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INVERSE HEAT CONDUCTION PROBLEM IN A SOLID SPHERE AND ITS THERMAL STRESSES

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ABSTRACT

This paper discusses the solution of an inverse heat conduction problem of one dimensional temperature distribution and stress field for a solid sphere. The sphere is subjected to arbitrary temperature within it under unsteady state condition. Initially the sphere is maintained at constant temperature. The governing heat conduction equation has been solved by the integral transform technique. The temperature distribution,unknown temperature and thermal stresses are obtained in the form of trigonometric function. The numerical example is presented for Titanium alloy to discuss the results.

Keywords

Unknown temperature; thermal stresses; Laplace transform; solid sphere.

Academic Discipline And Sub-Disciplines

Mathematics- solid mechanics;

SUBJECT CLASSIFICATION

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INTRODUCTION

In most of heat conduction problems the temperature and thermal stress fields are considered due to heating of the body. Whereas the consideration of the inverse problems when determination of the values of the causes like temperature of surface of the body, temperature outside the medium and so on that give rise to known temperature field or stress field within the body are of great importance. Inverse problems are encountered in various branches of science, engineering and space technology. To solve the inverse heat conduction problems (IHCP) it is extremely important in determining unknown surface temperature and heat flux from the known values inside the body which are usually measured as a function of space and time. Specifically under several surface conditions such as re-entry of space vehicle and accidents involving coolant breaks in the plasma facing components, a direct measurement of surface temperature change or heat flux on the surface is almost impossible, so that prediction of these values can help, depending on the solution of IHCP. The space exploration program has gained a remarkable importance and application in the advancement of solution techniques for inverse heat conduction problem since 1950s.

The related literature is available in the monographs of Beck et al [1], Hansel [2] and Ozisik and Orlande [3]. In the review of recent literature Dhawan and Paliwal [4] solved an interior value problem of transient heat conduction in a finite circular cylinder using the technique of integral transform. Cialkowski and Grysa [5] discussed an inverse temperature field problem of theory of thermal stress. Zabaras and Liu [6] investigated the two dimensional linear inverse heat transfers using the boundary element method in conjunction with Beck's sensitivity analysis. Noda [7] studied inverse problem of coupled thermal stress field in a circular cylinder. Ashida et. al [8] studied the inverse transient thermoelastic problem for a composite circular disk. Deshmukh et. al [9] determined the unknown temperature and thermal stresses on curved surface of a semi-infinite circular cylinder. Very recently Hetmoniok et. al [10] presented a solution of the inverse heat conduction problem with the Neumann boundary condition using the homotopy perturbation method.

In the present problem the solution of an inverse heat conduction of one dimensional transient temperature field in a solid sphere is discussed. The sphere is subjected to arbitrary temperature within it under unsteady state condition. Initially the sphere is maintained at constant temperature. The governing heat conduction equation has been solved by the integral transform technique. The temperature distribution and thermal stresses are obtained in the form of trigonometric function. The numerical example is presented for Titanium alloy to discuss the results.

1. FORMULATION OF PROBLEM

Temperature distribution

Consider radial heat flow in a solid sphere $0 \le r \le a$ which is at initial temperature T_i. The sphere is subjected to arbitrary temperature $f(t)$ within the region $0 \le r \le a$. The centre of the sphere is assumed to be at finite temperature. The thermophysical properties of the sphere material are constant.

Mathematically the temperature distribution problem is defined as [11]

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad 0 \le r \le a \tag{1.1}$$

subjected to the following boundary conditions

$$T(0,t) = finite \tag{1.2}$$

$$T(a,t) \text{ is unknown} \tag{1.3}$$

$$T(\xi,t) = f(t) \text{ is known} \quad , \qquad 0 < \xi < a \tag{1.4}$$

Initial condition

$$T(r,t) = T_i \qquad t = 0 \tag{1.5}$$

where $\alpha = \dfrac{k}{c_p \rho}$ is thermal diffusivity of the sphere material, k is thermal conductivity,

c_p is specific heat and ρ is density of the material of sphere.

Thermoelastic problem

For one dimensional problem in spherical coordinates system, which means spherically symmetric problem. The stress and strain components in the ϕ and θ direction are identical and shearing stress and strain components are zero. u denotes the displacement in the radial direction, the strain-displacement relations are as in Noda et al [12]

$$\varepsilon_{rr} = \frac{du}{dr} \quad \varepsilon_{\theta\theta} = \frac{u}{r} \tag{1.6}$$

The corresponding thermoelastic stress-strain relation or Hooke's relations are

$$\sigma_{rr} = \lambda e + 2\mu\varepsilon_{rr} - (3\lambda + 2\mu)a_t \vartheta \tag{1.7}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \lambda e + 2\mu\varepsilon_{\theta\theta} - (3\lambda + 2\mu)a_t \vartheta \tag{1.8}$$

where, σ_{rr} , $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ are the stresses in the radial and tangential direction and ε_{rr} and $\varepsilon_{\theta\theta}$ are strains in radial and tangential direction. $\vartheta(r,t)$ is the temperature change obtained from the heat conduction equation (1.1), a_t is the coefficient of thermal expansion, e is the strain dilatation and λ and μ are the Lame constants related to the modulus of elasticity E and the Poisson's ratio ν as,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)} \tag{1.9}$$

The equilibrium equation in the radial direction, excluding the body force and the inertia term is,

$$r\frac{d\sigma_{rr}}{dr} + 2(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{1.10}$$

Using (1.6) to (1.10), the radial displacement and thermal stresses for solid sphere are obtained as

$$u = \frac{a_t}{(1-\nu)}\left[(1+\nu)\frac{1}{r^2}\int_0^r \vartheta r^2 dr + 2(1-2\nu)\frac{r}{a^3}\int_0^a \vartheta r^2 dr\right] \tag{1.11}$$

$$\sigma_{rr} = \frac{a_t E}{(1-\nu)}\left[\frac{2}{a^3}\int_0^a \vartheta r^2 dr - \frac{2}{r^3}\int_0^r \vartheta r^2 dr\right] \tag{1.12}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{a_t E}{(1-\nu)}\left[\frac{2}{a^3}\int_0^a \vartheta r^2 dr + \frac{1}{r^3}\int_0^r \vartheta r^2 dr - \vartheta(r,t)\right] \tag{1.13}$$

The sphere is subjected to the traction free boundary conditions

$$\sigma_{rr} = 0 \text{ at } r = a \tag{1.14}$$

Equations (1.1) to (1.14) constitute the mathematical formulation of the problem.

2. SOLUTIONS

Temperature distribution

Letting $\vartheta = T - T_i$ the problem (1.1) to (1.5) transformed as,

$$\frac{\partial^2 \vartheta}{\partial r^2} + \frac{2}{r}\frac{\partial \vartheta}{\partial r} = \frac{1}{\alpha}\frac{\partial \vartheta}{\partial t} \tag{2.1}$$

$$\vartheta(0,t) = finite \tag{2.2}$$

$$\vartheta(a,t) = unknown \tag{2.3}$$

$$\vartheta(\xi,t) = f(t) - T_i, \quad 0 < \xi < a \tag{2.4}$$

$$\vartheta(r,t) = 0, \quad t = 0 \tag{2.5}$$

where, $\vartheta(r,t)$ gives temperature change.

Applying the Laplace transform to (2.1) to (2.5) and solving , one obtains temperature change $\vartheta = T - T_i$ as ,

$$\vartheta(r,t) = \frac{2\pi}{r\xi}\sum_{n=1}^{\infty}(-1)^n n \sin\frac{n\pi r}{\xi}\int_0^t [f(t) - T_i] e^{-n^2\pi^2\alpha\frac{(t-\tau)}{\xi^2}} d\tau \tag{2.6}$$

Hence the temperature distribution $T(r,t)$ obtained as

$$T(r,t) = T_i + \frac{2\pi}{r\xi}\sum_{n=1}^{\infty}(-1)^n n \sin\frac{n\pi r}{\xi}\int_0^t [f(\tau) - T_i] e^{-n^2\pi^2\alpha\frac{(t-\tau)}{\xi^2}} d\tau \tag{2.7}$$

The unknown temperature on the surface of the sphere for fixed time t is obtained by substituting $r = a$ in (2.7),

$$T(a,t) = T_i + \frac{2\pi}{a\xi}\sum_{n=1}^{\infty}(-1)^n n \sin\frac{n\pi a}{\xi}\int_0^t [f(\tau) - T_i] e^{-n^2\pi^2\alpha\frac{(t-\tau)}{\xi^2}} d\tau \tag{2.8}$$

Thermoelastic problem

Using (2.6) in (1.12) and (1.13) one gets radial and tangential stress functions as,

$$\sigma_{rr} = \frac{4a_t E}{(1-\nu)}\sum_{n=1}^{\infty}(-1)^n \left\{\frac{1}{a^3}\left[-a\cos\frac{n\pi a}{\xi} + \frac{\xi}{n\pi}\sin\frac{n\pi a}{\xi}\right] - \frac{1}{r^3}\left[-r\cos\frac{n\pi r}{\xi} + \frac{\xi}{n\pi}\sin\frac{n\pi r}{\xi}\right]\right\}\int_0^t [f(\tau) - T_i] e^{-n^2\pi^2\alpha\frac{(t-\tau)}{\xi^2}} d\tau \tag{2.9}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{2a_t E}{(1-\nu)}\sum_{n=1}^{\infty}(-1)^n \left\{\frac{2}{a^3}\left[-a\cos\frac{n\pi a}{\xi} + \frac{\xi}{n\pi}\sin\frac{n\pi a}{\xi}\right] + \frac{1}{r^3}\left[-r\cos\frac{n\pi r}{\xi} + \frac{\xi}{n\pi}\sin\frac{n\pi r}{\xi}\right] - \frac{n\pi}{r\xi}\sin\frac{n\pi r}{\xi}\right\}$$
$$\int_0^t [f(\tau) - T_i] e^{-n^2\pi^2\alpha\frac{(t-\tau)}{\xi^2}} d\tau \tag{2.10}$$

Equation (2.9) and (2.10) gives general solution for thermal stresses which satisfy the equilibrium equation (1.10).

3. NUMERICAL AND GRAPHICAL ANALYSIS

The exact analytical solutions are obtained for transient temperature distribution, unknown temperature on the surface of the sphere and thermal stress functions. The mathematical software MATLAB is used for further numerical. The numerical and graphical analysis for Titanium alloy sphere (T_i6A1-4V) is presented with following thermomechanical properties,

Material constants

Thermal conductivity $\quad k = 7.5 W/mK$

Coefficient of thermal expansion $\quad a_t = 9.5\times10^{-6} K^{-1}$

Thermal diffusivity $\quad \alpha = 5\times10^{-6} m^2/s$

Modulus of elasticity $\quad E = 116.7 GPa$ $\quad$ Poisson's constant $\quad \nu = 1/3$

Setting $\quad a = 1m, \xi = 0.8m, T_i = 300K, \; f(t) = e^t$

and for $f(t) = e^t$, the expressions for temperature distribution, unknown temperature and thermal stresses are obtained as,

$$T(r,t)=T_i+\frac{2\pi}{r\xi}\sum_{n=1}^{\infty}(-1)^n n\sin\frac{n\pi r}{\xi}\left\{\left[\frac{e^t-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}}{1+\frac{n^2\pi^2\alpha}{\xi^2}}\right]+\frac{T_i\xi^2}{n^2\pi^2\alpha}\left[1-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}\right]\right\} \tag{3.1}$$

$$T(a,t)=T_i+\frac{2\pi}{r\xi}\sum_{n=1}^{\infty}(-1)^n n\sin\frac{n\pi b}{\xi}\left\{\left[\frac{e^t-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}}{1+\frac{n^2\pi^2\alpha}{\xi^2}}\right]+\frac{T_i\xi^2}{n^2\pi^2\alpha}\left[1-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}\right]\right\} \tag{3.2}$$

$$\frac{(1-\nu)\sigma_{rr}}{4a_tE}=\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{a^3}\left[-a\cos\frac{n\pi a}{\xi}+\frac{\xi}{n\pi}\sin\frac{n\pi a}{\xi}\right]-\frac{1}{r^3}\left[-r\cos\frac{n\pi r}{\xi}+\frac{\xi}{n\pi}\sin\frac{n\pi r}{\xi}\right]\right\}$$
$$\left\{\left[\frac{e^t-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}}{1+\frac{n^2\pi^2\alpha}{\xi^2}}\right]+\frac{T_i\xi^2}{n^2\pi^2\alpha}\left[1-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}\right]\right\} \tag{3.3}$$

$$\frac{(1-\nu)\sigma_{\theta\theta}}{2a_tE}=\sum_{n=1}^{\infty}(-1)^n\left\{\frac{2}{a^3}\left[-a\cos\frac{n\pi a}{\xi}+\frac{\xi}{n\pi}\sin\frac{n\pi a}{\xi}\right]+\frac{1}{r^3}\left[-r\cos\frac{n\pi r}{\xi}+\frac{\xi}{n\pi}\sin\frac{n\pi r}{\xi}\right]-\frac{n\pi}{r\xi}\sin\frac{n\pi r}{\xi}\right\}$$
$$\left\{\frac{e^t-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}}{1+\frac{n^2\pi^2\alpha}{\xi^2}}+\frac{T\xi^2{}_i}{n^2\pi^2\alpha}\left[1-e^{-\frac{n^2\pi^2\alpha t}{\xi^2}}\right]\right\} \tag{3.4}$$

Fig 1 represents the temperature distribution with radius for time $t=0.001, 0.002(s)$ with initial temperature $300(K)$.The temperature increases from the centre to the surface of the solid sphere with prescribed interior transient temperature distribution. There are some radial positions where the temperature is identical irrespective of time. The temperature increases in the central region from $r=0.15$ to the surface while near the centre of the sphere it decreases. The variation on the surface is very small.

Fig. 2 shows the unknown temperature on the surface for the time duration $t=0.001\ to\ 0.002(s)$.It is observed that the variation on the surface temperature is from $302\ to\ 306.2(K)$.

Fig 3 shows the radial stress distribution with radius.The stress is equal to zero on the surface, due to induced boundary condition. The stress function develops tensile stresses at the centre and it increases with small variation in time. There are some radial positions in the interior of the sphere where the stress shows identical value for different times. The tension at the centre increases with the time.

Fig. 4 represents the variation of the tangential stress with time. It is observed that the centre is tensile tangentially while the surface is compressive tangentially. There are five radial positions which are obtained where the tangential stress is identical for different times. The stress value at the surface is very small.

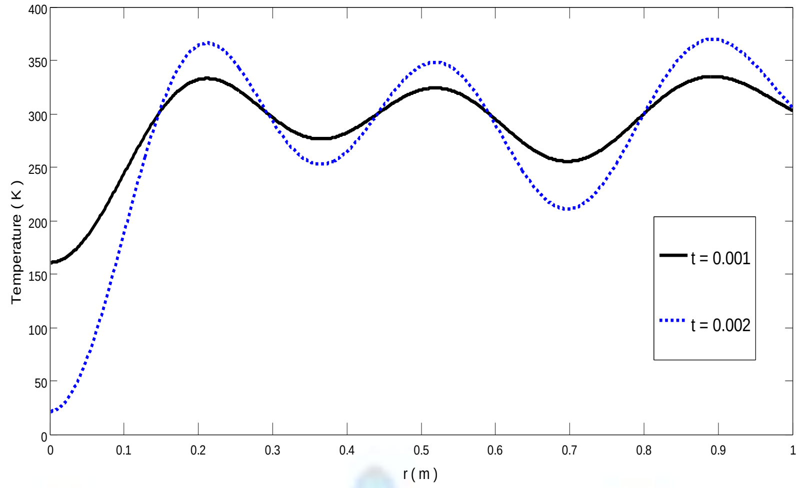

Figure 1: Temperature distribution with radius for fixed time $t = 0.001, 0.002(s)$

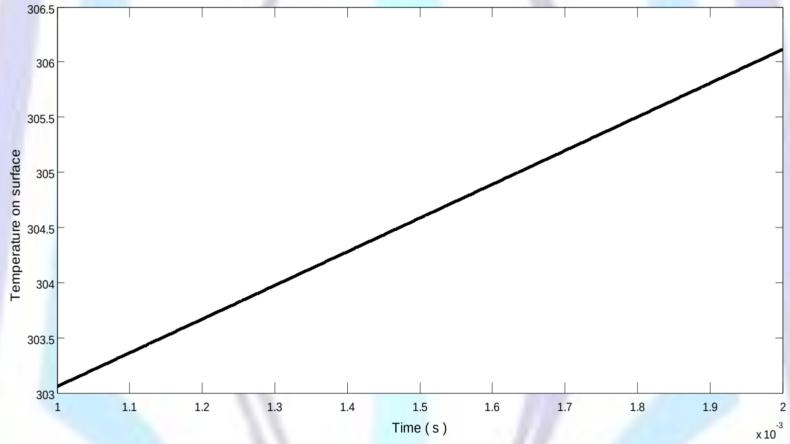

Figure 1: Unknown temperature at $r = a$ for time duration $t = 0.001\ to\ 0.002(s)$

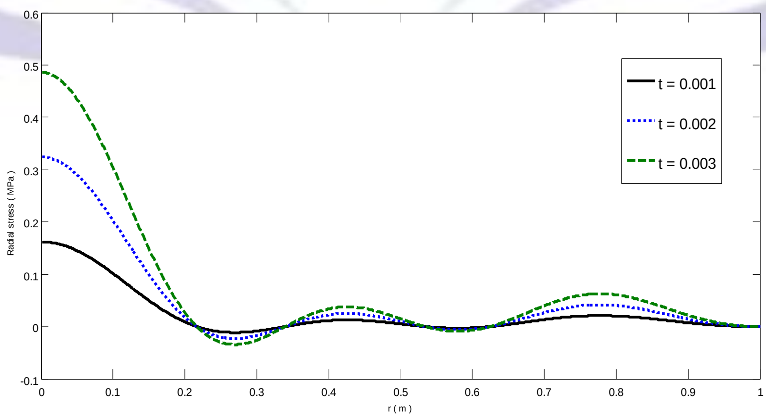

Figure 2: Radial stress distribution with radius for $t = 0.001, 0.002(s)$

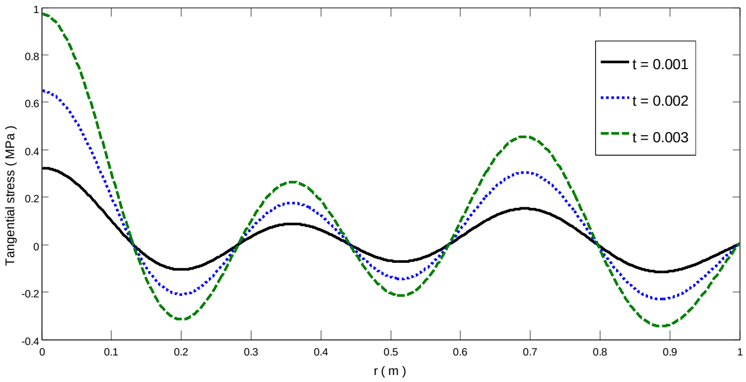

Figure 3: Tangential stress distribution with radius for $t = 0.001, 0.002(s)$

4. CONCLUSION

In this problem the analytical solutions are obtained for the transient temperature and thermal stresses due to internal transient heating. The unknown temperature on the surface is obtained. As a special case, mathematical model is constructed for Titanium alloy sphere (T_i6A1-4V) with the material properties specified in the numerical calculations.

Here one considers a solid sphere subjected to interior transient heating described in the form $f(t) = e^t$ within in $0 < r < a$.The temperature distribution is shown for the times $t = 0.001, 0.002(s)$.The temperature in the sphere varies from $25\ to\ 375(K)$. The radial positions at $r = 0.15, 0.28, 0.48, 0.59, 0.8(m)$ show the identical temperatures within a sphere. The unknown temperature on the surface is obtained. It is observed that the variation on the surface temperature is from $302\ to\ 306.2(K)$ in prescribed time.

The radial stress increases from surface to the centre and the maximum values occur at the centre. The radial stress is compressive at the centre and it equals zero on the surface, due to induced mechanical condition. There are some positions where the stress is identical for different times.

The tangential stress is tensile at the centre, while the surface is compressive tangentially. The tangential stress decreases from centre to surface. The tangential stress at the radial positions $r = 0.15, 0.28, 0.48, 0.59\ and\ 0.8(m)$are identical and independent of time.

The results obtained here are useful in engineering problems particularly in the determination of state of stress in a solid sphere subjected to transient heating inside it. Any particular case of special interest can also be derived by assigning suitable values to the parameters and functions in the problem. the results presented in this article are new and not discussed previously in the open literature.

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