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C'IRIC' TYPE $\Phi-\Psi$ - CONTRACTION RESULTS IN PARTIALLY ORDERED FUZZY METRIC SPACES

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Abstract: In this paper we establish a coupled coincidence point results for compatible mappings in a fuzzy metric spaces having a partial order relation defined on the space. The fuzzy metric space we use has a Hausdorff topology. The t-norm is used here is a Hadžic' type t-norm. Here we use C'iric' type inequality which has been considered in a good number of papers. Two control functions are also used. The main theorems have several corollaries. An illustrative example is given. The example establishes that the corollaries are actually contained the main theorem. We use a methodology which is a combination of order theoretic and analytic methods.

AMS Subject Classification: $47H10; 54H25$.

Keywords: Mixed monotone property; Hadžic' type t-norm; Φ-function; Ψ-function; Cauchy sequence; coupled coincidence point.

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1. INTRODUCTION

The purpose of this paper is to establish a coupled coincidence point theorem results in fuzzy metric spaces. Amongst various definitions of the fuzzy metric spaces we consider here the metric introduced by George et al [6]. Fuzzy fixed point theory has developed in the perspective of the metric space introduced in [6] in a very large way. This is supposed to have taken place due to the fact that the topology in this space is Hausdorff topology. A large number of papers have been written out of which [2, 4, 8, 9, 14, 15, 16] are some important examples. Fuzzy coupled fixed point result was successfully established first in the work[21]. After that, there have been a good number of papers on this subject, some examples being [11, 18, 19, 20].

Recently C′iri c′ [4] had introduced a fixed point theorem in which a new type of inequality was introduced. Later there appeared a number of works where this type of inequality has been considered.

In this paper we work out a coupled coincidence point theorem using the following concepts:

(a) C′iri c′ type inequality.

(b) Two control functions are used in the inequality.

(c) Compatibility condition.

(d) Hadži c′ type t-norm for fuzzy metric space.

We have several corollaries and an illustrative example. The example shows that the main theorem properly contains all its corollaries.

2. MATHEMATICALPRELIMINARIES

Throughout this paper $(X,^{\circ})$ stands for a partially ordered set with partial order . By '$x \pm y$ holds',we mean that '$x^{\circ}y$ holds' and by '$x \prec y$ holds', we mean that '$x^{\circ}y$ holds and $x \neq y$'.

Definition 2.1[17] A binary operation $*:[0,1]^2 \to 0,1]$ is called a t-norm if the following properties are satisfied:

- Is associative and commuative,
  - $a*1 = a$ for all $a \in [0,1]$,
  - $a*b \le c*d$ whenever $a \le c$ and $b \le d$, for all $a,b,c,d \in [0,1]$.

Typical examples of $t-$norms are $a*_1 b = \min\{a,b\}$, $a*_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $0<\lambda<1$, $a*_3 b = ab$ and $a*_4 b = \max\{a+b-1,0\}$.

Definition 2.2[6] The 3-tuple $(X,M,*)$ is called a fuzzy metric space in the sense of George and Veeramani if X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times 0,\infty)$ satisfying the following conditions for each $x,y,z \in X$ and $t,s>0$:

- $M(x,y,t) > 0$,
- $M(x,y,t) = 1$ if and only if $x = y$,
- $M(x,y,t) = M(y,x,t)$,
- $M(x,y,t)*M(y,z,s) \le M(x,z,t+s)$ and
- $M(x,y,.):(0,\infty) \to 0,1]$ is continuous.

The following details of this space are described in the introductory paper [6].

Let $(X,M,*)$ be a fuzzy metric space. For $t>0$, $0<r<1$, the open ball $B(x,r,t)$ with center $x \in X$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}.$$

A subset $A \subset X$ is open if for each $x \in A$, there exist $t>0$ and $0<r<1$ such that $B(x,r,t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X induced by the fuzzy metric M. This topology is

Hausdorff and first countable.

Example 2.3[6] Let $X = \mathrm{R}$. Let $a*b = a.b$ for all $a,b \in 0,1]$. For each $t \in (0,\infty)$ $x, y \in X$, let

$$M(x,y,t) = \frac{t}{t+|x-y|}.$$

Then $(\mathrm{R}, M,*)$ is a fuzzy metric space.

Example 2.4 Let (X,d) be a metric space and ψ be an increasing and continuous function of R_+ into such that $\lim_{t\to\infty}\psi(t)=1$. Let be any continuous t-norm For each $t \in (0,\infty)$, let

$$M(x,y,t) = \psi(t)^{d(x,y)}$$

for all $x, y \in X$. Then $(X,M,*)$ is a fuzzy metric space. Examples of the function ψ are

$$\psi(t) = \frac{t}{t+1}, \ \psi(x) = \sin(\frac{\pi t}{2t+1}) \text{ and } \psi(t) = 1-e^{-t}.$$

Lemma 2.5[7] Let $(X,M,*)$ be a fuzzy metric space. Then $M(x,y,.)$ is nondecreasing for all $x, y \in X$.

Definition 2.6[6] Let $(X,M,*)$ be a fuzzy metric space.

- A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
- A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathrm{N}$ such that $M(x_n, x_m, t) \geq 1-\varepsilon$ for each $n, m \geq n_0$.
- A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.7[16] M is a continuous function on $X^2 \times (0,\infty)$.

Let $(X,^\circ)$ be a partially ordered set and $F: X \to X$ be a mapping from X to itself. The mapping F is said to be non-decreasing if, for all $x_1, x_2 \in X$, $x_1{}^\circ x_2$ implies $F(x_1)^\circ F(x_2)$ and non-increasing if, for all $x_1, x_2 \in X$, $x_1{}^\circ x_2$ implies $F(x_1) \pm F(x_2)$ [1].

Definition 2.8[1] Let $(X,^\circ)$ be a partially ordered set and $F: X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is non-decreasing in its first argument and is non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $x_1{}^\circ x_2$ implies $F(x_1, y)^\circ F(x_2, y)$ for fixed $y \in X$ and if, for all $y_1, y_2 \in X$, $y_1{}^\circ y_2$ implies $F(x, y_1) \pm F(x, y_2)$, for fixed $x \in X$.

Definition 2.9[13] Let $(X,^\circ)$ be a partially ordered set and $F: X \times X \to X$ and $g: X \to X$ be two mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $gx_1{}^\circ gx_2$ implies $F(x_1, y)^\circ F(x_2, y)$ for all $y \in X$ and if, for all $y_1, y_2 \in X$, $gy_1{}^\circ gy_2$ implies $F(x, y_1) \pm F(x, y_2)$, for any $x \in X$.

Definition 2.10[1] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if

$$F(x,y) = x, F(y,x) = y.$$

Further Lakshmikantham and C'iric' have introduced the concept of coupled coincidence point.

Definition 2.11[13] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F: X \times X \to X$ and $g: X \to X$ if

$$F(x,y)=gx, F(y,x)=gy.$$

If further $x=gx=F(x,y)$ and $y=gy=F(y,x)$ then (x,y) is a common coupled fixed point of g and F.

Definition 2.12[13] Let X be a nonempty set and the mappings $F:X\times X\to X$ and $g:X\to X$ are commuting if for all $x,y\in X$

$$gF(x,y)=F(gx,gy).$$

Compatibility between two mappings $F:X\times X\to X$ and $g:X\to X$, where (X,d) is a metric space was defined in [3]. It is an extension of the commuting condition. Compatibility was used to obtained a coupled coincidence point result in the same work.

Definition 2.13[3] Let (X,d) be a metric space. The mappings F and g where $F:X\times X\to X$ and $g:X\to X$, are said to be compatible if

$$\lim_{n\to\infty} d(g(F(x_n,y_n)),F(g(x_n),g(y_n)))=0$$

and

$$\lim_{n\to\infty} d(g(F(y_n,x_n)),F(g(y_n),g(x_n)))=0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n)$ = $\lim_{n\to\infty} g(x_n)=x$ and $\lim_{n\to\infty} F(y_n,x_n)$ = $\lim_{n\to\infty} g(y_n)=y$ for some $x,y\in X$.

Intuitively we can think that the functions F and g commute in the limit in the situations where the functional values tend to the same point.

This notion of compatibility was introduced in fuzzy metric spaces by Hu in [11].

Definition 2.14[11] Let $(X,M,*)$ be a fuzzy metric space. The mappings F and g where $F:X\times X\to X$ and $g:X\to X$, are said to be compatible if for all $t>0$

$$\lim_{n\to\infty} M(g(F(x_n,y_n)),F(g(x_n),g(y_n),t)=1$$

and

$$\lim_{n\to\infty} M(g(F(y_n,x_n)),F(g(y_n),g(x_n),t)=1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n)$ = $\lim_{n\to\infty} g(x_n)=x$ and $\lim_{n\to\infty} F(y_n,x_n)$ = $\lim_{n\to\infty} g(y_n)=y$ for some $x,y\in X$.

We next give the following definition.

Definition 2.15[5] Two maps $F:X\times X\to X$ and $g:X\to X$, where X is a nonempty set, are weakly compatible pair if they commute at their coincidence point, that is, for any $x,y\in X$, $g(x)=F(x,y)$ and $g(y)=F(y,x)$ implies that $g(F(x,y))=F(g(x),g(y))$ and $g(F(y,x))=F(g(y),g(x))$

Definition 2.16[10] A t-norm is said to be Hadžic' type t-norm if the family $\{*^p\}_{p\in N}$ of its iterates defined for each $s\in(0,1)$ by

$*^0(s)=1$, $*^{p+1}(s)=*(*^p(s),s)$ for all $p\geq 0$ is equi continuous at $s=1$, that is, given $\lambda>0$ thereexists $\eta(\lambda)\in(0,1)$ such that

$$1\geq s>\eta(\lambda)\Rightarrow *^{(p)}(s)>1-\lambda \text{ for all } p\in\mathbb{N}.$$

We will use the following class of real mappings.

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Definition 2.17 (Ψ-function)[19] A function $\psi:[0,1]\times[0,1]\to[0,1]$ is said to be a Ψ-function if

- ψ continuous and monotone increasing in both the variables,
- $\psi(t,t)\geq t$ for all $0\leq t\leq 1$.

An example of ψ-function is

$$\psi(x,y)=\frac{p\sqrt{x}+q\sqrt{y}}{p+q}$$, p and q being positive numbers.

Definition 2.18 (Φ-function)[12] A function $\Phi=\{\phi:R^{+}\to R^{+}\}$, where $R^{+}=[0,\infty)$ and each $\phi\in\Phi$ satisfies the following conditions:

(i) ϕ is non-decreasing,

(ii) ϕ is upper semi-continuous from the right,

(iii) $\sum_{n=0}^{\infty}\phi^{n}(t)<+\infty$ for all $t>0$, where $\phi(\phi^{n}(t)), n\in N$.

It is easy to prove that if $\phi\in\Phi$ then $\phi(t)<t$ for all $t>0$.

Lemma 2.19[5] Let $(X,M,*)$ be a fuzzy metric space, where $*$ is a continuous t-norm of H-type. If there exists $\phi\in\Phi$ such that,

$M(x,y,\phi(t))\geq M(x,y,t),$ then $x=y.$

3. Main Results.

Theorem 3.1 Let $(X,M,*)$ be a complete fuzzy metric space with a Hadži c$'$ type t-norm where $M(x,y,t)$ is strictly increasing in the variable t and $M(x,y,t)\to 1$ as $t\to\infty$ for all $x,y\in X$. Let a partial order be defined on X. Let $F:X\times X\to X$ and $g:X\to X$ be two mappings such that F has the mixed g-monotone property. Let there exist $\phi\in\Phi$, $\psi\in\Psi$ and $q\geq 0$ such that,

$$M(F(x,y),F(u,v),\phi(t))+q(1-\max\{M(g(x),F(u,v),\phi(t)),M(g(u),F(x,y),\phi(t))\}) \geq\psi(M(g(x),F(x,y),t),M$$, (3.1)

for all $t>0$ and $x,y,u,v\in X$, with $gx^{\circ}gu$ and $gy\pm gv$. Let g be continuous, monotonic increasing $F(X\times X)\subseteq g(X)$ and (g,F) is a compatible pair. Also suppose that X has the following properties:

(a) if a non-decreasing sequence $\{x_n\}\to x$, then $x_n{}^{\circ}x$ for all $n\geq 0$, (3.2)

(b) if a non-increasing sequence $\{y_n\}\to y$, then $y_n\pm y$ for all $n\geq 0$. (3.3)

If there exist $x_0,y_0\in X$ such that $g(x_0)^{\circ}F(x_0,y_0)$ and $g(y_0)\pm F(y_0,x_0)$, then there exist $x,y\in X$ such that $g(x)=F(x,y)$ and $g(y)=F(y,x)$, that is, g and F have a coupled coincidence point in X.

Proof. Let x_0,y_0 be two points in X such that $g(x_0)^{\circ}F(x_0,y_0)$ and $g(y_0)\pm F(y_0,x_0)$. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$g(x_1)=F(x_0,y_0)$ and $g(y_1)=F(y_0,x_0)$

$g(x_2)=F(x_1,y_1)$ and $g(y_2)=F(y_1,x_1)$

and, in general, for all $n\geq 0$,

$g(x_{n+1})=F(x_n,y_n)$ and $g(y_{n+1})=F(y_n,x_n)$. (3.4)

This construction is possible by the condition $F(X \times X) \subseteq g(X)$.

Next, for all $n \geq 0$, we prove that

$$g(x_n)^\circ g(x_{n+1}) \qquad (3.5)$$

and

$$g(y_n) \pm g(y_{n+1}). \qquad (3.6)$$

Since $g(x_0)^\circ F(x_0, y_0)$ and $g(y_0) \pm F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0)^\circ g(x_1)$ and $g(y_0) \pm g(y_1)$. Therefore (3.5) and (3.6) hold for $n = 0$.

Let (3.5) and (3.6) hold for some $n = m$. As F has the mixed g-monotone property, $g(x_m)^\circ g(x_{m+1})$ and $g(y_m) \pm g(y_{m+1})$, from (3.4), we get

$$g(x_{m+1}) = F(x_m, y_m)^\circ F(x_{m+1}, y_m) \text{ and } F(y_{m+1}, x_m)^\circ F(y_m, x_m) = g(y_{m+1}). \qquad (3.7)$$

Also, for the same reason, we have

$$g(x_{m+2}) = F(x_{m+1}, y_{m+1}) \pm F(x_{m+1}, y_m) \text{ and } F(y_{m+1}, x_m) \pm F(y_{m+1}, x_{m+1}) = g(y_{m+2}). \qquad (3.8)$$

Then, from (3.7) and (3.8), we have

$$g(x_{m+1})^\circ g(x_{m+2}) \text{ and } g(y_{m+1}) \pm g(y_{m+2}).$$

Then, by induction, it follows that (3.5) and (3.6) hold for all $n \geq 0$.

Since $*$ is a t-norm of Hadži c$'$ type, for any $\lambda > 0$, there exists an $\mu > 0$ such that

$$(1-\mu) * (1-\mu) * \ldots\ldots\ldots * (1-\mu)(k \ \ times) \geq 1-\lambda \text{ for all } k \in N.$$

Since $M(x, y, .)$ is continuous and $\lim_{n\to\infty} M(x, y, t) = 1$ for all $x, y \in X$, there exists $t_0 > 0$ such that

$$M(g(x_0), g(x_1), t_0) > 1-\mu, \ M(g(y_0), g(y_1), t_0) \geq 1-\mu. \qquad (3.9)$$

On the other hand, since $\phi \in \Phi$, by property (iii) of Φ-function, we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$, such that

$$t > \sum_{n=n_0}^{\infty} \phi^n(t_0). \qquad (3.10)$$

Now, for all $t > 0$, $n \geq 1$, we have,

$$M(g(x_n), g(x_{n+1}), \phi(t_0)) = M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), \phi(t_0))$$

$$\geq \psi(M(g(x_{n-1}), F(x_{n-1}, y_{n-1}), t_0), M(g(x_n), F(x_n, y_n), t_0))$$

$$-q(1-\max\{M(g(x_{n-1}), F(x_n, y_n), \phi(t_0)), M(g(x_n), F(x_{n-1}, y_{n-1}), \phi(t_0))\})$$

(by(3.1), (3.5) and (3.6))

$$\geq \psi(M(g(x_{n-1}), g(x_n), t_0), M(g(x_n), g(x_{n+1}), t_0))$$

$$-q(1-\max\{M(g(x_{n-1}), g(x_{n+1}), \phi(t_0)), M(g(x_n), g(x_n), \phi(t_0))\})$$

$$\geq \psi(M(g(x_{n-1}), g(x_n), t_0), M(g(x_n), g(x_{n+1}), t_0)) \ -q(1-1)$$

$$\geq \psi(M(g(x_{n-1}), g(x_n), t_0), M(g(x_n), g(x_{n+1}), t_0)). \qquad (3.11)$$

If, for some $s>0$, and for some n, $M(g(x_{n-1}),g(x_n),s)\geq M(g(x_n),g(x_{n+1}),s)$, then, from the above inequality, and using the properties of Ψ, we obtain

$$M(g(x_{n+1}),g(x_n),\phi(s))\geq\psi(M(g(x_n),g(x_{n+1}),s),M(g(x_{n+1}),g(x_n),s))$$

$$\geq M(g(x_n),g(x_{n+1}),s).$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$M(g(x_n),g(x_{n+1}),\phi(s))>M(g(x_{n-1}),g(x_n),s)$ for all $n>0$.

Thus, for all $n>0$ and $t>0$, we have

$M(g(x_n),g(x_{n+1}),\phi(t_0))\geq\psi(M(g(x_{n-1}),g(x_n),t_0),M(g(x_{n-1}),g(x_n),t_0))$,

that is, for all $n>0$, $t_0>0$, we have

$M(g(x_n),g(x_{n+1}),\phi(t_0))\geq M(g(x_{n-1}),g(x_n),t_0)$, (using the properties of Ψ - function). (3.12)

Therefore for $n=1$, we have

$M(g(x_1),g(x_2),\phi(t_0))\geq M(g(x_0),g(x_1),t_0)$. (by using (3.12)) (3.13)

Now,

$$M(g(x_2),g(x_3),\phi^2(t_0))=M(F(x_1,y_1),F(x_2,y_2),\phi^2(t_0))$$

$$\geq\psi\{M(g(x_1),F(x_1,y_1),\phi(t_0)),M(g(x_2),F(x_2,y_2),\phi(t_0))\}$$

$$\geq\psi\{M(g(x_1),g(x_2),\phi(t_0)),M(g(x_2),g(x_3),\phi(t_0))\}$$

$$-q\{1-\max(M(g(x_1),F(x_2,y_2),\phi^2(t_0))),M(g(x_2),F(x_1,y_1),\phi^2(t_0))))\}$$

$$=\psi\{M(g(x_1),g(x_2),\phi(t_0)),M(g(x_2),g(x_3),\phi(t_0))\}$$

$$-q\{1-\max(M(g(x_1),F(x_2,y_2),\phi^2(t_0))),M(g(x_2),g(x_2),\phi^2(t_0))))\}$$

$$=\psi\{M(g(x_1),g(x_2),\phi(t_0)),M(g(x_2),g(x_3),\phi(t_0))\}$$

$$-q\{1-1\}$$

$$\geq\psi\{M(g(x_0),g(x_1),t_0),M(g(x_0),g(x_1),t_0)\}(by\ \ using\ \ (3.12)\ \ and\ \ (3.13))$$

$$\geq M(g(x_0),g(x_1),t_0).$$

By repeated application of the above inequality, it is easy to prove that

$M(g(x_n),g(x_{n+1}),\phi^n(t_0))\geq M(g(x_0),g(x_1),t_0)$. (3.14)

So, from (3.9) and (3.10) for $m>n\geq n_0$, we have,

$$M(g(x_n),g(x_m),t)\geq M(g(x_n),g(x_m),\sum_{k=n_0}^{\infty}\phi^k(t_0))$$

$$\geq M(g(x_n),g(x_m),\sum_{k=n}^{m-1}\phi^k(t_0))$$

$$\geq M(g(x_n),g(x_m),\phi^k(t_0))$$

$\geq M(g(x_n), g(x_{n-1}), \phi^n(t_0)) * M(g(x_{n-1}), g(x_{n-2}), \phi^{(n-1)}(t_0))$

$* M(g(x_{n-2}), g(x_{n-3}), \phi^{(n-2)}(t_0)) * .... * M(g(x_{m-1}), g(x_{m-2}), \phi^{(m-1)}(t_0))$

$\geq M(g(x_0), g(x_1), t_0) * M(g(x_0), g(x_1), t_0) * .... * M(g(x_0), g(x_1), t_0)\ (by\ (3.14))$

$\geq (1-\mu) * (1-\mu) * .... * (1-\mu)$

$\geq 1-\lambda,$ which implies for all $m, n \in N$ with $m > n \geq n_0$ and $t > 0$,

$M(g(x_n), g(x_m), t) \geq 1-\lambda$

So, $\{g(x_n)\}$ is a Cauchy sequence.

Similarly we can show that $\{g(y_n)\}$ is a Cauchy sequence.

Since X is complete, there exist $x, y \in X$ such that

$\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} g(y_n) = y$,

that is, $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y.$ (3.15)

Now we show that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Since (g, F) is a compatible pair, using continuity of g and Definition 2.14, we have

$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(g(x_n), g(y_n))$ (3.16)

and

$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(g(y_n), g(x_n))$. (3.17)

By (3.5), (3.6) and (3.15), we have that $\{g(x_n)\}$ is a non-decreasing sequence with $g(x_n) \to x$ and $\{g(y_n)\}$ is a non-increasing sequence with $g(y_n) \to y$ as $n \to \infty$. Then, by (3.2) and (3.3), it follows that, for all $n \geq 0$,

$g(x_n)^{\circ} x$ and $g(y_n) \pm y$.

Since g is monotonic increasing, we have from the above inequality,

$g(g(x_n))^{\circ} g(x)$ and $g(g(y_n)) \pm g(y)$. (3.18)

Now, for all $t > 0$, $n \geq 0$, we have

$M(g(x), F(x, y), \phi(t)) \geq M(g(x), g(g(x_{n+1})), (\phi(t) - \phi(kt)))$

$* M(g(g(x_{n+1})), F(x, y), \phi(kt)).$

Taking $n \to \infty$ on the both sides of the above inequality, for all $t > 0$,

$M(F(x, y), g(x), \phi(t)) \geq \lim_{n\to\infty} [M(g(x), g(g(x_{n+1})), (\phi(t) - \phi(kt)))$

$* M(g(g(x_{n+1})), F(x, y), \phi(kt))]$

$= M(g(x), \lim_{n\to\infty} g(g(x_{n+1})), (\phi(t) - \phi(kt)))$

$* M(\lim_{n\to\infty} g(F(x_n, y_n)), F(x, y), \phi(kt))$

(by lemma 2.7)

$= M(g(x), g(x), (\phi(t) - \phi(kt)))$

$* M(\lim_{n\to\infty}(F(g(x_n), g(y_n))), F(x, y), \phi(kt))$

(by (3.16))

= $\lim_{n\to\infty}[1 * M(F(g(x_n), g(y_n)), F(x, y), \phi(kt))]$ (by lemma 2.7)

= $\lim_{n\to\infty} M(F(g(x_n), g(y_n)), F(x, y), \phi(kt))$

$\geq \lim_{n\to\infty}[\psi(M(g(g(x_n)), F(g(x_n), g(y_n)), kt), M(gx, F(x, y), kt))$

$- q(1 - \max\{M(g(gx_n), F(x, y), \phi(kt)), M(gx, F(g(x_n), g(y_n)), \phi(kt))\})].$

(by (3.1) and (3.18))

$= [\psi(M(\lim_{n\to\infty} g(g(x_n)), \lim_{n\to\infty} F(g(x_n), g(y_n), kt), M(gx, F(x, y), kt))$

$- q(1 - \max\{M(\lim_{n\to\infty} g(gx_n), F(x, y), \phi(kt)), M(gx, \lim_{n\to\infty} F(g(x_n), g(y_n)), \phi(kt))\})].$

$= \psi(M(g(x), g(x), t), M(g(x), F(x, y), kt)$

$- q(1 - \max\{M(g(x), F(x, y), \phi(kt)), M(g(x), g(x), \phi(kt))\})$

$= \psi(1, M(g(x), F(x, y), kt)) - q(1 - 1)$

$\geq \psi(M(g(x), F(x, y), kt), M(g(x), F(x, y), kt))$

(since ψ is monotone increasing)

$\geq M(g(x), F(x, y), kt).$ (by the property of Ψ -function)

Taking $k \to 1$ and by an application of lemma 2.19, we have

$g(x) = F(x, y)$.

Similarly, we can prove $g(y) = F(y, x)$.

This completes the proof of the theorem.

Theorem 3.2 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadžić type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for

all $x, y \in X$. Let a partial order   be defined on X. Let $F: X \times X \to X$ and $g: X \to X$ be

two mappings such that F has the mixed g -monotone property. Let there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that,

$M(F(x, y), F(u, v), \phi(t)) \geq \psi(M(g(x), F(x, y), t), M(g(u), F(u, v), t))$,

for all $t > 0$ and $x, y, u, v \in X$, with $g(x) ° g(u)$ and $g(y) \pm g(v)$. Let g be continuous, monotonic increasing $F(X \times X) \subseteq g(X)$ and (g, F) is a compatible pair. Also suppose that X has the following properties:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n ° x$ for all $n \geq 0$

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \pm y$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) ° F(x_0, y_0)$ and $g(y_0) \pm F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, g and F have a coupled coincidence point in X. Further, if (g, F) is a weakly compatible pair and $g(x) ° g(g(x))$, $g(y) \pm g(g(y))$ whenever (x, y) is a coincidence point of g and F, then g and F have a common coupled fixed point.

Proof. Putting $q=0$ in (3.1), by an application of the theorem 3.1, we have that g and F have a coupled coincidence point in X, that is, there exist $x, y \in X$ such that $g(x)=F(x,y)$ and $g(y)=F(y,x)$.

Suppose $z_1 = g(x) = F(x,y)$ and $z_2 = g(y) = F(y,x)$.

Then $g(z_1) = g(g(x)) = g(F(x,y))$ and $g(z_2) = g(g(y)) = g(F(y,x))$.

Since g and F commute at (x,y), we have

$g(z_1) = g(g(x)) = F(g(x), g(y))$ and $g(z_2) = g(g(y)) = F(g(y), g(x))$,

that is, $g(z_1) = g(g(x)) = F(z_1, z_2)$ and $g(z_2) = g(g(y)) = F(z_2, z_1)$.

Since (x,y) is a coupled coincidence point, by our assumption, $g(x)^{\circ}g(g(x))$, that is, $g(x)^{\circ}g(z_1)$ and $g(y) \pm g(g(y))$, that is, $g(y) \pm g(z_2)$, it follows that, for all $t>0$,

$M(z_1, F(z_1,z_2), \phi(t)) = M(F(x,y), F(z_1,z_2), \phi(t)) \quad \geq \psi(M(g(x), F(x,y), t), M(g(z)_1, F(z_1,z_2), t)) \quad (\mathit{since}\ g(x)^{\circ}$
(by the property of ψ -function),

which implies that, for all $t>0$, $M(z_1, F(z_1,z_2), \phi(t)) = 1$,

that is, $z_1 = F(z_1, z_2)$.

Therefore $z_1 = g(z_1) = F(z_1, z_2)$.

Similarly, $z_2 = g(z_2) = F(z_2, z_1)$.

This proves g and F have a common coupled fixed point.

If we take $\phi(t) = kt$ in the theorem 3.1, then we obtain the following corollary.

Corollary 3.3 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadžic$'$ type t-norm where $M(x,y,t)$ is strictly increasing in the variable t and $M(x,y,t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let a partial

order be defined on X. Let $F: X \times X \to X$ and $g: X \to X$ be two mappings such that F

F has the mixed g -monotone property. Let there exists $\psi \in \Psi$ and $q \geq 0$ such that,

$$M(F(x,y), F(u,v), kt) + q(1 - \max\{M(g(x), F(u,v), kt), M(g(u), F(x,y), kt)\}) \quad \geq \psi(M(g(x), F(x,y), t), M(g(u), F(u,v), t))$$

for all $t>0$ and $x, y, u, v \in X$, with $g(x)^{\circ}g(u)$ and $g(y) \pm g(v)$ and $0<k<1$. Let g be continuous, monotonic increasing $F(X \times X) \subseteq g(X)$ and (g, F) is a compatible pair. Also suppose that X has the following properties:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n{}^{\circ}x$ for all $n \geq 0$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \pm y$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $gx_0{}^{\circ}F(x_0, y_0)$ and $gy_0 \pm F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$, that is, g and F have a coupled coincidence point in X.

If we take $\phi(t) = kt$ in theorem 3.2, then we obtain the following corollary.

Corollary 3.4 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadžic$'$ type t-norm where $M(x,y,t)$ is strictly increasing in the variable t and $M(x,y,t) \to 1$ as $t \to \infty$ for all

$x, y \in X$. Let a partial order be defined on X. Let $F: X \times X \to X$ and $g: X \to X$ be two

mappings such that F has the mixed g -monotone property. Let there exists $\psi \in \Psi$ such that,

$$M(F(x,y), F(u,v), kt) \geq \psi(M(gx, F(x,y), t), M(gu, F(u,v), t)),$$

for all $t > 0$ and $x, y, u, v \in X$, with $gx^\circ gu$ and $gy \pm gv$ and $0 < k < 1$. Let g be continuous, monotonic increasing $F(X \times X) \subseteq g(X)$ and (g, F) is a compatible pair. Also suppose that X has the following properties:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n{}^\circ x$ for all $n \geq 0$

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \pm y$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $gx_0{}^\circ F(x_0, y_0)$ and $gy_0 \pm F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, g and F have a coupled coincidence point in X. Further, if (g, F) is a weakly compatible pair and $gx^\circ g(g(x))$, $gy \pm g(g(y))$ whenever (x, y) is a coincidence point of g and F, then g and F have a common coupled fixed point.

Corollary 3.5 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadžic′ type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for

all $x, y \in X$. Let a partial order be define on X. Let $F : X \times X \to X$ be a mapping such

that F has the mixed monotone property and satisfies the following condition:

- $M(F(x, y), F(u, v), \phi(t)) \geq \psi(M(x, F(x, y), t), M(u, F(u, v), t))$,

for all $t > 0, x, y, u, v \in X$ with $x^\circ u$ and $y \pm v$, where $\phi \in \Phi$ and ψ is a ψ-function. Also suppose that X has the following properties:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n{}^\circ x$ for all $n \geq 0$,

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \pm y$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $x_0{}^\circ F(x_0, y_0)$ and $y_0 \pm F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in X.

Proof. The proof follows by putting $q = 0$ and $g(x) = I$, the identity mapping, in the theorem 3.1.

Example 3.6 Let $(X, {}^\circ)$ be the partially ordered set with $X = [0,1]$ with the natural ordering

$\leq$ of the real numbers as the partially ordering. Let $M(x, y, t)(t) = e^{-\frac{|x-y|}{t}}$ for all $x, y \in X$ and $a * b = \min\{a, b\}$, then $(X, M, *)$ is a complete fuzzy metric space.

Let the mapping $g : X \to X$ be defined as

$gx = x^2$ for all $x \in X$

and the mapping $F : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} \dfrac{x^2 - y^2}{8}, & \text{if} x, y \in [0,1], x \geq y, \\ 0, & \text{otherwise.} \end{cases}$$

Here F satisfies the mixed g-monotone property. Also, $F(X \times X) \subseteq g(X)$. Let $\psi\{x, y\} = \min\{x, y\}$ and $\phi(t) = \frac{1}{2} t$.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$\lim_{n \to \infty} F(x_n, y_n) = a$, $\lim_{n \to \infty} g(x_n) = a$,

$\lim_{n \to \infty} F(y_n, x_n) = b$ and $\lim_{n \to \infty} g(y_n) = b$.

Now, for all $n \geq 0$,

$$g(x_n) = x_n^2, \quad g(y_n) = y_n^2,$$

$$F(x_n, y_n) = \frac{x_n^2 - y_n^2}{8}$$

and

$$F(y_n, x_n) = \frac{y_n^2 - x_n^2}{8}.$$

Then necessarily $a = 0$ and $b = 0$.

It then follows from lemma 2.7 that, for all $t > 0$,

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(gy_n), g(x_n)), t) = 1.$$

Therefore the mappings F and g are compatible in X.

Let $x_0 = 0$ and $y_0 = c > 0$.

Then $gx_0 = g0 = 0 = F(0,c) = F(x_0, y_0)$ and $gy_0 = gc = c^2 > \frac{c^2}{8} = F(c,0) = F(y_0, x_0)$.

Thus x_0 and y_0 satisfy their requirements in theorem 3.1. Let $x, y, u, v \in X$ be such that $gx \leq gu$ and $gy \geq gv$, that is, such that $x^2 \leq u^2$ and $y^2 \geq v^2$.

We next show that the inequality (3.1) is satisfied for $q = 0$, for all $t > 0$ and x, y, u, v chosen to satisfy the above requirements.

The following three cases are possible.

Case I. $x \geq y$ and $u \geq v$,

$$M(F(x,y), F(u,v), \phi(t)) = e^{-\frac{|F(x,y)-F(u,v)|}{\phi(t)}}$$

$$= e^{-\frac{\frac{|x^2-y^2-u^2+v^2|}{8}}{\frac{1}{2}t}}$$

$$= e^{-\frac{|x^2-y^2-u^2+v^2|}{4t}}$$

$$= e^{-\frac{|(x^2-u^2)-(y^2-v^2)|}{4t}}$$

$$\geq e^{-\frac{|x^2-u^2|+|y^2-v^2|}{4t}} \qquad \text{(since } |x-y| < |x| + |y|\text{)}$$

$$\geq e^{-\frac{|x^2-v^2|+|x^2-v^2|}{4t}} \qquad \text{(since } x^2 \geq y^2 \text{ and } u^2 \geq v^2\text{)}$$

$$\geq e^{-\frac{|x^2-v^2|}{2t}}$$

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$$\geq e^{-(\frac{x^2}{t}-\frac{x^2-y^2}{8t})} \qquad \text{(since } (\frac{x^2}{t}-\frac{x^2-y^2}{8t}-\frac{|x^2-v^2|}{2t})>0)$$

$$\geq \min\{e^{-\frac{|x^2-F(x,y)|}{t}}, e^{-\frac{|u^2-F(u,v)|}{t}}\}$$

$$=\psi\{e^{-\frac{|x^2-F(x,y)|}{t}}, e^{-\frac{|u^2-F(u,v)|}{t}}\}.$$

Case II. $x<y$ and $u \geq v$,

$$M(F(x,y),F(u,v),\phi(t)) = e^{-\frac{|0-F(u,v)|}{\phi(t)}}$$

$$=e^{-\frac{|u^2-v^2|}{4t}}$$

$$=e^{-\frac{|(u^2-v^2)|}{4t}}$$

$$\geq e^{-(\frac{u^2}{t}-\frac{u^2-v^2}{8t})} \qquad \text{(since } (\frac{u^2}{t}-\frac{u^2-v^2}{6t}-\frac{u^2-v^2}{4t})>0)$$

$$\geq \min\{e^{-\frac{|x^2-F(x,y)|}{t}}, e^{-\frac{|u^2-F(u,v)|}{t}}\}$$

$$=\psi\{e^{-\frac{|x^2-F(x,y)|}{t}}, e^{-\frac{|u^2-F(u,v)|}{t}}\}.$$

Case III. $x<y$ and $u<v$.

In this case the inequality (3.1) for $q=0$ is trivially satisfied.

Taking into account all the three cases mentioned above, we conclude that the inequality (3.1) with any value of $q \geq 0$ is satisfied by x,y,u,v chosen according to the conditions given in theorem 3.1 and for all $t>0$. Thus all the conditions of theorem 3.1 are satisfied. Then, by an application of the theorem 3.1, we conclude that g and F have a coupled coincidence point. Here $(0,0)$ is a coupled coincidence point of g and F in X.

Remark 3.7 The result in theorems 3.1 and 3.2 remains valid if we omit the condition that $M(x,y,t)$ is strictly monotonic increasing in t and at the same time modify the definition of ψ by requiring that $\psi(t,t)>t$ for all $0<t<1$. Then, by lemma 2.5, $M(x,y,t)$ is nondecreasing in t which is enough requirement for our purposes.

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