ISSN 2347-1921

COMMON FIXED POINT THEOREM FOR A PAIR OF WEAKLY COMPATIBLE SELF-MAPPINGS IN FUZZY METRIC SPACE USING (CLRG) PROPERTY

Dr. V. Dharmaiah
Department of Mathematics Osmania University Hyderabad (TS)
N. Appa Rao
Department of Mathematics Dr. B.R. Ambedkar Open University
Hyderabad (TS)
apparaonemaala1968@gmail.com

ABSTRACT

In this paper we prove a common fixed point theorem for a pair of weakly compatible self-mappings in fuzzy metric space by using (CLRg) property. The result is extended for two finite families of self-mappings in fuzzy metric space by using the concept of pairwise commuting. An example is provided which demonstrates the validity of main theorem.

Key Words: Fuzzy Metric Space; Weakly Compatible Mappings; E.A Property and (CLRg) Property.

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol .10, No.5
www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

1. INTRODUCTION

In 1965, Zadeh [33] introduced the concept of fuzzy sets which opened an avenue for further development of analysis. In 1975, Kramosil and Michalek [10] introduced the concept of fuzzy metric space. George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10]. In 2002, Aamri and El-Moutawakil [1] defined the notion of E.A property for self-mappings. E.A property allows replacing completeness requirement of the space with a more natural condition of closedness of the range in the proof of fixed point theorems. Many authors have proved common fixed point theorems in fuzzy metric spaces using E.A. property. For example we referer to [2, 7, 8, 11-13, 15, 19-21, 23, 25-27, 29, 30].

Recently, Sintunavarat and Kumam [28] defined the notion of (CLRg) property and proved the results of Mihet [14] without any requirement of the closedness of the range subspace.

In this paper, we prove a common fixed point theorem for a pair of weakly compatible self-mappings by using (CLRg) property in fuzzy metric space. We give an example satisfying the theorem. We also extend the theorem for two finite families of self-mappings in fuzzy metric space by using the notion of pairwise commuting due to Imdad, Ali and Tanveer [9].

Our results improved the results of Sedghi et al. [22].

2. Preliminaries

2.1 Fuzzy Metric Space

A fuzzy metric space is a triple $(X,\ M,\ T)$ where *X* is a nonempty set, *T* is a continuous *t*-norm and *M* is a fuzzy set on $X^2 \times (0,\ \infty)$ and the following conditions are satisfied for all $x,\ y \in X$ and $t,\ s > 0$:

(FM-1) $M(x,\ y,\ t) > 0$;

(FM-2) $M(x,\ y,\ t) = 1 \Leftrightarrow x = y$;

(FM-3) $M(x,\ y,\ t) = M(y,\ x,\ t)$;

(FM-4) $M(x,\ y,\ \cdot): (0,\ \infty) \to [0,\ 1]$ is continuous;

(FM-5) $M(x,\ z,\ t+s) \geq T\big(M(x,\ y,\ t),\ M(y,\ z,\ s)\big)$. □

2.2 Weakly Compatible Mappings

Two self-mappings *f* and *g* of a non-empty set *X* are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $fz = gz$ for some $z \in X$, then $fgz = gfz$. □

2.3 E.A. Property

Let *f* and *g* be self-mappings on a fuzzy metric space $(X,\ M,\ T)$. Then the pair $(f,\ g)$ is said to satisfy E.A. property, if there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} = gx_n = u$ for some $u \in X$.

2.4 (CLRg) Property

Let *f* and *g* be self-mappings on a fuzzy metric space $(X,\ M,\ T)$. Then the pair $(f,\ g)$ is said to satisfy CLRg property (common limit in the range of *g* property) if $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} = gx_n = gx$ for some $x \in X$. □

2.5 Pairwise Commuting

Two families of self-mappings $\{f_i\}_{i=1}^m$ and $\{g_k\}_{k=1}^n$ are said to be pairwise commuting if

1. $f_i f_j = f_j f_i$ for all $i,\ j \in \{1,\ 2, \dots, m\}$,
2. $g_k g_l = g_l g_k$ for all $k,\ l \in \{1,\ 2, \dots, n\}$,
3. $f_i g_k = g_k f_i$ for all $i \in \{1,\ 2, \dots, m\}$ and $k \in \{1,\ 2, \dots, n\}$. □

3 Main Result

In 2010, Sedghi et al. [22] proved a common fixed point theorem for a pair of weakly compatible self-mappings with E.A property in fuzzy metric space by using an increasing and continuous function:

$$\phi: (0,1] \to (0,1] \text{ such that } \phi(t) > t \text{ for every } t \in (0,1).$$

An example of increasing and continuous function satisfying the above condition is given by

$$\phi: (0,1] \to (0,1] \text{ defined by } \phi(t) = t^{\frac{1}{2}}.$$

3.1 Theorem

Let $(X,\ M,\ T)$ be a fuzzy metric space and f and g be self-mappings of X satisfying the following conditions:

1. $f(X) \subseteq g(X)$
2. $f(X)$ or $g(X)$ is a closed subset of X
3. $M(fx,\ fy,\ t) \geq$

$$\phi\left(\min\left\{\begin{array}{c} M(gx,gy,\ t), \\ \sup_{t_1+t_2=\frac{2}{k}t} \min\{M(gx,fx,\ t_1),\ M(gy,fy,t_2)\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \min\{M(gx,fy,\ t_3),\ M(gy,fx,t_4)\} \end{array}\right\}\right) \quad \text{-- (1)}$$

for all $x,\ y \in X,$ and $t > 0$ and for some $1 \leq k < 2$.

Suppose that the pair $(f,\ g)$ is weakly compatible and satisfies E.A property. Then f and g have a unique common fixed point in X. □

3.2 Theorem

Let $(X,\ M,\ T)$ be a fuzzy metric space. Let $f,\ g$ be self-mappings of X satisfying inequality (1) of Theorem 3.1. If the pair $(f,\ g)$ is weakly compatible and satisfies the (CLRg) property, then f and g have a unique common fixed point.

Proof **:** Since the pair $(f,\ g)$ satisfies the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} = qx_n = gu$$

for some $u \in X$. Now we assert that $fu = gu$ Let, on the contrary, $fu \neq gu,$ then there exists $t_0 > 0$ such that

$$M\left(fu,\ gu, \tfrac{2}{k}t_0\right) > M(fu,\ gu,\ t_0) \quad \text{--} \quad (2)$$

To support the claim, let it be untrue. Then we have

$$M\left(fu,\ gu, \tfrac{2}{k}t_0\right) = M(fu,\ gu,\ t_0), \text{for all } t > 0.$$

Repeatedly using this equality, we obtain

$$M(fu,\ gu,\ t) = M\left(fu,\ gu, \tfrac{2}{k}t\right) = \cdots = M\left(fu,\ gu, \left(\tfrac{2}{k}\right)^n t\right) \to 1,$$

as $n \to \infty$. This shows that $M(fu,\ gu,\ t) = 1$ for all $t > 0$ which contradicts $fu \neq gu$ and hence $fu = gu$.

Putting $x = x_n,\ y = u,$ in inequality (1), we get, for some $t_0 > 0$

$M(fx_n,\ fu,\ t_0) \geq$

$$\phi\left(\min\left\{\begin{array}{c} M(gx_n,gu,\ t_0), \\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min\{M(gx_n,fx_n,\ t_1),\ M(gu,fu,t_2)\}, \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max\{M(gx_n,fu,\ t_3),\ M(gu,fx_n,t_4)\} \end{array}\right\}\right).$$

For all $\varepsilon \in \left(0,\ \frac{2}{k}t_0\right)$ and $n \to \infty,$ it follows that

$M(gu,\ fu,\ t_0) \geq$

$$\phi\left(\min\left\{\begin{array}{c} M(gu,\ gu,\ t_0), \\ \min\left\{M(gu,\ gu,\ \varepsilon),\ M\left(gu,fu,\ \frac{2}{k}t_0 - \varepsilon\right)\right\}, \\ \max\left\{M\left(gu,fu,\ \frac{2}{k}t_0 - \varepsilon\right),\ M(gu,\ gu,\ \varepsilon)\right\} \end{array}\right\}\right).$$

$$= \phi\left(M\left(gu,fu,\ \tfrac{2}{k}t_0 - \varepsilon\right)\right)$$

$$> M\left(gu,fu,\ \tfrac{2}{k}t_0 - \varepsilon\right).$$

As $\varepsilon \to 0,$ we have

$$M(gu, fu, t_0) \geq M\left(gu, fu, \frac{2}{k}t_0\right)$$

which contradicts (2). Thus we have $gu = fu$. Next, we let $z = fu = gu$. Since the pair (f, g) is weakly compatible $fgu = gfu$ which implies that $fz = fgu = gfu = gz$.

Now, we show that $z = fz$. Suppose that $z \neq fz$, then on using (1) with $x = z,\ y = u$, we get, for some $t_0 > 0$ and for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$

$$M(fu, fu, t_0) \geq$$

$$\phi\left(\min\begin{Bmatrix} M(gz, gu, t_0), \\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min\{M(gz, fz, t_1), M(gu, fu, t_2)\}, \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max\{M(gz, fu, t_3), M(gu, fz, t_4)\}\end{Bmatrix}\right).$$

$$M(fz, z, t_0) \geq$$

$$\phi\left(\min\begin{Bmatrix} M(fz, z, t_0), \\ \min\left\{M(fz, fz, \varepsilon), M\left(z, z, \frac{2}{k}t_0 - \varepsilon\right)\right\}, \\ \max\left\{M(fz, z, \varepsilon), M\left(z, fz, \frac{2}{k}t_0 - \varepsilon\right)\right\}\end{Bmatrix}\right).$$

As $\varepsilon \to 0$, we have

$$M(fz, z, t_0) \geq \phi\left(\min\left\{M(fz, z, t_0), M\left(z, z, \frac{2}{k}t_0\right)\right\}\right)$$

$$= \phi(M(fz, z, t_0)) > M(fz, z, t_0)$$

which is a contradiction. Hence $z = fz = gz$. Therefore z is a common fixed point of f and g. □

Uniqueness Let $w(\neq z)$ be another common fixed point of f and g. On using inequality (1) with $x = z,\ y = w$, we get, for some $t_0 > 0$

$$M(fz, fw, t_0) \geq$$

$$\phi\left(\min\begin{Bmatrix} M(gz, gw, t_0), \\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min\{M(gz, fz, t_1), M(gw, fw, t_2)\}, \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max\{M(gz, fw, t_3), M(gw, fz, t_4)\}\end{Bmatrix}\right).$$

And for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$, we have

$$M(z, w, t_0) \geq$$

$$\phi\left(\min\begin{Bmatrix} M(z, w, t_0), \\ \min\left\{M(z, z, \varepsilon), M\left(w, w, \frac{2}{k}t_0 - \varepsilon\right)\right\}, \\ \max\left\{M(z, w, \varepsilon), M\left(w, z, \frac{2}{k}t_0 - \varepsilon\right)\right\}\end{Bmatrix}\right).$$

As $\varepsilon \to 0$, we have

$$M(z, w, t_0) \geq \phi\left(\min\left\{M(z, w, t_0), M\left(w, z, \frac{2}{k}t_0\right)\right\}\right)$$

$$= \phi(M(z, w, t_0)) > M(z, w, t_0)$$

which is a contradiction. Therefore $fz = z = gz$. It implies that f and g have a unique a common fixed point. □

3.3 Remark

Theorem 3.2 improves the main result of Sedghi, Shobe and Aliouche ([22] Theorem 1) without any requirement on containment of ranges amongst the involved mappings and closedness of one or more subspaces.

The following example illustrates Theorem 3.2.

3.4 Example

Let (X, M, T) be a fuzzy metric space, where $X = [3, 19)$, with t-norm T defined by $T(a, b) = ab$ for all $a, b \in [0, 1]$ and

$$M(x, \quad y, \quad t) = \begin{cases} \dfrac{t}{t + |x - y|}, & \text{if } t > 0 \\ 0 \quad , & \text{if } t = 0 \end{cases}$$

for all $x, \ y \in X$. Let the function $\phi: (0, \ 1] \to (0, 1]$ be defined by $\phi(t) = t^{\frac{1}{2}}$. Define the self-mappings f and g by

$$f(x) = \begin{cases} 3, & \text{if } x \in \{3\} \cup (5, 19) \\ 12, & \text{if } x \in (3, 5]. \end{cases}$$

$$g(x) = \begin{cases} 3, & \text{if } x = 3 \\ 11, & \text{if } x \in (3, 5] \\ \dfrac{x+1}{2}, & \text{if } x \in (5, \quad 19). \end{cases}$$

Taking $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$ or $\{x_n\} = \{3\}$, it is clear that the pair $(f, \ g)$ satisfies the (CLRg) property

$$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = 3 = g(3) \in X.$$

It is noted that $f(X) = \{3, 12\} \not\subseteq [3, 10) \cup \{11\} = g(X)$. Thus, all the conditions of Theorem 3.2 are satisfied and 3 is a unique common fixed point of the pair $(f, \ g)$. Also, all the involved mappings are even discontinuous at their unique common fixed point 3. Here, it may be pointed out that $g(X)$ is not a closed subspace of X. □

Now we utilize the notion of pairwise commuting due to Imdad, Ali and Tanveer [9] and extend Theorem 3.2 to two finite families of self-mappings in fuzzy metric space.

3.5 Corollary

Let $(X, \ M, \ T)$ be a fuzzy metric space. Let $\{f_1, f_2, \ \dots, \ f_m\}$ and $\{g_1, g_2, \ \dots, \ g_n\}$ be two finite families of self-mappings of X. Suppose $f = f_1 f_2 \dots f_m$ and $g = g_1 g_2 \dots g_n$ and satisfy inequality (1) of Theorem 3.1. Assume that the pair $(f, \ g)$ satisfies the (CLRg) property.

If the family $\{f_i\}_{i=1}^{m}$ commutes pairwise with the family $\{g_j\}_{j=1}^{n}$, then for all $i \in \{1, 2, \ \dots, \ m\}$ and $j \in \{1, 2, \ \dots, \ n\}$ f_i and g_j have a unique common fixed point.

Proof **:** The proof of this theorem is similar to that of Theorem 3.1 contained in Imdad, Ali and Tanveer [9], hence details are avoided. □

3.6 Remark Corollary 3.5 improves the result of Sedghi, Shobe and Aliouche ([22] Theorem 2). □

By setting $f_1 = f_2 = \dots = f_m = f$ and $g_1 = g_2 = \dots = g_n = g$ in Corollary 3.5, we deduce the following:

3.7 Corollary

Let $(X, \ M, \ T)$ be a fuzzy metric space. Let f and g be self-mappings of X such that the pair $(f^m, \ g^n)$ satisfies the (CLRg) property and

$$M(f^m x, \ f^m y, \ t) \geq \phi\left(\min \begin{Bmatrix} M(g^n x, \ g^n y, \ t), \\ \sup_{t_1 + t_2 = \frac{2}{k} t_0} \min\{M(g^n x, \ f^m x, \ t_1), \ M(g^n y, \ f^m y, \ t_2)\}, \\ \sup_{t_3 + t_4 = \frac{2}{k} t_0} \max\{M(g^n x, \ f^m y, \ t_3), \ M(g^n y, \ f^m x, \ t_4)\} \end{Bmatrix} \right)$$

holds for all $x, \ y \in X$ and $t > 0$, for some $1 \leq k < 2$ and $m, \ n$ are fixed positive integers. If $(f^m, \ g^m)$ commutes pairwise, then f and g have a unique common fixed point. □

REFERENCES

[1] M. Aamri - D. El Moutawakil, "*Some New Common Fixed Point Theorems under Strict Contractive Conditions*", J. Math. Anal., Appl., 270 (1) (2002), pp. 181-188.

[2] S. Chauhan - B.D. Pant, "*Common Fixed Point Theorems in Fuzzy Metric Spaces*", Bull. Allahabad Math. Soc. 27(2012) in press.

[3] Y.J. Cho, "*Fixed Points in Fuzzy Metric Spaces*", J. Fuzzy Math. 5 (4) (1997), 949-962.

[4] Y.J. Cho, "*Fixed Points for Compatible Mappings of Type (A)*", Math. Japon, Vol.38 (3) (1993), 497-508.

[5] A. George - P. Veeramani, "*On Some Results in Fuzzy Metric Space*", Fuzzy Sets and Systems, Vol.64 (1994), 395-399.

[6] M. Grabiec, "*Fixed Point in Fuzzy Metric Spaces*", Fuzzy Sets and Systems, Vo. 27, no.3, (1988), 385-389.

[7] M. Imdad - J. Ali, "*Some Common Fixed Point Theorems in Fuzzy Metric spaces*", Math. Commun., Vol.11 (2006), 153-163.

[8] M. Imdad - J. Ali, "*A General Fixed point Theorem in Fuzzy Metric Spaces via an Implicit Function*", J. Appl. Math. Inform. Vol.26 (3-4) (2008), 591-603.

[9] M. Imdad - J. Ali - M. Tanveer, "*Coincidence and Common Fixed Point Theorems for Nonlinear Contractions in Menger PM Spaces*", Chaos, Solitons Fractals Vol. 42(5) (2009), 3121-3129.

[10] I. Kramosil - J. Michalek, "*Fuzzy Metrics and Statistical Metric Spaces*", Kybernetica, Vol.11 (5) (1975), 336-344.

[11] S. Kumar - S. Chauhan, "*Common Fixed Point Theorems using Implicit Relation and Property (E.A) in Fuzzy Metric Spaces*", Ann. Fuzzy Math. Inform. (2012), in Press.

[12] S. Kumar, "*Fixed Point Theorems for Weakly Compatible Maps under E.A. Property in Fuzzy Metric Spaces*", J. Appl. Math. Inform. Vol.29 (2011), 395-405.

[13] S. Kumar - B. Fisher, "*A Common Fixed Point Theorem in Fuzzy Metric Space using Property (E.A) and Implicit Relation*", Thai J. Math. Vol.8 (3) (2010), 439-446.

[14] D. Mihet, "*Fixed Point Theorems in Fuzzy Metric Spaces using Property E.A., Non-linear Analysis Theory, Methods & Applications*", Int. J. Math. Math. Sci., Vol.73 (7) (2010), 2184-2188.

[15] P.P. Murthy - S. Kumr - K. Tas, "*Common Fixed Points of Self Maps Satisfying an Integral Type Contractive Condition in Fuzzy Metric Spaces*", Math. Commun. 15(2) (2010), 521-537..

[16] M.S. El Naschie, "*On the Uncertainty of Cantorian Geometry and two-slit Experiment*", Chaos, Solitons and Fractals, 9 (1998), 517-529.

[17] M.S. El Naschie, "*A Review of E-infinity Theory and the Mass Spectrum of High Energy Particle Physics*", Chaos, Solitons and Fractals, 19 (2004), 209-236.

[18] M.S. El Naschie, "*A Review of Applications and Results of E-infinity Theory*", Int. J. Nonlinear Sci. Numer. Simul. 8(2007), 11-20.

[19] B.D. Pant - S. Chauhan, "*Common Fixed Point Theorems for two pairs of Weakly Compatible Mappings in Menger Spaces and Fuzzy Metric Spaces*", Vasile Alecsandri Univ. Bacau Sci. Stud. Res. Ser. Math. Inform. 21 (2) (2011), 81-96.

[20] V.Pant - R.P. Pant, "*Fixed Points in Fuzzy Metric Space for Non-Compatible Maps*", Soochow J. Math. 33 (4) (2007), 647-655.

[21] D.O. Regan - M. Abbas, "*Necessary and Sufficient Conditions for Common Fixed Point Theorems in Fuzzy Metric Space*", Demonstratio Math. Vol.42 (4) (2009), 887-900.

[22] S. Sedghi - N. Shobe - A. Aliouche, "*A Common Fixed Point Theorem for Weakly Compatible Mappings in Fuzzy Metric Spaces*", Gen. Math. Vol.18 (3) (2010), 3-12.

[23] S. Sedghi - N. Shobe, "*Common Fixed Point Theorems for Weakly Compatible Mappings Satisfying Contractive Condition of Integral Type*", J. Adv. Res. Appl. Math. Vol.3 (4) (2011), 67-68.

[24] S. Sedghi - C. Alaca - N. Shobe, "*On Fixed Points of Weakly Commuting Mappings with Property (E.A)*", J. Adv. Stud. Topol, Vol.3 (3) (2012), 11-17.

[25] Y. Shen - D. Qiu - W. Chen, "*Fixed Point Theorems in Fuzzy Metric Spaces*", Appl. Math. Lett. Vol. 25 (2) (2012), 138-141.

[26] S.L. Singh - A. Tomar, "*Fixed Point Theorems in FM-Spaces*", J. Fuzzy Math. Vol.12 (4) (2004), 845-859.

[27] W. Sintunavarat - Y.J. Cho - P. Kumam, "*Coupled Coincidence Point Theorems for Contractions without Commutative Condition in Intuitionistic Fuzzy Normed Spaces*", Fixed Point Theory Appl. Vol. 2011.

[28] W. Sintunavarat - P. Kumam, "*Common Fixed Point Theorems for a Pair of Weakly Compatible Mappings in Fuzzy Metric Spaces*", J. Appl. Math. Vol. 2011, Article ID 637958, 14 pages, 2011.

[29] W. Sintunavarat - P. Kumam "*Fixed Point Theorems for a Generalized Intuitionistic Fuzzy Contraction in Intuitionistic Fuzzy Metric Spaces*", Thai J. Math. Vol.10 (1) (2012), 123-135.

[30] W. Sintunavarat - P. Kumam "*Common Fixed Points for R-weakly Commuting in Fuzzy Metric Spaces*", Annali dell' Universitâ di Ferr., in Press.

[31] P.V. Subrahmanyam, "*Common Fixed Point Theorems in Fuzzy Metric Spaces*", Inform. Sci., Vol. 83 (1995), 109-112.

[32] R. Vasuki, "*Common Fixed Points for R-weakly Commuting Maps in Fuzzy Metric Spaces*", Indian J. Pure Appl. Math., Vol.30 (4) (1999), 419-423.

[33] L.A. Zadeh, "*Fuzzy Sets*", Inform and Control, Vol.8, No. 3, (1965), 338-353.