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On Properties of UBAC Class of Life Distributions

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Abstract. Some properties of the used better than aged in convex ordering(UBAC) class of life distributions are given. These properties include moment inequalities and moment generating functions behavior. Nonparametric estimation of UBAC survival function are discussed. In addition testing of the survival function of this class based on moment inequalities are introduced.

Key Words: UBAC class of life distribution; Moment inequalities; Moment generating function; exponentiality; Pitman's efficiency; Asymptotic normality.

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1 INTRODUCTION

Let X be a nonnegative continuous random variable with distribution function $F(x)$, survival function $\overline{F}=1-F$, at age t, we define the random residual life by X_t with survival function $\overline{F}_t=\frac{\overline{F}(t+x)}{\overline{F}(t)}, x,t\geq 0$ and assume that X has a finite mean $\mu=E(X)=\int_0^\infty \overline{F}(u)du$. Some properties concerning the asymptotic behavior of X_t as $t\to\infty$ will be used. Bhattacharjee(1982) gave the following definition.

Definition(1.1). If X is nonnegative random variable, its distribution function F is said to be finitely and positively smooth if a number $\gamma\in(0,\infty)$ exists such that:

$$\lim_{t\to\infty}\frac{\overline{F}(t+x)}{\overline{F}(t)}=e^{-x\gamma}, \tag{1}$$

where γ is called to b a asymptotic decay coefficient of X. Denoting X_e be a random variable exponentially distributed by mean $\frac{1}{\gamma}$, the following definitions imply that X_t converges to X_e in distribution written as $X\xrightarrow{d}X_e$. This property is useful for description of random life times of devices of unknown age.

Definition(1.2). The distribution F is said to be used better than age UBA if for all $x,t\geq 0$

$$\overline{F}(x+t)\geq\overline{F}(x)e^{-\gamma t}, \tag{2}$$

where γ called is the asymptotic decay of X.

Definition(1.3). The distribution F is said to be used better than aged in convex ordering (UBAC) if for all $x,t\geq 0$

$$\int_t^\infty \overline{F}(x+u)du\geq\overline{F}(x)\int_t^\infty e^{-\gamma u}du, \tag{3}$$

or

$$\nu(x+t)\geq\frac{1}{\gamma}\overline{F}(x)e^{-\gamma t}, \tag{4}$$

where $\nu(x+t)=\int_{x+t}^\infty \overline{F}(u)du$.

We observe that the equality of (1.3) is achieved when $F(x)$ has an exponential distribution with mean μ equal to the coefficient of the asymptotic decay γ, where the exponential distribution is the only distribution which has the lack of memory property.

Willmot and Cai (2000) showed that the UBA class includes the decreasing mean residual life (DMRL) class. While Al-Nachawati and Alwasel(1997) showed that UBAC class includes the UBA class of life distributions. Thus we have

$$IHR\subset DMRL\subset UBA\subset UBAC.$$

For definitions and discussions of the classes IHR and DMRL see Barlow and Prochan (1981), Ahmad (1992, 1994) and among others. Ahmad (2004) discussed some properties of UBA class of life distribution. Abu-Youssef presented nonparametric test for UBAC class based on U-test statistic. Based on the goodness of fit approach, Ali and Abu-Youssef (2012) found a new test for testing exponentiality against UBAC Class of life distributions.

The present work discuses probability and inferential properties of UBAC class of life distributions. In section 2 we give moment inequalities of the class. In section 3 we obtain upper bounds of the moment generating function that guarantee its existence and finiteness. In section 4, we estimate the survival function $\overline{F}(t)$ whenever assumed to be UBAC class of life distributions. Testing in this class based on a moment inequality is introduced in section 5. Finally we apply the proposed test to real practical data in medical science.

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2 Moment Inequalities

Several authoress derived moments inequalities of different families of life distributions such as IHR, IHRA (increasing hazared rate in average), NBU (new better than used), UBA and UBAE (used better than aged in expectation), cf. Ahmad(2001, 2004) and Ahmad and Mugadi (2004), among others. The following theorem gives moments inequality for UBAC class of life distributions.

Theorem 2.1. let F be UBAC class of life distributions such that for some integers $r,s \geq 0$, $\mu_{(r+s+3)} = E(X^{r+s+3}) \leq \infty$, then

$$\frac{\gamma^{s+2}\mu_{(r+s+3)}}{(r+s+3)!} \geq \frac{\mu_{(r+1)}}{(r+1)!}, \quad r \geq 0. \tag{1}$$

Proof. Since F is UBAC, then

$$\gamma\nu(x+t) \geq \overline{F}(x)e^{-\gamma t}, \tag{2}$$

Multiplying both sides by $x^r t^s$, $r,s \geq 0$, and integrating over $(0,\infty)$, w.r.t. x,t , then

$$\gamma\int_0^\infty\int_0^\infty x^r t^s \nu(x+t)dtdx \geq \int_0^\infty\int_0^\infty x^r t^s \overline{F}(x)e^{-t\gamma}dxdt \tag{3}$$

By taking $w=t, x=u-w$, the left hand side of (2.3) is

$$\gamma\int_0^\infty\int_0^u (u-w)^r w^s \nu(u)dwdu = \gamma\int_0^\infty u^{r+s+1}\nu(u)\int_0^u (1-\frac{w}{u})^r(\frac{w}{u})^s dwdu$$

$$= \gamma B(r+1,s+1)\int_0^\infty u^{r+s+1}\nu(u)du, \tag{4}$$

where $B(r+1,S+1) = \int_0^1 (1-u)^r (u)^s du$.

But

$$\int_o^\infty u^{r+s+1}\nu(u)du = \int_o^\infty u^{r+s+1}E(U-u)I(U>u)du$$

$$= E(\int_o^U u^{r+s+1}(U-u)du = \frac{\mu_{(r+s+3)}}{(r+s+2)(r+s+3)}. \tag{5}$$

Than (2.4) becomes as the following:

$$\gamma\int_0^\infty\int_0^\infty x^r t^s \nu(x+t)dtdx = \gamma B(r+1,s+1)\frac{\mu_{(r+s+3)}}{(r+s+2)(r+s+3)}. \tag{6}$$

The right hand side of (2.3) is equal to

$$\int_0^\infty t^s e^{-t\gamma}dt\int_0^\infty x^r \overline{F}(x)dx = \frac{\Gamma(s+1)\mu_{(r+1)}}{\gamma^{s+1}(r+1)!}. \tag{7}$$

By using (2.6) and (2.7) in (2.3), (2.1) is obtained.

Remark 2.1. Theorem 2.1 above may be extended as follows: The definitions of UBAC is equivalent to the following: let $x_1, x_2, ..., x_{k+1}$ be nonnegative random variables and integers $r \geq 0$, then F is UBAC if and only if

$$\gamma\nu(\sum_{i=1}^{k+1} x_i) \geq \overline{F}(x_1)e^{-\gamma\sum_{i=2}^{k+1} x_i}. \tag{8}$$

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Using the same methodology, we thus have : If F is UBAC then

$$\gamma^{\sum_{i=2}^{k+1} r_i+k+1} \frac{\mu_{(\sum_{i=1}^{k+1} r_i+k+2)}}{(\sum_{i=1}^{k+1} r_i+k+2)} \geq \frac{\mu_{(r_1+1)}}{(r_1+1)} e^{-\gamma \sum_{i=2}^{k+1} x_i}. \tag{9}$$

Corollary 2.1. let $r=s=0$, then $\gamma^2 \mu_{(3)} \geq 6\mu.$

Corollary 2.2. let $r=0$, then $\gamma^{s+2} \mu_{(s+3)} \geq (s+3)!\mu.$

Corollary 2.3. let $s=0$, then $\gamma^2 \mu_{(r+3)} \geq (r+3)(r+2)\mu.$

3 Existence of moment generating functions

In this section we show that the moment generating function of X exists and is finite for the UBAC class of life distributions if μ exists. Actually, upper bounds of the moment generating functions are given. We have the following theorem

Theorem 3.1. If F is UBAC and $\mu < \infty$, then

$$M(\theta) \leq (1 - \frac{\nu^2 \theta(\mu + \mu_{(2)}\theta/2)}{(\nu^2 - \theta^2)}), \nu \neq \theta, \tag{10}$$

where $M(\theta) = E(e^{\theta x})$.

proof. Note that:

$$M(\theta) = 1 + \theta \int_0^\infty e^{\theta x} \overline{F} dx. \tag{11}$$

Since F is UBAC, then

$$\gamma \int_0^\infty \int_0^\infty e^{\theta x} \nu(x+t) dt dx \geq \int_0^\infty e^{\theta t} \overline{F}(t) \int_0^\infty e^{-x\gamma} dx dt. \tag{12}$$

The left hand side of (3.12)

$$\gamma \int_0^\infty \int_0^u e^{\theta(u-v)} \nu(u) dv du = \frac{\gamma}{\theta} [\int_0^\infty e^{\theta u} \nu(u) du - \int_0^\infty \nu(u) du]. \tag{13}$$

Since $\frac{d\nu(t)}{dt} = -\overline{F}(t)$ then,

$$\int_0^\infty e^{\theta u} \nu(u) du = \frac{1}{\theta} [\frac{M(\theta - 1)}{\theta} - \mu] \text{ and } \int_0^\infty e^{\theta u} \nu(u) du = \frac{\mu_{(2)}}{2}.$$

Thus8 (3-13) becomes as the following:

$$\gamma \int_0^\infty \int_0^u e^{\theta(u-v)} \nu(u) dv du = \frac{\gamma}{\theta} [\frac{(M(\theta - 1)}{\theta^2} - \frac{\mu}{\theta} - \frac{\mu_{(2)}}{2\theta}]. \tag{14}$$

The right hand side of (3.12)

$$\int_0^\infty e^{\theta t} \overline{F}(t) dt \int_0^\infty e^{-x\gamma} dx = \frac{(M(\theta - 1)}{\gamma \theta}. \tag{15}$$

Now the result now follows from (3.13) and (3.14).

ISSN 2347-1921

4 Estimating UBAC survival function

Let $X_1, X_2, \ldots, X_n$ represent a random sample from a population with survival distribution function $\overline{F}$. With no restriction on $\overline{F}$, the emperical distribution $\overline{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \geq x)$, where $I(X_i \geq x) = 1$ if $X_i \geq x$ and is 0 otherwise, is a widly used nonparametric estimate of $\overline{F}$. When $\overline{F}$ is UBAC, we will modify $\overline{F}_n(x)$ to $\overline{F^*}(x) = inf_{t>0}\overline{F}(x+t)e^{t\gamma}$ (See Ahmad (2004)).

lemma 4.1. let $F(x)$ be a distribution function. Then $F^* = 1 - \overline{F^*}$ is UBAC.

proof. we have

$$\int_t^{\infty} \overline{F}^*(x+y)dy = \int_t^{\infty} inf_{z>0}\overline{F}(x+y+z)e^{z\gamma}dy$$

$$= \int_t^{\infty} inf_{z>0}\overline{F}(x+y+z)e^{(z+y)\gamma}e^{-y\gamma}dy$$

$$= \int_t^{\infty} inf_{u>y}\overline{F}(x+u)e^{u\gamma}e^{-y\gamma}dy$$

$$\geq \int_t^{\infty} inf_{u>0}\overline{F}(x+u)e^{u\gamma}e^{-y\gamma}dy$$

$$= \int_t^{\infty} \overline{F^*}(x)e^{-(y\gamma}dy = \frac{1}{\gamma}\overline{F}^*(x)e^{-y\gamma}. \qquad (1)$$

We then propose to estimate $\overline{F}(x)$ by

$$\overline{F}^*(x) = inf_{t>0}\overline{F}_n(x+t)e^{t\gamma}. \qquad (2)$$

To show the consistency of $\overline{F}^*(x)$ and its rate, see Ahmad (2004).

5 Testing against UBAC alternatives

Testing exponentially against the classes of life distribution has seen a good deal of attention. For testing against IHR, we refer to Barlow and Proschan(1981) and Ahmad (1994), among others. While testing against DMRL see Ahmad(1992). Finally testing against UBA see Ahmad (2004).

Let $X_1, X_2, \ldots, X_n$ represent a random sample from a population with distribution F . We wish to test the null hypothesis $H_0 : \overline{F}$ is exponential with mean μ against $H_1 : \overline{F}$ is UBAC and not exponential. Using theorem (2.1), we may use the following δ_M as a measure of departure from H_0 against of H_1:

$$\delta_M(r) = \frac{\gamma^{r+2}\mu_{(2r+3)}}{(2r+3)!} - \frac{\mu_{(r+1)}}{(r+1)!}. \qquad (1)$$

Note that under $H_0 : \delta_M = 0$, while under $H_1 : \delta_M > 0$. Thus to estimate δ_M by $\hat{\delta}_{M_n}$, let $X_1, X_2, \ldots, X_n$ be a random sample from F , $\hat{\gamma} = \frac{n}{\sum X_i}$ is the estimate of γ and μ is estimated by $\overline{X}$, where $\overline{X} = \frac{1}{n}\sum X_i$ is the usual sample mean . Then $\hat{\delta}_{M_n}$ is given by using (5.1) as

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$$\hat{\delta}_{M_n} = \frac{1}{n}\sum_i \left\{ \gamma^{r+2} \frac{X_i^{2r+3}}{(2r+3)!} - \frac{X_i^{r+1}}{(r+1)!} \right\}. \quad (2)$$

to make the test statistic scale invariant, we use

$$\Delta_{M_n} = \frac{\hat{\delta}_{M_n}}{\mu^{2r+3}}.$$

which is estimated by

$$\hat{\Delta}_{M_n} = \frac{\hat{\delta}_{M_n}}{\overline{X}^{2r+3}}. \quad (3)$$

Setting $\phi(X_1) = \frac{\gamma^{r+2} X_1^{2r+3}}{(2r+3)!} - \frac{X_1^{r+1}}{(r+1)!}$, then $\hat{\Delta}_{M_n}$ in (5.3) is a U-statistic, cf. Lee (1990). The following theorem summarizes the large sample properties of $\hat{\Delta}_{M_n}$.

Theorem 5.1. As $n \to \infty, \sqrt{n}(\hat{\Delta}_{M_n} - \Delta_M)$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = var[\gamma^{r+2} \frac{X_1^{2r+3}}{(2r+3)!} - \frac{X_1^{r+1}}{(r+1)!}]. \quad (4)$$

Under $H_0 : \Delta_M = 0$ and variance σ_0^2 is given by

$$\sigma_0^2(r) = \frac{(4r+6)!}{((2r+3)!)^2} + \frac{(2r+2)!}{((r+1)!)^2} - \frac{2(3r+4)!}{(2r+3)!(r+1)!}. \quad (5)$$

Proof: Since $\hat{\Delta}_{M_n}$ and $\frac{\hat{\delta}_{M_n}}{\mu}$ have the same limiting distribution, we use $\sqrt{n}(\hat{\delta}_{M_n} - \delta_{M_n})$. Now this is asymptotically normal with mean 0 and variance

$$\sigma^2 = var[\phi(X_1)]$$

, where

$$\phi(X_1) = \frac{\gamma^{r+2} X_1^{2r+3}}{(2r+3)!} - \frac{X_1^{r+1}}{(r+1)!}.$$

Than (5.4) follows. Under H_0: $\Delta_M = E(\phi(X_1)) = 0$ and

$$\sigma_0^2(r) = E[\frac{X_1^{2r+3}}{(2r+3)!} - \frac{X_1^{r+1}}{(r+1)!}]^2. \quad (6)$$

Hence (3.5) follows. The Theorem is proved. When $r = 0$,

$$\delta_M = \frac{\gamma^2 \mu_{(3)}}{6} - \mu, \quad (7)$$

in this case $\sigma_0^2 = 14$ and the test statistic

$$\hat{\delta}_{M_n} = \frac{1}{6n}\sum_i \left\{ X_i^3 - 6X_i \right\} \quad (8)$$

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and

$$\hat{\Delta}_{M_n} = \frac{\hat{\delta}_{M_n}}{\overline{X}^3}, \tag{9}$$

which is quite simple statistics. When $r = 1$,

$$\delta_M(1) = \frac{\gamma^3 \mu_{(5)}}{5!} - \frac{\mu}{2}, \tag{10}$$

in this case $\sigma_0^2 = 196$ and the test statistic

$$\hat{\delta}_{M_n}(1) = \frac{1}{60n} \sum_i \left\{ X_i^5 - 30 X_i^2 \right\} \tag{11}$$

and

$$\hat{\Delta}_{M_n} = \frac{\hat{\delta}_{M_n}}{\overline{X}^5}, \tag{12}$$

which is quite simple statistics.

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{M_n}\sigma_0$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$. To illustrate the test, we calculate, via Monte Carlo Method, the empirical critical points of $\hat{\Delta}_{M_n}$ in (5.9) for sample sizes 5(5)50. Tables (5.1) gives the upper percentile points for 95%, 98%, 99% . The calculations are based on 10000 simulated samples sizes $n =$ 5(1)50.

Table (5.1) Critical Values of $\hat{\Delta}_{M_n}$ in(5.9)

n	95%	98%	99%
5	0.5089	0.8445	1.1248
10	0.7473	1.2414	1.7054
15	0 .7782	1.2563	1.7017
20	0.80250	1.3375	1.7829
25	0.7678	1.2503	1.6892
30	0.7434	1.2715	1.7535
35	0.7231	1.1741	1.5095
40	0.7048	1.1054	1.4436
45	0.6898	1.0726	1.3674
50	0.6943	1.1478	1.3963

To asses how good this procedure is relative to others in the literatures, we use the concept of Pitman's asymptotic efficiency (PAE). To do this we need to evalute PAE of the proposed test and compare it with other tests. We may compare it with smaller classes such as (DMRL), and UBA . Here we choose the tests K^*, $\hat{\delta}_1$ and $\hat{\Delta}_{u_n}$ were presented by Hollander and Prochan (1975) and Ahmad (2004) respectively for decreasing mean residual life class (DMRL) and used better than aged (UBA) classes. Also we may compare with $\hat{\Delta}_{u_n}$ was presented by Abu-Youssef (2009) for UBAC class based on U-test statistics. Note that PAE of $\hat{\Delta}_{M_n}$

$$PAE(\Delta_u(\theta)) = \left\{ \frac{d}{d\theta} \Delta_M(\theta)|_{\theta \to \theta_0} \right\} / \sigma_0. \tag{13}$$

Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i) Linear failure rate family : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$

ii) Makeham family $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makham families. The PAE's of these alternatives of our procedure are, respectively:

$$PAE(\Delta_h, LFR) = -\frac{1}{2}(3r^2 + 11r + 10), \quad r \geq 0 \tag{14}$$

$$PAE(\Delta_u, Makeham) = -(r+2) - \frac{1}{2^{r+3}} + \frac{1}{2^{r+1}} \tag{15}$$

Direct calculations of PAE of K^*, $\hat{\Delta}_2$,$\hat{\Delta}_{u_n}$ and $\hat{\Delta}_{M_n}$ are summarized in

Table(5.2)

Distribution	K^*	$\hat{\delta}_2$	$\hat{\Delta}_{u_n}$	$\hat{\Delta}_{M_n}$
F_1 Linear failure rate	0.81	0.92	1.92	1.76
F_2 Makeham	0.29	0.51	0.57	0.43

From Table (5.2), the test statistic $\hat{\Delta}_{M_n}$ is more efficient than $\hat{\Delta}_2$ and K^* for linear failure rate family and Makeham family. But the test statistic $\hat{\Delta}_{M_n}$ is more efficient than $\hat{\Delta}_{u_n}$ for linear failure rate family. Hence our test which deals much larger classes UBAC is better and simpler.

Note that: Since $\hat{\Delta}_{M_n}$ defines a class (with parameter) r of test statistic, we choose r that the maximizes the PAE of that alternatives. If we take $r = 0$ then our test will have more efficiency than others. Finally, the power of the test statistics $\hat{\Delta}_{M_n}$ is considered for 95% percentiles in Table 5.3 for two of the most commonly used alternatives [see Hollander and Proschan (1975)], they are

(i) Linear failure rate family : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$

ii) Makeham family $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$

These distributions are reduced to exponential distribution for appropriate values of θ .

Table 5.3 Power Estimate of $\hat{\Delta}_{M_n}$

Distribuations	θ	Sample size		
		n=10	n=20	n=30
F_1 Linear failure rate	1	0.933	1.000	1.000
	2	1.000	1.000	1.000
	3	1.000	1.000	1.000
F_2 Makeham	1	0.544	0.524	0.920
	2	0.966	1.000	0.929
	3	1.000	1.000	1.000
		1.000	1.000	1.000

6 Applying the test

Consider real data representing 40 patients suffering from blood cancer.We use the data as given in Abu-Youssef (2009). The ordered life times (in day) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852. Using equation (5.9), the value of test statistics, based on the above data is $\hat{\Delta}_{M_n} = 0.0025$. This value leads to the acceptance of H_0 at the significance level $\alpha = 0.05$ see Table (5.1). Therefore the data has't UBAC Property. This is agreeing with the result of Abu-Youssef (2009) and the result of Ali and Abu-Youssef(2012).

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