimal central series of $F$ determined by a subalgebra $A$ is the decreasing series of ideals whose terms are given by the recursive formulas $[1]$:

$_0(A)=\langle A\rangle,\quad {}_lA=[F,{}_{l-1}A]$ for $l>0,$ where $\langle A\rangle$ is ideal of $F$ generated by the subalgebra $A$. The lower central series of $F$ is a well known example of a minimal central series of $F$.

The lower central series of a free Lie algebra has been studied by T.C. Hurley $[3,4]$ and the minimal central series has been defined for groups by Baer $[1]$. The terms of these series serve as a tool for the construction of new classes of Lie algebras.

Is it possible to define and classify all operations on a set of Lie algebras which share some properties with the free product and yet different from this operation? Investigation of the possibility of defining new constructions similar to those of the free product which would be a suplementary means in the study of various classes of Lie algebras.

Solvable and nilpotent operations were studied by Levi $[6]$, Golovin $[2]$ and Struik $[7]$ for groups.

In this work we determine the structure of the terms of the lower central series and minimal central series of nilpotent product of free abelian Lie algebras.

All Lie algebras considered in this work will be over a field $K$ of characteristic zero.

A Hall set of a free Lie algebra containing the elements of weight $k$ is denoted by $H_k$.

## 2 Lower Central Series

We need the following theorems.

**Theorem 1** $[8]$ Let $F$ be the free Lie algebra freely generated by the set $\{x_1,...x_n\}$ and let $H$ be a Hall basis for $F$. If $f$ is any element in $F$ then

$$f = \sum_{i=1}^{l} \alpha_i h_i \left(mod\gamma_{n+1}(F)\right),$$

where $\alpha_1,...\alpha_l \in K$ and $h_1,...h_k$ are elements of $H_1 \cup ... H_n$.

For $i=1,...,m$ let $A_i$ be the free abelian Lie algebra generated by the set $\left\{a_{i_1},...a_{i_{r_i}}\right\}$ and let $A=A_1 * ... * A_m$ be the free product of $A_1,...,A_m$. By $D$ we denote the cartesian subalgebra of $A$.

**Definition 2** The $k-th$ nilpotent product of the free abelian Lie algebras $A_1,...,A_m$ is defined to be the Lie algebra

$$L = *_k A_i = A \square \gamma_{k+1}(A),$$

where $k \geq 1.$

**Theorem 3** Suppose that $u_1,...,u_t$ are basic commutators of weight less than $k+1$, on the letters $a_{i1},...,a_{ir_i}$, where $i=1,...,m,\ r_i \geq 1.$ Then every element $u$ of $L$ can be uniquely expressed as

$$u = \sum \alpha_i u_i,$$

where $\alpha_i \in K, i=1,...,t$

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**Proof.** Let $A = A_1 * ... * A_m$ be the free product of $A_1,..., A_m$. Then $L = A/\gamma_{k+1}(A)$. Clearly every element $u$ of $L$ can be written as

$$u =_{\substack{1\le i\le m\\ 1\le i\le j}} \beta_{ij} a_{ij} + w,$$

where $\beta_{ij} \in K, w \in D.$

By lemma1 of $[5]$ the set of basic commutators of weight $\geq 2$ on the set $\{a_{ij} \mid 1 \leq i \leq m, 1 \leq i \leq r_i, r_i \geq 2\}. \ \phi$ is the set of free generators of $D$.

Thus the element $w$ can be expressed as

$$w =_{l=1}^{s} \gamma_l w_l,$$

where $w_1,..., w_s$ are basic elements constructed on free generators of $D$. Therefore the element

$$u = \sum \beta_{ij} a_{ij} + \sum \gamma_l w_l$$

is an expression of basic commutators of weight $1,2,...k$. This completes the proof.

**Theorem 4** Let $L =_k^* A_i$ be the $k-th$ nilpotent product of the free abelian Lie algebras $A_1,..., A_m$. Then

i) if $n \geq k$ then $\gamma_{n+1}(L) = \{0\},$

ii) if $n < k$ then $\gamma_{n+1}(L) = \gamma_{n+1}(A) \square \gamma_{k+1}(A),$

and every element $u$ of $\gamma_{n+1}(L)$ can be expressed as $u = \sum \alpha_i u_i$, where $\alpha_i \in K, u_1,...u_t$ are basic elements of weight $n+1, n+2,..., k$ on the letters $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq r_i, r_i \geq 2.$

**Proof.** $i)$ If $L =_k^* A_i$ then $L = A \square \gamma_{k+1}(A),$ where $A = A_1 * ... * A_m$. Hence $\gamma_{k+1}(L) = \{0\}.$ Therefore $\gamma_{n+1}(L) = \{0\}$ for $n \geq k.$

$ii)$ If $L =_k^* A_i$ then

$$\begin{aligned}\gamma_{n+1}(L) &= \gamma_{n+1}(A \square \gamma_{k+1}(A)) \\ &= (\gamma_{n+1}(A) + \gamma_{k+1}(A)) \square \gamma_{k+1}(A) \\ &\cong \gamma_{n+1}(A) \square \gamma_{n+1}(A) \cap \gamma_{k+1}(A) \\ &= \gamma_{n+1}(A) \square \gamma_{k+1}(A),\end{aligned}$$

where $n < k$ and $A = A_1 * ... * A_m$ be the free product of $A_1,..., A_m.$

Let $A_i = \left\langle a_{i_1},...a_{i_{r_i}} \right\rangle, i = 1,...m.$ Also let $F$ be the free Lie algebra generated by the set $X = \{x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq r_i\}$ and $Y$ be the set of basic elements of the form $[x_{ij_1}, x_{ij_2}],$ where $x_{ij_1} > x_{ij_2}, 1 \leq i \leq m, 1 \leq j \leq r_i.$ Then Theorem 1 implies that

$$L = F\langle Y \rangle_{id} + \gamma_{k+1}(F).$$

So

$$\gamma_{n+1}(L) = \gamma_{n+1}(F\langle Y\rangle + \gamma_{k+1}(F))$$

$$= (\gamma_{n+1}(F) + \langle Y\rangle + \gamma_{k+1}(F))\langle Y\rangle + \gamma_{k+1}(F)$$

$$\cong \gamma_{n+1}(F) \square \gamma_{n+1}(F) \cap (\langle Y\rangle + \gamma_{k+1}(F))$$

$$\cong (\gamma_{n+1}(F) \square \gamma_{k+1}(F))(\gamma_{n+1}(F) \cap (\langle Y\rangle + \gamma_{k+1}(F)) \square \gamma_{k+1}(F))$$

$$\cong (\gamma_{n+1}(F) \square \gamma_{k+1}(F))(\gamma_{n+1}(F) \cap \langle Y\rangle) + \gamma_{k+1}(F) \square \gamma_{k+1}(F)$$

By Theorem 1, $\gamma_{n+1}(F) \square \gamma_{k+1}(F)$ is a free nilpotent Lie algebra with basis $H_{n+1} \cup \ldots \cup H_k$, where $H_1, \ldots, H_k$ are Hall sets constructed on the set $X$ .

Set $M = (\gamma_{n+1}(F) \cap \langle Y\rangle + \gamma_{k+1}(F)) \square \gamma_{k+1}(F).$

Clearly every element $g$ of $(\gamma_{n+1}(F) \square \gamma_{k+1}(F)) \square M$ can be expressed as

$$g = \sum c_t h_t + M,$$

where $c_t \in K, \quad h_t \in_{s=n+1}^{k} H_s$. Consider the isomorphism

$$\phi : (\gamma_{n+1}(F) \square \gamma_{k+1}(F)) \square M \to \gamma_{n+1}(L)$$

defined as $\phi(h_t) = 0,$ if $h_t$ contains the words $\lfloor x_{ij_1}, x_{ij_2} \rfloor$ as subwords, where $1 \le i \le m, 1 \le j_1, j_2 \le r_i, h_t \in_{s=n+1}^{k} H_s.$

$\phi(h_t) = h_t(a_{ij}),$ if $h_t$ does not contains the words $\lfloor x_{ij_1}, x_{ij_2} \rfloor$ as subwords, where $h_t(a_{ij})$ is a basic word on the letters $a_{ij}, 1 \le i \le m, 1 \le j_1, j_2 \le r_i, h_t \in_{s=n+1}^{k} H_s.$

Then the image of any element $g = \sum c_t h_t + M$ of $(\gamma_{n+1}(F) \square \gamma_{k+1}(F)) \square M$ is $\phi(g) = \sum c_t h_t(a_{ij}).$

Therefore any element of $\gamma_{n+1}(L)$ can be expressed uniquely as a linear combination of basic elements of weight $n+1, \ldots, k$ on the letters $a_{ij}, 1 \le i \le m, 1 \le j \le r_i.$

## 3 Minimal Central Series

Let $A_1, \ldots, A_m$ be free abelian Lie algebras of finite rank, and let $A = A_1 * \ldots * A_m$ be the free product of $A_1, \ldots, A_m.$

The terms of the minimal central series of the free product $A = A_1 * \ldots * A_m$ defined by the commutator subalgebra $A_i^m = ((A_1 A_2), \ldots, A_m)$ of $A$ are called free commutator subalgebras of $A,$ namely, the $k-th$ term ${}_k(A_i^m)$ is called the $k-th$ free commutator subalgebra.

The $0-th$ commutator subalgebra of $A$ is the commutator subalgebra $A_i^m$ of $A.$

**Definition 5** *The* $k-th$ *nilpotent product of the free abelian Lie algebras* $A_1, \ldots, A_m$ *is defined as*

$$N = A_1 *^k \ldots *^k A_m = A \square_k (A_i^m),$$

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where $k \geq 0,\ A = A_1 * \ldots * A_m.$

The commutator subalgebra $((A_1 A_2), \ldots, A_m)$ in $N$ is called the $k-th$ nilpotent commutator subalgebra. We shall denote it by $(A_i^m)_N$.

Note that the $0-th$ nilpotent product of t $A_1, \ldots, A_m$ is the direct sum $A_1 \oplus \ldots \oplus A_m.$

By ${}_n(A_i^m)_N$ we denote the $n-th$ term of the minimal central series of $(A_i^m)_N$. We have for an arbitrary $n \leq k$

$$ {}_n(A_i^m)_N = {}_n(A_i^m) / {}_k(A_i^m). $$

Therefore

$$ 0(A_i^m)_N = (A_i^m)_N \supseteq_1 (A_i^m)_N \supseteq \ldots \supseteq_k (A_i^m)_N = \{0\} $$

**Theorem 6** The factor algebra $N / {}_n(A_i^m)_N$ is isomorphic to the $n-th$ nilpotent product $A_1 *^n \ldots *^n A_m.$ Namely,

$$ N / {}_n(A_i^m)_N \cong A_1 *^n \ldots *^n A_m $$

**Proof.** The proof is a direct consequence of the second isomorphism theorem:

$$ \begin{aligned} N / {}_n(A_i^m)_N &= A / {}_k(A_i^m) / {}_n(A_i^m) / {}_k(A_i^m) \\ &\cong A / {}_n(A_i^m) \\ &= A_1 *^n \ldots *^n A_m \end{aligned} $$

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