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Some Properties of Certain subclass of Meromorphically Multivalent Functions Defined by Convolution and Integral Operator involving I-Function

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Abstract: In the present paper, we introduce a certain subclass of meromorphic functions $I_p^{*,\alpha,\beta}(\theta,\delta,v)$. We obtain some results, like, Coefficient inequality, Modified Hadamard Product, Integral means and Inclusion properties for this class.

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1.INTRODUCTION:

Let $\mathcal{R}_p^*$ denote the class of functions of the form :

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_{n-p}z^{n-p}\ ,\ (p\in N=\{1,2,3,\dots\}),\qquad (1.1)$$

which are meromorphic multivalent in the punctured unit disk $U^*=\{z:z\in\mathbb{C},0<|z|<1\}$. Consider a subclass J_p of the class $\mathcal{R}_p^*$ consisting of function of the form:

$$f(z)=z^{-p}-\sum_{n=1}^{\infty}a_{n-p}z^{n-p}(a_{n-p}\geq 0),\qquad (1.2)$$

The Hadamard product of two functions, f is given by (1.2) and

$$g(z)=z^{-p}-\sum_{n=1}^{\infty}b_{n-p}z^{n-p},\ \ (b_{n-p}\geq 0).\qquad (1.3)$$

is defined by

$$(f*g)(z)=z^{-p}-\sum_{n=1}^{\infty}a_{n-p}b_{n-p}z^{n-p}=(g*f)(z).$$

The I-function which was introduced by Saxena [14] is an extension of Fox's H-function. On specializing the parameters, I-function can be reduced to almost all the known as well as unknown special function.

The definition of I-functions given by Saxena [14] is as follows:

$$I(z)=I_{p_i,q_i:r}^{m,n}[z]=I_{p_i,q_i:r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(a_j,\alpha_j)_{n+1},p_i\\(b_j,\beta_j)_{1,m}(b_j,\beta_j)_{m+1},q_i\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_c t(s)\,z^s ds,$$

where

$$t(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_j+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}$$

$p_i(i=1,2,3,\dots,r),q_i(i=1,2,3,\dots,r),m,n$ are integers satisfying $0\leq n\leq p_i,0\leq m\leq q_i(i=1,2,3,\dots,r),r$ is finite $\alpha_i,\beta_i,\alpha_{ij},\beta_{ij}$ are real and positive and a_i,b_i,a_{ij},b_{ij} are complex numbers such that

$$\alpha_j(b_h+v)\neq\beta_j(a_h-1-k).$$

With all necessary conditions for existence as given by Saxena [14]. If the integral operator of $f\in\mathcal{R}_p^*$ for $\alpha,\beta>0$ is denoted by

$I_p^{\alpha,\beta}$ and defined as following:

$$I_p^{\alpha,\beta}f(z)=\frac{z^{\beta-p}}{\Gamma(\alpha-\beta)I_{p_i^{+1},q_i^{+1}:r}^{m,n+1}[z]}\int_0^z t^{p-\alpha}(z-t)^{\alpha-\beta-1}I_{p_i^{+1},q_i^{+1}:r}^{m,n}[t]f(t)dt\ ,\qquad (1.4)$$

where

$$I_{p_i^{+1},q_i^{+1}:r}^{m,n+1}[z]=I_{p_i^{+1},q_i^{+1}:r}^{m,n+1}\left[z\left|\begin{matrix}(\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})(\beta,1)\end{matrix}\right.\right],$$

valid when

$$\left(Re(b_j)<Re(a_j)<1+\min_{1\leq j\leq m}Re\left(\frac{b_j}{\beta_j}\right)\right).$$

Then $I_p^{\alpha,\beta}f(z)$ can be expressed for

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p},$$

as given below,

$$I_p^{\alpha,\beta} f(z) = z^{-p} + \sum_{n=1}^{\infty} \left[\frac{I_{p_i^{+1},q_i^{+1}:r}^{m,n+1}\left[z \left| \begin{matrix} (\alpha-p-n,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta-p-n,1) \end{matrix} \right. \right]}{I_{p_i^{+1},q_i^{+1}:r}^{m,n+1}\left[z \left| \begin{matrix} (\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta,1) \end{matrix} \right. \right]} \right] a_{n-p} z^{n-p}$$

$$= z^{-p} + \sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta) a_{n-p} z^{n-p}, \qquad (1.5)$$

where

$$\emptyset_p(m,n,\alpha,\beta) = \frac{I_{p_i^{+1},q_i^{+1}:r}^{m,n+1}\left[z \left| \begin{matrix} (\alpha-p-n,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta-p-n,1) \end{matrix} \right. \right]}{I_{p_i^{+1},q_i^{+1}:r}^{m,n+1}\left[z \left| \begin{matrix} (\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta,1) \end{matrix} \right. \right]} \qquad (1.6)$$

Definition (1.1): Let $f \in \mathcal{R}_p^*$ given by (1.1). Then f be in the class $J_p^{\alpha,\beta}(\theta,\delta,v)$ if it satisfies the following condition:

$$\left| \frac{\frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} - \theta \left| \frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} + p \right| + p}{\delta \left[\frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} - \theta \left| \frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} + p \right| \right] + p[\delta - \beta(1-v)]} \right| < 1, \quad (1.7)$$

where θ, δ, β belong to $(0,1]$ and v belong to $[0,1)$.

We define the subclass $J_p^{*,\alpha,\beta}(\theta,\delta,v) = J_p \cap J_p^{\alpha,\beta}(\theta,\delta,v)$.

2.Main results:

In the first theorem, we provide sufficient condition for functions to be in the class $J_p^{\alpha,\beta}(\theta,\delta,v)$.

Theorem (2.1): Let the function $f(z)$ defined by (1.1) be in the class $J_p^{\alpha,\beta}(\theta,\delta,v)$. Then

$$\sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta) \left(n(1+\theta)(1+\delta) - p\beta(1-v)\right) |a_{n-p}||b_{n-p}| \le p\beta(1-v). \; (2.1)$$

where θ, δ, β belong to $(0,1]$, v belong to $[0,1)$and $\emptyset_p(m,n,\alpha,\beta)$is given by (1.6).

proof: Let the condition (2.1) hold true, then we have

$$\left| z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' - \theta \left| z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' + pI_p^{\alpha,\beta}(f*g)(z) \right| + pI_p^{\alpha,\beta}(f*g)(z) \right|$$

$$- \left| p\beta(1-v)I_p^{\alpha,\beta}(f*g)(z) - \delta \left[z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' - \theta \left| z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' + pI_p^{\alpha,\beta}(f*g)(z) \right| \right] - p\delta\left(pI_p^{\alpha,\beta}(f*g)(z)\right) \right|$$

$$= \left| \sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta) n a_{n-p} b_{n-p} z^{n-p} - \theta \left| \sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)\, n a_{n-p} b_{n-p} z^{n-p} \right| \right| -$$

$$\left| p\beta(1-v)z^{-p} + p\beta(1-v) \sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta) a_{n-p} b_{n-p} z^{n-p} \right.$$
$$\left. - \delta \left[\sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta) n a_{n-p} b_{n-p} z^{n-p} - \theta \left| \sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)\, n a_{n-p} b_{n-p} z^{n-p} \right| \right] \right|$$

$$\leq (1+\theta)\sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)\, n a_{n-p} b_{n-p}|z|^{n-p} - p\beta(1-v)|z|^{-p} - p\beta(1-v)\sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)|a_{n-p}||b_{n-p}||z|^{n-p} + \delta(1+\theta)\sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)\, n|a_{n-p}||b_{n-p}||z|^{n-p}$$

$$\leq \sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)|a_{n-p}||b_{n-p}| - p\beta(1-v) \leq 0,$$

By hepothesis. Then by Maximum modulus theorem, we have $f \in J_p^{\alpha,\beta}(\theta,\delta,v)$.

Theorem (2.2): The function $f(z)$ defined by (1.2) is said to be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. If and only if

$$\sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right) a_{n-p} b_{n-p} \leq p\beta(1-v), \qquad (2.2)$$

where θ, δ, β belong to $(0,1]$, v belong to $[0,1)$ and $\emptyset_p(m,n,\alpha,\beta)$ is given by (1.6).

proof: We only need to prove the "only if" part of Theorem (2.1). For functions $f(z) \in J_p$, we can write

$$\left|\frac{\frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} - \theta\left|\frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} + p\right| + p}{\delta\left[\frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} - \theta\left|\frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)'}{I_p^{\alpha,\beta}(f*g)(z)} + p\right| + p[\delta - \beta(1-v)]\right]}\right|$$

$$= \left|\frac{z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' - \theta\left|z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' + pI_p^{\alpha,\beta}(f*g)(z)\right| + pI_p^{\alpha,\beta}(f*g)(z)}{\delta\left[z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' - \theta\left|z\left(I_p^{\alpha,\beta}(f*g)(z)\right)' + pI_p^{\alpha,\beta}(f*g)(z)\right| + p[\delta - \beta(1-v)]I_p^{\alpha,\beta}(f*g)(z)\right]}\right|$$

$$\leq \left|\frac{(1+\theta)\sum_{n=1}^{\infty}\emptyset_p(m,n,\alpha,\beta)\, n a_{n-p} b_{n-p} z^{n-p-1}}{p\beta(1-v) + p\beta(1-v)\sum_{n=1}^{\infty}\emptyset_p(m,n,\alpha,\beta)\, a_{n-p} b_{n-p} z^{n-p-1} - \delta(1+\theta)\sum_{n=1}^{\infty}\emptyset_p(m,n,\alpha,\beta)\, n a_{n-p} b_{n-p} z^{n-p-1}}\right| < 1,$$

since $Re(z) \leq |z|$, $(z \in U^*)$, we thus find that

$$Re\left(\frac{(1+\theta)\sum_{n=1}^{\infty}\emptyset_p(m,n,\alpha,\beta)\, n a_{n-p} b_{n-p} z^{n-p-1}}{p\beta(1-v) + p\beta(1-v)\sum_{n=1}^{\infty}\emptyset_p(m,n,\alpha,\beta)\, a_{n-p} b_{n-p} z^{n-p-1} - \delta(1+\theta)\sum_{n=1}^{\infty}\emptyset_p(m,n,\alpha,\beta)\, n a_{n-p} b_{n-p} z^{n-p-1}}\right) < 1.$$

If we now choose z to be real and let $z \to 1^-$, we get

$$\sum_{n=1}^{\infty} \emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right) a_{n-p} b_{n-p} \leq p\beta(1-v)$$

which is equivalent to (2.2).

Corollary (2.1): Let the function $f(z)$ defined by (1.2) be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. Then

$$a_{n-p} \leq \frac{p\beta(1-v)}{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right) b_{n-p}},$$

The result is sharp for the function

$$f(z) = z^{-p} - \frac{p\beta(1-v)}{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right) b_{n-p}} z^{n-p}. \qquad (2.3)$$

Let the function $f_j(z) (j = 1,2)$ be defined by

$$f_j(z) = z^{-p} - \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p}, \quad (a_{n-p,j} \geq 0). \qquad (2.4)$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^{-p} - \sum_{n=1}^{\infty} a_{n-p,1}a_{n-p,2}z^{n-p} = (f_2 * f_1)(z).$$

Theorem (2.3): Let the function $f_j(z)(j = 1,2)$ be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. Then $(f_1 * f_2)(z) \in J_p^{*,\alpha,\beta}(\theta,\delta,v)$, where

$$\eta = 1 - \frac{np\beta(1-v)^2(1+\theta)(1+\delta)}{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)^2 b_{n-p}}.$$

The result is sharp for the functions $f_j(z)(j = 1,2)$ given by

$$f_j(z) = z^{-p} - \frac{p\beta(1-v)}{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)b_{n-p}}z^{n-p}, (j = 1,2), \quad (2.5)$$

where $\emptyset_p(m,n,\alpha,\beta)$ given by (1.6).

proof: Employing the technique used earlier by Shild and Silverman [15], we need to find the largest η such that

$$\sum_{n=1}^{\infty} \frac{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-\eta)\right)}{p\beta(1-\eta)} a_{n-p,1}a_{n-p,2} \le 1.$$

Since $f_j(z) \in J_p^{*,\alpha,\beta}(\theta,\delta,v)$, $(j = 1,2)$, we readily see that

$$\sum_{n=1}^{\infty} \frac{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)b_{n-p}}{p\beta(1-v)} a_{n-p,1} \le 1,$$

and

$$\sum_{n=1}^{\infty} \frac{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)b_{n-p}}{p\beta(1-v)} a_{n-p,2} \le 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)b_{n-p}}{p\beta(1-v)} \sqrt{a_{n-p,1}a_{n-p,2}} \le 1. \quad (2.6)$$

Thus it is sufficient to show that

$$\frac{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-\eta)\right)}{p\beta(1-\eta)} a_{n-p,1}a_{n-p,2} \le \frac{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)}{p\beta(1-v)} \sqrt{a_{n-p,1}a_{n-p,2}}$$

or equivalently, that

$$\sqrt{a_{n-p,1}a_{n-p,2}} \le \frac{\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)(1-\eta)}{\left(n(1+\theta)(1+\delta) - p\beta(1-\eta)\right)(1-v)}.$$

Hence, in the right of inequality (2.6), it is sufficient to prove that

$$\frac{p\beta(1-v)}{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)b_{n-p}} \le \frac{\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)(1-\eta)}{\left(n(1+\theta)(1+\delta) - p\beta(1-\eta)\right)(1-v)}. \quad (2.7)$$

It follows from (2.6) that

$$\eta \le 1 - \frac{np\beta(1-v)^2(1+\theta)(1+\delta)}{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)^2 b_{n-p}}.$$

Using similar arguments to those in the proof of Theorem (2.3), we obtain the following theorem.

Theorem (2.4): Let the function $f_1(z)$ defined by (2.5) be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. Suppose also that the function $f_2(z)$ defined by (2.5) be in the class $J_p^{*,\$,\beta}(\theta,\delta,v)$. Then $(f_1 * f_2)(z) \in J_p^{*,\alpha,\beta}(\theta,\delta,v)$, where

$$¥ = 1 - \frac{np\beta(1+\theta)(1+\delta)(1-v)(1-\$)}{\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta) - p\beta(1-v)\right)\left(n(1+\theta)(1+\delta) - p\beta(1-\$)\right)}.$$

the result is sharp for the function $f_j(z)(j=1,2)$ given by

$$f_1(z)=z^{-p}-\frac{np\beta(1+\theta)(1+\delta)(1-v)}{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)b_{2-p}}z^{2-p},$$

and

$$f_2(z)=z^{-p}-\frac{np\beta(1+\theta)(1+\delta)(1-\$)}{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-\$)\big)b_{2-p}}z^{2-p}.$$

Theorem (2.5): Let the functions $f_j(z),(j=1,2)$ defined by (2.5) be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. Then the function

$$h(z)=z^{-p}+\sum_{n=1}^{\infty}\left(a_{n-p,1}^2+a_{n-p,2}^2\right)z^{n-p}$$

belong to the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$, where

$$\varphi=1-\frac{2np\beta(1-v)^2(1+\theta)(1+\delta)}{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)^2b_{n-p}+p\beta(1-v)^2}.$$

The result is sharp for the function $f_j(z)(j=1,2)$ given by(2.7).

Proof: By using Theorem (2.1), we obtain

$$\sum_{n=1}^{\infty}\left[\frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)b_{n-p}}{p\beta(1-v)}\right]^2 a^2{}_{n-p,1}$$
$$\leq\left[\sum_{n=1}^{\infty}\frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)b_{n-p}}{p\beta(1-v)}a_{n-p,1}\right]^2\leq 1,\quad (2.8)$$

and

$$\sum_{n=1}^{\infty}\left[\frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)b_{n-p}}{p\beta(1-v)}\right]^2 a^2{}_{n-p,2}$$
$$\leq\left[\sum_{n=1}^{\infty}\frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)b_{n-p}}{p\beta(1-v)}a_{n-p,2}\right]^2\leq 1.\quad (2.9)$$

It follows from (2.8) and (2.9) that

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)b_{n-p}}{p\beta(1-v)}\right]^2\left(a^2{}_{(n-p)_1}+a^2{}_{(n-p)_2}\right)\leq 1.$$

Therefore, we need to find the largest φ such that

$$\frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-\varphi)\big)b_{n-p}}{p\beta(1-\varphi)}\leq\frac{1}{2}\left[\frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)b_{n-p}}{p\beta(1-v)}\right]^2.$$

That is

$$\varphi\leq 1-\frac{2np\beta(1-v)^2(1+\theta)(1+\delta)}{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)^2b_{2-p}+p\beta(1-v)^2}\ \blacksquare$$

In the following theorem, we consider integral transforms of the functions in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$.

Theorem (2.6): Let the function f defined by (1.2) be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. Then the integral transforms

$$F_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)du,\quad (0<u\leq 1, 0<c<\infty)\qquad (2.10)$$

is in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$, where

$$\mu=1-\frac{cn(1-v)(1+\theta)(1+\delta)}{(c+1)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)+cp\beta(1-v)}.$$

The result is sharp for the function f given by

$$f(z)=\frac{1}{z^p}+\frac{p\beta(1-v)}{\emptyset_p(m,n,\alpha,\beta)\big((1+\theta)(1+\delta)-p\beta(1-v)\big)}z^{1-p},\quad (p\in N, n\in N)\quad (2.11)$$

Proof: Suppose $f(z)=z^{-p}+\sum_{n=1}^{\infty}a_{n-p}z^{n-p}$ be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. Then we have

$$f_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)du,$$

$$=c\int_0^1\left[u^{c-1}z^{-p}+\sum_{n=1}^{\infty}a_{n-p}u^{c+n-1}z^{n-p}\right]du$$

$$=z^{-p}+\sum_{n=1}^{\infty}\frac{c}{c+n}a_{n-p}z^{n-p}.$$

In view of Theorem (2.1), it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{c\big[\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-\mu)\big)\big]b_{n-p}}{(c+n)p\beta(1-\mu)}a_{n-p}\le 1.\qquad (2.12)$$

Since $f\in J_p^{*,\alpha,\beta}(\theta,\delta,v)$, we have

$$\sum_{n=1}^{\infty}\frac{\big[\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)\big]b_{n-p}}{p\beta(1-v)}a_{n-p}\le 1.$$

Note that (2.12) is satisfies if

$$\frac{c\big[\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-\mu)\big)\big]}{(c+n)p\beta(1-\mu)}\le\frac{\big[\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)\big]}{p\beta(1-v)}.$$

Rewriting the inequality, we have

$$\frac{\big(n(1+\theta)(1+\delta)-p\beta(1-\mu)\big)}{(1-\mu)}\le\frac{(c+n)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)}{c(1-v)}.$$

Solving for μ, we have

$$\mu=1-\frac{cn(1-v)(1+\theta)(1+\delta)}{(c+1)\big(n(1+\theta)(1+\delta)-p\beta(1-v)\big)+cp\beta(1-v)}=F(n).\qquad (2.13)$$

A simple computation shows that $F(n)$ is increasing $F(n)\ge F(1)$. Using this result follows:

Theorem (2.7): Let the function f defined by (1.2) is in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$. Then the integral transforms

$$F_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)du,\quad (0<u\le 1, 0<c<\infty)$$

is in the class $J_p^{*,\alpha,\beta}\left(\theta,\delta,\frac{1+\emptyset(c+p-1)}{1+c+p}\right)$, the result is sharp for function f given by

$$f(z)=\frac{1}{z^p}+\frac{p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)}{\emptyset_p(m,n,\alpha,\beta)\left((1+\theta)(1+\delta)-p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)\right)}z^{1-p},\quad (p\in N, n\in N)\quad (2.14)$$

Proof: By Definition of F_{c+p-1}, we get

$$F_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)du$$

$$=z^{-p}+\sum_{n=1}^{\infty}\frac{c}{c+n}a_{n-p}z^{n-p}.$$

In view of Theorem (2.1), it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{c\left[\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta)-p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)\right)\right]b_{n-p}}{(c+n)p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)}a_{n-p}\leq 1. \qquad (2.15)$$

Since $f\in J_p^{*,\alpha,\beta}(\theta,\delta,v)$, we have

$$\sum_{n=1}^{\infty}\frac{[\emptyset_p(m,n,\alpha,\beta)(n(1+\theta)(1+\delta)-p\beta(1-v))]b_{n-p}}{p\beta(1-v)}a_{n-p}\leq 1.$$

Note that (2.15) is satisfies if

$$\frac{c\left[\emptyset_p(m,n,\alpha,\beta)\left(n(1+\theta)(1+\delta)-p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)\right)\right]}{(c+n)p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)}\leq\frac{[\emptyset_p(m,n,\alpha,\beta)(n(1+\theta)(1+\delta)-p\beta(1-v))]}{p\beta(1-v)},$$

or equivalently, when

$$\mathcal{H}(n,c,\alpha,\beta,p,\theta,v,\delta)=\frac{c(1-v)\left[\left(n(1+\theta)(1+\delta)-p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)\right)\right]}{(c+n)\left[p\beta\left(1-\frac{1+\emptyset(c+p-1)}{1+c+p}\right)\right](n(1+\theta)(1+\delta)-p\beta(1-v))}\leq 1,$$

since $\mathcal{H}(n,c,\alpha,\beta,p,\theta,v,\delta)$ is decreasing of n $(n\geq 1)$. Then the proof is complete. ■

Theorem (2.8): Let the function f defined by (1.2) be in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$, and

$$F(z)=\frac{1}{c}[(c+p)f(z)+zf'(z)]=z^{-p}+\sum_{n=1}^{\infty}\frac{c+n}{c}a_{n-p}z^{n-p}, c>0 \quad (2.16)$$

then F is in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$ for $|z|\leq r(\alpha,\beta,\theta,\delta,c,\varepsilon)$, where

$$r(\alpha,\beta,\theta,\delta,c,\varepsilon)=\inf_n\left[\frac{c(1-\varepsilon)(n(1+\theta)(1+\delta)-p\beta(1-v))}{(n+c)(1-v)(n(1+\theta)(1+\delta)-p\beta(1-\varepsilon))}\right]^{\frac{1}{n-p}}, n\in N.$$

The result is sharp for the function given by (2.14).

Proof: Let $T=[z^pf(z)+(1+\delta)z^{p+1}f'(z)]$. Then it is sufficient to show that

$|T-1|<|T+1-2\varepsilon|$

A computation shows that is satisfied if

$$\sum_{n=1}^{\infty}\frac{(n+c)[\emptyset_p(m,n,\alpha,\beta)(n(1+\theta)(1+\delta)-p\beta(1-\varepsilon))]}{cp\beta(1-\varepsilon)}a_{n-p}|z|^{n-p}\leq 1. \qquad (2.17)$$

Since $f\in J_p^{*,\alpha,\beta}(\theta,\delta,v)$, then by Theorem (2.1), we have

$$\sum_{n=1}^{\infty}\emptyset_p(m,n,\alpha,\beta)(n(1+\theta)(1+\delta)-p\beta(1-v))\leq p\beta(1-v).$$

The inequality (2.17) is satisfied if

$$\frac{(n+c)[\emptyset_p(m,n,\alpha,\beta)(n(1+\theta)(1+\delta)-p\beta(1-\varepsilon))]}{cp\beta(1-\varepsilon)}a_{n-p}|z|^{n-p}\leq\frac{\emptyset_p(m,n,\alpha,\beta)(n(1+\theta)(1+\delta)-p\beta(1-v))a_{n-p}b_{n-p}}{p\beta(1-v)}$$

Solving for $|z|$, we get

$$|z|^{n-p}\leq\frac{c(1-\varepsilon)(n(1+\theta)(1+\delta)-p\beta(1-v))}{(n+c)(1-v)(n(1+\theta)(1+\delta)-p\beta(1-\varepsilon))b_{n-p}}.$$

Therefore,

$$|z|\leq\left[\frac{c(1-\varepsilon)(n(1+\theta)(1+\delta)-p\beta(1-v))}{(n+c)(1-v)(n(1+\theta)(1+\delta)-p\beta(1-\varepsilon))b_{n-p}}\right]^{\frac{1}{n-p}}.$$

Solving for $|z|$ we get the result.

Now, we obtain the inclusion properties of the class $J_p^{*,\alpha,\beta}(\theta,\delta,v)$.

Theorem (2.9): Let θ,φ,δ belong to (0,1], v belong to [0,1) and $\lambda \geq 0$. Then

$J_p^{*,\alpha,\beta}(\theta,\delta,v+1) \subseteq J_p^{*,\alpha,\beta}(\theta,\delta,v)$, where

$$\lambda = \frac{\big(n(1+\theta)(1+\delta) - p\beta(2-v)\big)(1-v) + (2-v)\big(p\beta(1-v) - n(1+\theta)\big)}{n(2-v)(1+\theta)}. \qquad (2.18)$$

Proof: Let the function defined by (1.2) is in the class $J_p^{*,\alpha,\beta}(\theta,\delta,v+1)$. Then by using Theorem (2.1), we have

$$\sum_{n=1}^{\infty} \frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta) - p\beta(2-v)\big)}{p\beta(2-v)} a_{n-p} b_{n-p} \leq 1. \qquad (2.19)$$

In order to prove that $f \in J_p^{*,\alpha,\beta}(\theta,\delta,v)$, we must have

$$\sum_{n=1}^{\infty} \frac{\emptyset_p(m,n,\alpha,\beta)\big(n(1+\theta)(1+\delta) - p\beta(1-v)\big)}{p\beta(1-v)} a_{n-p} b_{n-p} \leq 1. \qquad (2.20)$$

Not that (2.20) is satisfies if

$$\frac{\big(n(1+\theta)(1+\delta) - p\beta(1-v)\big)}{p\beta(1-v)} \leq \frac{\big(n(1+\theta)(1+\delta) - p\beta(2-v)\big)}{p\beta(2-v)}. \qquad (2.21)$$

Rewriting the inequality, we have

$$\lambda \leq \frac{\big(n(1+\theta)(1+\delta) - p\beta(2-v)\big)(1-v) + (2-v)\big(p\beta(1-v) - n(1+\theta)\big)}{n(2-v)(1+\theta)}. \qquad (2.22)$$

Since the right-hand side of (2.22) is an increasing function of n, thus we get (2.18). ∎

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