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Max-fully cancellation modules

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Abstract:-

Let R be a commutative ring with identity and let M be a unital an R`-module. We introduce the concept of max-fully cancellation R-module , where an R-module M is called max-fully cancellation if for every nonzero maximal ideal I of R and every two submodulesN_1 andN_2 , of M such that IN_1=IN_2 , implies $N_1 = N_2$. some characterization of this concept is given and some properties of this concept are proved. The direct sum and the trace of module with max-fully cancellation modules are studied , also the localization of max-fully cancellation module are discussed..

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INTRODUCTION:-

Throughout this thesis all rings are commutative rings with unity and all modules are unital modules. Gilmer in [14] introduced the concept of cancellation ideal , where an ideal I of a ring R is said to be cancellation if whenever AI=BI with A and B are ideals of R , implies A=B . Also ,D.D. Anderson and D.F. Anderson in[3] ,studied the concept of cancellation ideals . In 1992 A.S. Mijbass in [16], gave the generalization of this concept namely cancellation module (weakly cancellation module) , where an R-module M is called cancellation (weakly cancellation) if wherever I and J are two ideals of R , with IM=JM implies I=J (I+M =J+M)

Inaam, M .A .Hadi, A.A. Elewi in [5], introduced the concept of fully cancellation module , where an R-module M is called fully cancellation module if for each ideal I of R and for each submodules N_1, N_2 , of M such that $IN_1=IN_2$, implies $N_1=N_2$.

In section One , we introduce the definition of max-fully cancellation module and we give some characterizations for a module to be max-fully cancellation module , see proposition(1.7), also many propositions and results related with this concept are given.

In section two, we study the direct sum of max-fully cancellation modules and many of important results are given , see proposition (2.2) , proposition(2.7) and proposition(2.8).

In section three , we study the behavior of max-fully cancellation modules under localization . we show that : M is max-fully cancellation R-module if and only if M is locally max-fully cancellation, see proposition (3.5).

In section four , we discuss the relationship between max-fully cancellation module and its trace T(M). However in class of multiplication and projective module we give a condition on T(M) under which M be max-fully cancellation module , see proposition (4.10),also we prove that the max-fully cancellation module and its trace are equivalent under certain condition , see proposition (4.11)

§1 max-fully cancellation modules

In this section ,we introduce the concept of max-fully cancellation module as a generalization of fully cancellation module .we give some characterizations and establish some basic properties of this concept .

We introduce the following definition

Definition(1.1)[4]:-

Let I be a proper ideal of a ring .Then I is said to be **maximal ideal** of R , if there exists an ideal J of R such that $I \subsetneq J \subseteq R$ then $J = R$.

proposition (1.2)[4] :-

(1) Every proper ideal is contained in a maximal ideal .

(2) Every commutative ring with identity contains maximal ideal.

Definition(1.3):-

An R-module M is called **faithful** , if ann_R(M)=0,where ann_R(M)={r∈ R:rm=0 ∀m∈ M} .

Definitio(1.4)[15]:-

An R –module M is called **cancellation R-module**, if AM=BM ,where A and B two ideals of R ,then A=B

Proposition(1.5) [15]:-

(1) Every cancellation R-module is faithful .

(2) If M is multiplication faithful finite generated Then M is cancellation.

Defi1nition (1.6):-

An R-module M is called **max-fully cancellation module** if for every non zero maximal ideal Iof R and for every submodules N_1 and N_2 of M such that $IN_1 = IN_2$,then $N_1 = N_2$.

Remarks and Examples (1.7):-

(1) Z as a Z-module is max-fully cancellation module .

Since if we take I=pZ ;p is prime number

also , N_1=< x_1 >and N_2 =< x_2 >where x_1 ,$x_2 \in Z$.

Assume that $IN_1 = IN_2$, then$px_1Z = px_2Z$.

and hence p x_1 =p x_2a and $px_1 = pm_1b$,where a ,b ∈ Z.

Therefore $px_1 = px_1ab$,then ab =1 and hence either ab=1 or ab =-1 .

In each case we get $px_1 = px_2$ which implies $x_1=x_2$ and hence $N_1 = N_2$.

2) The Z-module Z_6 is not max-fully cancellation

Since ,if we take I=2Z ,$N_1=(\bar{2})$and $N_2 = Z_6$.

Then (2Z) $(\bar{2})$ =(2Z) Z_6 .But$(\bar{2}) \neq Z_6$

(3) every fully cancellation R-module is max-fully cancellation R-module .But the converse is not true in general .

For examples :

Consider $(\bar{3})$ as an R-module and R=Z_{24} .

Then$(\bar{3})$ is max-fully cancellation R-module .

Since , $(\bar{2})$ is maximal ideal of R and$(\bar{9})$, $(\overline{21})$ are two

submodules of $(\bar{3})$ such that $(\bar{2})$ $(\bar{9})$= $(\bar{2})$ $(\overline{21})$= $(\overline{18})$.

Then$(\bar{9})$ =$(\overline{21})$.

But it is not fully cancellation R-module .since , $(\bar{8})$ is an ideal of R and $(\bar{3})$, $(\bar{0})$ are two submodules of$(\bar{3})$ such that $(\bar{8})$ $(\bar{3})$ =$(\bar{8})$ $(\bar{0})$= $(\bar{0})$,but $(\bar{3}) \neq (\bar{0})$.

(4) The Z-module $Z_p^{\infty}$is not max-fully cancellation module.

Since , Q_p ={$\frac{m}{n}$,g.c.d(m ,n)=1 ;n=p^i ,i=1,2,3,…….} is a submodule of Q containing Z .

also , Z_p^{∞}= Q_p /Z ={x∈ Q ; x=$\frac{m}{P^i}$ +Z ;m∈ Z ,i=1,2,3,……}.

Let (P) be a maximal ideal of Z and ($\frac{1}{p}$ +Z) ,(0) be two submodules of Z_p^{∞}, then we have (P) ($\frac{1}{p}$ +Z) = (P) (0) ,

But ($\frac{1}{p}$ +Z)≠ (0).

(5)$Z_{12}$is not max-fully cancellation Z_{12}-module .

Let$(\bar{6})$, $(\bar{0})$ be two submodules of Z_{12} and $(\bar{2})$ be maximal ideal of Z_{12} .

Since $(\bar{2})$ $(\bar{6})$ = $(\bar{2})$ $(\bar{0})$ =$(\bar{0})$, But $(\bar{6}) \neq (\bar{0})$.

(6) The homomorphic image of the max-fully cancellation need not be max-fully cancellation module ,for example :-

We have from (1) that the Z-module Z is max-fully cancellation module .But Z/6Z≅Z_6 is not max-fully-cancellation Z-module by (2) .

(7) Every submodule N of max-fully cancellation R-module M is also max-fully cancellation .

Proof:-

Let I be a non zero maximal ideal of R such that$IN_1 = IN_2$.

where N_1 ,N_2 are any two submodules of N ,since N_1, N_2 are submodules of M and M is max-fully cancellation module ,then $N_1=N_2$.

$which$ implies that N is max-fully cancellation .

As an application of (7) ,we get the following results in (8) and (9) .

(8) The intersection of two R-submodules of M which are at least one of them is max-fully cancellation R-submodule is also ,max-fully cancellation .

Proof:-

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Let N_1 and N_2 be two submodules of an R-module M .It is known that $N_1 \cap N_2 \subseteq N_1$.

Also $N_1 \cap N_2 \subseteq N_2$,so according to (7) , $N_1 \cap N_2$ is max-fully cancellation .

As a generalization of (8) ,we get:

(9) If $\{N_k\}_{k=1}^{n}$ is a finite collection of a submodules of an R-module M and N_k is max-fully cancellation submodule for some k ,Then $\cap_{k=1}^{n} N_k$ is also max-fully cancellation .

Proof:-

The proof is by induction on n .

The following theorem is a characterization of max-fully cancellation modules .

Theorem(1.8):-

Let M be an R-module ,let N_1 ,N_2 are two submodule of M ,let I be a non zero maximal ideal of R Then the following statement are equivalent

(1) M is max-fully cancellation module .

(2) if $IN_1 \subseteq IN_2$ then $N_1 \subseteq N_2$.

(3) if $I \prec a \succ \subseteq IN_2$ then $a \in N_2$ where $a \in M$.

(4) $(IN_1 :_R IN_2) = (N_1 :_R N_2)$.

Proof :-

(1)$\Rightarrow$ (2).If $IN_1 \subseteq IN_2$ then $IN_2 = IN_1 + IN_2$ Which Implies

$IN_2 = I(N_1 + N_2)$, But M is max-fully cancellation module, then

$N_2 = (N_1 + N_2)$ and hence $N_1 \subseteq N_2$

$\Rightarrow$ (3)(1) $\because$ if $I \prec a \succ \subseteq IN_2$ then $\prec a \succ \subseteq N_2$ by (2)

Which implies ,$a \in N_2$.

(3) $\Rightarrow$ (1): If $IN_1 = IN_2$,To prove that $N_1 = N_2$.

Let $a \in N_1$ then $I \prec a \succ \subseteq IN_1 \subseteq IN_2$.

And hence $a \in N_2$ by (3)

Similarly ,we can show $N_2 \subseteq N_1$.

Thus $N_1 = N_2$.

(1) $\Rightarrow$ (4): let $r \in (IN_1 :_R IN_2)$ .Then $r\, IN_2 \subseteq IN_1$

So , $IrN_2 \subseteq IN_1$ and since (1) implies (2) , we have

$rN_2 \subseteq N_2$.

Thus $r \in (N_1 :_R N_2)$ and hence $(IN_1 :_R IN_2) \subseteq (N_1 :_R N_2)$.

Let $r \in (N_1 :_R N_2)$. Then $rN_2 \subseteq N_1$ which implies $IrN_2 \subseteq IN_1$.

And hence $rIN_2 \subseteq IN_1$. Therefore $r \in (IN_1 :_R IN_2)$ and hence $(N_1 :_R N_2) \subseteq (IN_1 :_R IN_2)$

Then we get $(N_1 :_R N_2) = (IN_1 :_R IN_2)$.

(4) $\Rightarrow$ (1): Let $IN_1 = IN_2$ Then by (4) $(IN_1 :_R IN_2) = (N_1 :_R N_2)$.

But $(IN_1 :_R IN_2)$=R (since $IN_1 = IN_2$).

Then $(N_1 :_R N_2)$ =R. so $N_2 \subseteq N_1$. .

Similarly $(IN_2 :_R IN_1) = (N_2 :_R N_1)$ .Thus $(N_2 :_R N_1)$ =R.

Which implies $N_1 \subseteq N_2$. Therefore $N_1 = N_2$.

Definition(1.9)[10]:-

An ideal I of a ring R is called **cancellation ideal** if AI= BI ,then A=B ,where A and B are two ideals of R ."

Proposition(1.10):-

Let M be a max fully cancellation R-module . If M is a cancellation module ,then every non zero maximal ideal of R is cancellation ideal .

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Proof:-

Let Ibe a nonzero maximal ideal of R ,such that $AI=BI$,where A ,B are two ideal of R .

Now ,we have $AIM = BIM$,then $IAM = IBM$But M is max-fully cancellation module, therefore AM=BM

and also ,M is cancellation module , then A=B.

which is what we wanted.

However ,we have the following result **.**

Proposition(1.11):-

Let M ,N be two R-module .If M$\cong N$,then M is max-fully cancellation module if and only if N is max-fully cancellation module.

Proof:-

Let $\theta: M \longrightarrow N$ be an isomorphism . Suppose M is max-fully cancellation module .

To prove N is max-fully cancellation module.

For every non zero maximal ideal I of R and every submodules$\grave{N}_1$,$\grave{N}_2$ of N .

Let $I\grave{N}_1$**=**$I\grave{N}_2$ **.**

Now , there exists two submodules N_1 ,N_2 of M such that

$\theta(N_1)=\grave{N}_1$, $\theta(N_2)=\grave{N}_2$.

Then $I\,\theta(N_1)= I\,\theta(N_2)$, Which implies $\theta(I\,N_1)= \theta(I\,N_2)$

Therefore $IN_1 = IN_2$ (since θ is (1-1)).

But M is max-fully cancellation R-module . Then $N_1=N_2$ and hence $\theta(N_1)= \theta(N_2)$.

Therefore $\grave{N}_1 =\grave{N}_2$.

That is N is max –fully cancellation R-module .

Conversely:

Suppose that N is max- fully cancellation R-module .

Let $IN_1 = IN_2$ for every non zero maximal ideal I of R and every submodules N_1 , N_2 of M .

Now , $\theta(I\,N_1)= \theta(I\,N_2)$.

Which implies $I\,\theta(N_1)= I\,\theta(N_2)$,where $\theta(N_1)$, $\theta(N_2)$ are two submodule of N .

Also N is max-fully cancellation module. Then $\theta(N_1)= \theta(N_2)$

Which implies $N_1=N_2$ (since θ is (1-1))

Which completes the proof .

Proposition(1.12):-

Let R be a principle ideal ring and M be an R-module such that ann(I)=0 for each non zero ideal I of R .Then M is max-fully cancellation module .

Proof:-

Let I be a non zero maximal ideal of R and N_1 ,$N_2$are submodules of M such that $IN_1 = IN_2$.

By assumption I=(x) ,for some x≠o ,x∈ R.

Therefore $(x)N_1 =(x)N_2$ .To prove $N_1=N_2$.

Let a∈N_1 .Then xa∈$(x)N_1=(x)N_2$ and hence xa=xb , for some b∈N_2.

Which implies that x(a-b)=0 and hence a-b∈ann(I) =0.

Therefore a-b=0. Thus a=b and hence $N_1=N_2$.

Which implies , M is max-fully cancellation module .

The converse of proposition (1.12) is not true in general

,for examples

The Z_6 –module Z_6 is max-fully cancellation module by remarks and examples (1.7) ,since Z_6 is principle ideal ring and $(\bar{2})$ is an ideal of Z_6 ,but ann$(\bar{2})\neq 0$.

The following lemma is needed in our next proposition

Lemma(1.13):-

Let R be any ring . I be a proper ideal of R such that ann(M)⊆ I .If I is maximal ideal of R ,then $\frac{I}{ann\,(M)}$ is maximal ideal of $\frac{R}{ann\,(M)}$.

Proof:-

Suppose that I is maximal ideal of R .

We want To prove that $\frac{I}{ann\,(M)}$ is maximal ideal in $\frac{R}{ann\,(M)}$.

Assume that there exists an ideal $\frac{J}{ann\,(M)}$ of $\frac{R}{ann\,(M)}$ such that

$\frac{I}{ann\,(M)} \subsetneq \frac{J}{ann\,(M)}$.

Then there exists x+ann(M)∈ $\frac{J}{ann\,(M)}$and x+ann(M)∉ $\frac{I}{ann\,(M)}$ which implies x∉ I. But I maximal ideal of R and x∉ I , then R=(I,X).

Therefore 1=a+rx where a∈ I,r∈ R,

Hence$\theta(1)$=$\theta(a)+\theta(rx)$. where $\theta: R \longrightarrow R/ann(M)$ natural homomorphism .

Then 1+ann(M)=(a+ann(M))+(r+ann(M))(x+ann(M)).

Thus 1+ann(M)∈ $\frac{J}{ann\,(M)}$ and hence $\frac{J}{ann\,(M)}$ =$\frac{R}{ann\,(M)}$.

Therefore $\frac{I}{ann\,(M)}$ is maximal ideal of $\frac{R}{ann\,(M)}$.

Conversely:- To prove that I is maximal ideal in R

Suppose that there exists an ideal J of R such that $I \subsetneq J$.

Then there exists x∈ J ,x∉ I which implies x+ann(M)∉ $\frac{I}{ann\,(M)}$ .But $\frac{I}{ann\,(M)}$ is maximal ideal in $\frac{R}{ann\,(M)}$, then $\frac{R}{ann\,(M)}$=($\frac{I}{ann\,(M)}$,x+ann(M)).Therefore 1+ann(M) =$\bar{m}$+(r+ann(M))(x+ann(M)) ,where $\bar{m} \in \frac{I}{ann\,(M)}$,$\bar{m}$=a+ann(M) and a∈ I .

1+ann(M)=(a+ann(M))+(rx+ann(M))

1+ann(M)=(a+rx)+ann(M) which implies that 1-(a+rx)∈ $ann(M) \subseteq I$.

Then 1-(a+rx)∈ I.

Then 1-a-rx=n ,n∈ I.

Thus 1=n+a+rx ∈ J .

Therefore $J = R$ which completes proof .

Proposition(1.14):-

M is max-fully cancellation R-module if and only if M is max-fully cancellation $\bar{R} = \frac{R}{ann\,(M)}$ -module .

Proof:-

(⟹) let M be a max –fully cancellation R-module .

Let I be a non zero maximal ideal of $\bar{R} = \frac{R}{ann\,(M)}$,and N_1 ,N_2 are two $\bar{R}$-submodules.

Then I=$\frac{\acute{I}}{ann\,(M)}$,for some ann(M)⊆ $\acute{I}$ and N_1 ,N_2 are R-submodules .

Now ,suppose $IN_1 = IN_2$ and we have for any x∈ $\acute{I}$,x+ann(M)∈ I,then (x+ann(M))=x$n \in \acute{I}n$,for every n∈ N_1 .

But (x+ann(M))N_1∈ IN_1= IN_2 ,where x+ann(M)∈ I.

Thus xn∈ IN_2 ,then xn_1 =$\sum_{i=1}^{m} \bar{a}_i y_i$ where $\bar{a}_i \in I$, $y_i \in N_2$.

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But for every i , $1 \leq i \leq m$, $\bar{a}_i = a_i$+ann(M) and hence xn=$\sum_{i=1}^{m}(a_i + ann(M))\, y_i$=$\sum_{i=1}^{m} a_i y_i \in \acute{I}N_2$ .Therefore $\acute{I}N_1 \subseteq \acute{I}N_2$.similarly$\acute{I}N_2 \subseteq \acute{I}N_1$,thus $\acute{I}N_1 = \acute{I}N_2$and since $\acute{I}$ is maximal ideal of R by lemma(1.13) ,also M is max-fully cancellation R- module.

Then $N_1=N_2$ and hence M is max-fully cancellation $\bar{R}$-module .

($\Leftarrow$) The proof is similarly

§2 Direct Sum Of Max-Fully cancellation Modules

In this section , we discuss the direct sum of max-fully cancellation modules and show that the direct sum of max-fully cancellation R-module needs not to be max-fully cancellation .However ,we give some conditions under which the class of max-fully cancellation modules is closed under direct sum **.**

Definition(2.1)[6]:-

A submodule M_1 of M is **a direct summand** of M in case there is a submodule M_2 of M with $M=M_1 \oplus M_2$.

The following proposition proves that the direct summand of max-fully cancellation is also max-fully cancellation under the condition annM_1+annM_2=R .

Proposition(2.2):-

Let $M=M_1 \oplus M_2$ be an R-module ,where M_1 ,M_2 are two submodules of M such that annM_1+annM_2=R Then M_1 and M_2 are max-fully cancellation R-modules if and only if M is max-fully cancellation.

Proof:- ($\Rightarrow$)To prove M is max-fully cancellation .Let I be a non zero maximal ideal of R and N_1 ,N_2 are two submodules of M such that$IN_1=IN_2$.

Since annM_1+annM_2=R then by the [12] we get $N_1=A_1+A_2$ and $N_2=B_1+B_2$ for some,A_1,B_1 are submodule of M_1 andA_2,B_2 are submodules of M_2.

Thus $I(A_1+A_2)=I(B_1+B_2)$.

Then $IA_1+IA_2=IB_1+IB_2$.

Which implies $IA_1=IB_1$ and $IA_2=IB_2$.

But M_1 ,M_2 are max-fully cancellation R-module.

Then $A_1=B_1$ and $A_2=B_2$,Thus $N_1=N_2$.

($\Leftarrow$)

since $M_1 \subseteq M=M_1 \oplus M_2$,but M is max-fully cancellation Then M_1 is max-fully cancellation.

And $M_2 \subseteq M$,then M_2 is max-fully cancellation

Definition(2.3)[15]:-

A submodule N of an R-module M is called **invariant** if f(N) $\subseteq N$ for each f$\in END_R$(M)

Definition(2.4)[12]:-

An R-module M is called **fully invariant** if every submodule of M is an invariant .

Remark (2.5)[12]:-

Every invariant R-module is fully invariant and the converse is not true in general.

Remark (2.6):

Every submodule of invariant module is invariant.

The following proposition also shows that the direct sum of max-fully cancellation modules is also max-fully cancellation ,under another condition annM_1+ annM_2=R.

Proposition(2.7):-

Let $M=M_1\oplus M_2$ be an R-module where M_1 ,M_2 are two submodules of M such that M_1,M_2 are fully invariant submodules . Then M_1,M_2 are max-fully cancellation R-modules if and only if M is max-fully cancellation R-module .

Proof-:

($\Longrightarrow$) suppose that M_1 , M_2 are max-fully cancellation .

Now , let N_1 ,N_2 are submodules of M and let I be a non zero maximal ideal of R .

Suppose $IN_1=IN_2$ since M_1 ,M_2 are fully invariant submodule

Then $N_1=(N_1\cap M_1)\oplus(N_1\cap M_2)$ and $N_2=(N_2\cap M_1)\oplus(N_2\cap M_2)$ [26].

Therefore I $((N_1\cap M_1)\oplus(N_1\cap M_2)) = I((N_2\cap M_1)\oplus(N_2\cap M_2))$.

So $I(N_1\cap M_1) = I(N_2\cap M_1)$ and $I(N_1\cap M_2) = I(N_2\cap M_2)$.

Then $N_1\cap M_1= N_2\cap M_1$ and $N_1\cap M_2 = N_2\cap M_2$ since M_1 , M_2 are max-fully cancellation .

Then $N_1=N_2$.

($\Leftarrow$)suppose that M is max-fully cancellation module .

Since $M_1\subseteq M=M_1\oplus M_2$ and $M_2\subseteq M=M_1\oplus M_2$

But M is max-fully cancellation then by remarks and examples (1.7),we get , M_1 and M_2 are max fully cancellation module .

Proposition(2.8):-

Let M_1 ,M_2 be two R-modules and P_1 ,P_2 are two submodules of M_1 ,M_2 respectively such that $annM_1+annM_2=R$ Then P_1 ,P_2 are max-fully cancellation R-module if and only if $P_1\oplus P_2$ is max-fully cancellation R-module of $M_1\oplus M_2$.

Proof:-

($\Rightarrow$) For each non zero maximal ideal I of R and $K_1\oplus W_1$,$K_2\oplus W_2$ are submodules of $P_1\oplus P_2$.

Suppose$I(K_1\oplus W_1)= I(K_2\oplus W_2)$

Then$IK_1\oplus IW_1= IK_2\oplus IW_2$.

Which implies $IK_1= IK_2$ and $IW_1= IW_2$. But P_1 ,P_2 are max-fully cancellation R-modules

Then $K_1=K_2$ and $W_1=W_2$,hence $K_1\oplus W_1= K_2\oplus W_2$.

($\Leftarrow$) since $P_1\subseteq P_1\oplus P_2$, $P_2\subseteq P_1\oplus P_2$

The result follows from Remark (1.7).

Remark (2.9):-

A direct summand of R-module which is max-fully cancellation is also max-fully cancellation .

Proof:-

It is obvious from remark and examples(1.7).

Remark(2.10):-

The converse of remark(2.9) is not true in general

for example : The Z-module $M=Z\oplus Z$ is not max-fully cancellation Z-module , since $(2)(Z\oplus(0))=(2)((0)\oplus Z)=2Z$,where (2) is maximal ideal of Z and $(Z\oplus(0))$, $((0)\oplus Z)$ are two submodules of M .

But $(0)\oplus Z \neq Z\oplus(0)$,while$Z$as a Z-module is max-fully cancellation by remark and examples (1.7).

From remark (2.10) ,we obtain the following

Remark(2.11):

It is not necessary that $M^2=M\oplus M$ is max-fully cancellation module if M is max-fully cancellation R- module.

Definition (2.12)[6]:-.

A ring R is said to be **chained ring** if every non- empty set of ideals in R with respect to inclusion as ordering .

The following result is an immediate consequence of remark (2.9)

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Corollary (2.13):-

Let R be a chained ring .Then ,the direct summand of two max-fully cancellation R-module is also a max-fully cancellation R-module.

§3 max-fully cancellation modules and

A subset S of a ring R is called multiplicatively closed if 1∈S and ab∈S for every a , b∈ S. we know that every proper ideal P in R is prime if and only if R-P is multiplicatively closed[11]

Let M be a module on the ring R and S be a multiplicatively closed on R such that S≠0 and let R_s be the set of all fractional $\frac{r}{s}$ where r∈R and s∈S and M_s be the set of all fractional $\frac{x}{s}$ where x∈M , s∈S. For x_1, x_2 ∈M and s_1, s_2 ∈S , $\frac{x_1}{s_1}=\frac{x_2}{s_2}$ if and only if there exist t∈S such that t$(s_1x_2 - s_2x_1)$ =0.

So , we can make M_s into R_s -module by setting $\frac{x}{s}+\frac{y}{t}=\frac{tx+sy}{st}$, $\frac{r}{t}.\frac{x}{s}=\frac{rx}{ts}$, for every x,y∈M and every r∈R , s,t∈S.

If S=R-P where P is a prime ideal we used M_p instead of M_s and R_p instead of R_s . A ring in which there is only one maximal ideal is called a local ring . Hence , R_p is often called the localization of R at P , similar M_p is the localization of M at P. so we can define the two maps $\psi: R \longrightarrow R_s$ such that $\psi(r)$ =r/ 1 , ∀r∈R , $\theta: M \longrightarrow M_s$, such that $\theta(m)$ =m/1 , ∀m∈M.

Recall that if N be a submodule of an R-module M and S be a multiplicatively closed in R so N_s ={$\frac{n}{s}$: n∈N,s∈ S} be submodule on R_s –module M_s , see [11].

In this section we study the behavior of max-fully cancellation R-module under localization and many results are provided .

Definition(3.1)[13]:-

Let M be an R-module .For all submodules N of M we shall denote the extension N in M_p by N^e and for all submodules L in M_p we shall denote the contraction of L in M_p by L^c and L^c means $f^{-1}(L)$;where f:M→ M_p is the natural homomorphism

For our next proposition ,the following lemma is needed.

Lemma(3.2):-

Let R be a ring and let I be an ideal of R.Then I is maximal ideal of R if and only if I_p is maximal ideal of R_p,for every maximal ideal P of R.

Proof:-

Suppose that I is maximal ideal of R . Let J_p be an ideal of R_p such that $I_p \subsetneq J_p$,then there exists $\frac{as}{s} \in J_p$,$\frac{as}{s} \notin I$,Therefore a$\notin I$

,but I is maximal in R, then (I, a)=R and hence x+ra=1 for some r∈R,x∈ I.

Then $\frac{xs^2}{s^2}+\frac{rs}{s}.\frac{as}{s}=1_p \in J_p = R_p$.

Which implies I_p maximal ideal in R_p.

Now, suppose that I_p is maximal ideal in R_p

Let J be an ideal of R such that $I \subsetneq J$,

Then there exists x∈ J,x$\notin I$.

Which implies $\frac{xs}{s} \notin I_p$,but I_p is maximal ideal in R_p ,and $\frac{xs}{s} \notin I_p$.

Then $(I_p, \frac{xs}{s})=1_p$ and hence $\frac{as^2}{s^2}+\frac{rs}{s}.\frac{xs}{s}=1_p$.

Therefore $\frac{as^2}{s^2}+\frac{rxs^2}{s^2}=1_p$ which implies a+rx=1∈ $J = R$

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Then I is maximal ideal in R.

Lemma(3.4)[10]:-

Let M be an R-module ,and let A ,B are submodule of M Then A=B if and only if A_p=B_p ,for every maximal ideal of R.

The following proposition shows that the concept of max-fully cancellation modules is equivalent between a module M and locally of M**.**

Proposition(3.5):-

Let M be R-module then M_p is max- fully cancellation (for every maximal ideal P of R) if and only if M is max-fully cancellation R-module.

Proof:-

Suppose that IN= $IK$where $I$is anon zero maximal ideal of R and N ,K are any two submodules of M.

Then $(IN)_p$=$(IK)_p$ for every maximal ideal P of R by lemma(3.4).

ThenI_pN_p= I_pK_p [16]

But M_p is max-fully cancellation so N_p=K_p for every maximal ideal P of R.

Thus by lemma(3.4) we have N=K.

Conversely:

Let P be any maximal ideal , I bean maximal ideal of R and let A be a submodule of M ,

We have $I_p\frac{a}{s}\in I_pB_p$,where I_p is an maximal ideal of the ring R_p and ,A_pB_p are submodules of R_p –module M_p and $\frac{a}{s}\in A_p$.

Thus for any x∈ I we have $\frac{x}{1}\in I_p$ and $\frac{x}{1}.\frac{a}{s}\in I_p$. B_p and then

$\frac{xa}{s}=\sum_{i=1}^{n}\frac{K_i}{S_i}.\frac{b_i}{t_i}$ where $k_i\in I$,$b_i\in B$,S_i ,$t_i\notin P$.

Thus $\frac{xa}{s}=\sum_{i=1}^{n}\frac{k_ib_i}{\bar{s}_i}$ where $\bar{s}_i=s_it_i$

Therefore $\frac{xa}{s}=\frac{k_1b_1u_1+k_2b_2u_2+\cdots\ldots\ldots\ldots+k_nb_nu_n}{v}$.

Where $v=\bar{s}_1\bar{s}_2\bar{s}_3\ldots\ldots\ldots\bar{s}_n$,$u_1=\bar{s}_1\bar{s}_3\ldots\ldots\ldots\bar{s}_n$,$u_n=\bar{s}_1\bar{s}_2\ldots\bar{s}_{n\_1}$

Thus there exists K$\notin P$ such that Kxav=$(k_1b_1u_1+k_2b_2u_2+\ldots\ldots.+k_nb_nu_n)S_k$ But kxav∈ I_A ,

$(k_1b_1u_1+k_2b_2u_2+\ldots\ldots..+k_nb_nu_n)S_k\in IB$

But M is max-fully cancellation so by [5] we have a∈ B Thus $\frac{a}{s}\in B_p$

Therefore M_p is max-fully cancellation R_p-module.

Now ,we have the following proposition.

Proposition(3.6):-

Let M be an R-module and N,L be two finitely generated submodules of M .if N_p,L_p are max-fully cancellation then N∩ L is max-fully cancellation R-submodule.

Proof:-

Let N and L be two finitely generated submodules of M .

Then $(N_p:L_p)$+(L_p+N_p)=R_p for all maximal ideal P of R[15]

Therefore $L_p\cap N_p=N_p$ or $N_p\cap L_p = L_p$.

Which implies $L_p\cap N_p$ is max-fully cancellation , but $L_p\cap N_p$=$(L\cap N)_p$

then $(L\cap N)_p$ is max-fully cancellation and L∩ N is max –fully cancellation R-submodule by(3.5).

Proposition(3.7):-

Let M be an R-module and N,L be two finitely generated submodules of M .if N_p ,L_p are max-fully cancellationR_p-module then N+L is max-fully cancellation R-module.

Proof:-

Let N ,L be two finitely generated submodules of M.

Then (N_p:L_p)+(L_p:N_p)=R_P for all maximal ideal P of R[16].

Now ,let $r_1 \in (N_p:L_p)$ and $r_2 \in (L_p:N_p)$ such that $r_1+r_2=1$,

Then either r_1 is a unite element or r_2 is a unite element (since R_p is local ring)

Which implies (N_p:L_p)=R_p or (L_p:N_p)=R_p ,

Thus either$L_p \subseteq N_p$ or $N_p \subseteq L_p$.

Then L_p+N_p=N_p or N_p+L_p=L_p

Therefore N_p+L_p is max-fully cancellation R_p-submodule(since N_p and L_p are max-fully cancellation R_p-submodules).

Which implies ,$(L+N)_p$ is max-fully cancellation R_p-submodule and hence by (3.5),we get L+N is max- fully cancellation R-submodule.

§4 The relationship between Max-fully cancellation Modules and its trace

In this section we give some relationships between the modules having the max-fully cancellation module property and its trace , see proposition (4.4) , proposition(4.7) and proposition(4.8).

"Definition (4.1)[1]:

The **Dual** of M denote by M^* and defined by M^*=HOM_R(M,R) and the **dibual** of M denoted by M^{**} is defined M^{**}=HOM_R(M^*,R)."

"Definition(4.2)[8]:-

The **trace** of an R-module M denoted by T(M) is defined byT(M) =$\sum_{i\in\wedge} \theta_i(M)$ where the sunnruns over all θ_i In HOM_R (M,R)."

"Definition (4.3)[12]:-

An R-module M is said to be a **multiplication module** if for every submodule N of M there exists an ideal I of R such that N= IM ".

Now ,we state and prove the following result .

Proposition(4.4):

Let M,N be two R-modules such that M is multiplication R-modules and let L=$\sum_{finite} \psi_\lambda(M)$ be a cancellation submodule of N ,where the sum is taken as a subset of

HOM_R(M,N).Then M is max-fully cancellation R-module when L is fully cancellation submodule.

Proof:-

Let $I N_1$=$I N_2$,for every a non zero maximal ideal of R and N_1,N_2 be any two submodules of M .

Now ,there exists two ideals A,B of R such that N_1=AM,N_2=BM(since M is multiplication).

Then IAM= IBM and next$\psi_\lambda(IAM)$=ψ_λ(IBM) and hence $\sum_{finite} \psi_\lambda(IAM)$=$\sum_{finite} \psi_\lambda(IBM)$,which implies that

IA$\sum_{finite} \psi_\lambda(M) = IB\sum_{finite} \psi_\lambda(M)$.

Therefore IAL =IBL and hence A=B (since L is fully cancellation submodule and also,L is cancellation submodule)

Thus AM=BM and hence N_1=N_2

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Which completes the proof.

Definition(4.5)[11]:-

A fractional ideal A of a ring R is **invertible** if there exists a fractional ideal B of R such that AB=R. where A fractional ideal of a ring R is a subset A of the total quotient ring K of R such that

(1)A is an R-module , that is , if a,b$\in A$ and r$\in R$,then a-b, ra$\in A$;and

(2)there exists a regular element d of R such that dA$\subseteq R$.

Remark(4.6)[5]:-

An invertible ideal is a cancellation ideal.

The following corollaries is an immediately proposition (4.4)

Corollary(4.7):

let M be a multiplication R-module and T(M)is an invertible and fully-cancellation ideal of R. then M is max –fully cancellation module.

Proof:-

Directly from the definition of T(M).]and by remark (4.6) and by proposition (4.4).

Corollary(4.8):

let M be a multiplication R-module and N be a cancellation homomorphic image of M . If N is fully cancellation submodule ,then M is max-fully cancellation module

proof:-

Let I be a non-zero maximal ideal of R and N_1,N_2 are two submodules of M such that $IN_1=IN_2$ and θ(M)=N

Then N_1=AM,N_2=BM for some ideals A,B of M.

Therefore $IAM= IBM$ and hence $\theta(IAM)= \theta(IBM)$.

and next $IA\theta(M)= IB\theta(M)$.

Which implies that IAN=IBN .But N is fully cancellation and cancellation module.

Then A=B and hence AM=BM . finally we get $N_1=N_2$.

Definition(4.9)[7]:-

An R-module M called **projective** if for every R-epimorphisim h:A$\longrightarrow B$ and f$\in HOM_R$ (M,B) ,there exists g$\in HOM_R$(M,B) such that h∘g=f.

The following proposition gives a sufficient conditions for a module M to be max –fully cancellation.

Proposition(4.10):-

Let M be a multiplication projective R-module and T(M) is fully cancellation ideal .Then M is max-fully cancellation module.

.Proof:

let I be a nonzero maximal ideal $I$of R and N_1,N_2be two submodules of M such that

$IN_1=IN_2$

Let N_1=AM ,N_2=BM for some ideals A and B of R (since M is multiplication)

Now , IAM=IBM.

Then $\theta_i(I\text{AM})=IA\,\theta_i(\text{M})=\theta_i(IBM)=$, $IB\,\theta_i$(M)

And hence $IA\sum_{i\in\wedge}\theta_i(\text{M})= IB\sum_{i\in\wedge}\theta_i(\text{M})$

Which implies IAT(M)= IBT(M). but T(M) is fully cancellation

Then AT(M)=BT(M) ,we have M is projective ,then T(M)M=M

And hence AT(M)M=BT(M)M.

Therefore AM=BM and hence $N_1=N_2$.

The following proposition gives a characterization for max-fully cancellation module.

Proposition(4.11)

Let M be a cancellation R-module and Ker$\sum_{i=1}^{n}\theta_i$(M)=0,whereθ_i is taken as a subset of Hom_R(M,R). Then the following are equivalents:

(1)M is max-fully cancellation module.

(2)T(M) is max-fully cancellation ideal.

Proof:-

(1)⟹(2) :Assume that M is max-fully cancellation module .

To prove that T(M) is max-fully cancellation ideal for every a non zero maximal ideal I of R and two an ideals A and B of T(M).

+Let IA=IB .Then IAM=IBM. but M is max-fully cancellation module and AM ,BM are submodule of M.

Therefore AM=BM and hence A=B (since M is cancellation module)

Therefore T(M) is max-fully cancellation ideal .

(2)⟹(1) :Assume that T(M) is max-fully cancellation ideal

To show that M is max-fully cancellation module.

Let for every a non zero maximal ideal I of R and any two submodules W ,K of M such that IW= IK.

Now$\theta_i(IW)$=$\theta_i(I$K) and next $\sum_{i=1}^{n}\theta_i(IW)$=$\sum_{i=1}^{n}\theta_i(IK)$ But $\theta_i(IW)$=$I\ \theta_i(W)$=$\theta_i(IK)$=$I\ \theta_i$(K).

Therefore $I\sum_{i=1}^{n}\theta_i(W)$=$I\ \sum_{i=1}^{n}\theta_i(K)$ and hence I T(W)= I T(K).

But T(W) ,T(K) are subideals of T(M) and T(M) is max-fully cancellation ideal ,then T(W)=T(K) .

To prove W=K . let w_i ∈W. then$\theta_i(w_i) \in \theta_i(W)$.

$\sum_{i=1}^{n}\theta_i(w_i) \in \sum_{i=1}^{n}\theta_i(W)$=T(W)=T(K) And hence $\sum_{i=1}^{n}\theta_i(w_i)$ ∈T(K) = $\sum_{i=1}^{n}\theta_i(K)$.

Therefore $\sum_{i=1}^{n}\theta_i(w_i)$= $\sum_{i=1}^{n}\theta_i(k_i)$ And hence $\sum_{i=1}^{n}\theta_i(w_i-k_i)$=0

Which implies ,$w_i-k_i \in \mathrm{Ker}\sum_{i=1}^{n}\theta_i$=0 .

Then w_i-k_i=0 and hencew_i=k_i .

Thus W ⊆ K ,similarly we can show that K ⊆ W

And hence W= K. This end the proof .

Next ,we have the following proposition

Proposition(4.12):-

Let M be an R-module .M is max-fully cancellation module ,if T(M) is fully cancellation ideal such that $\sum\varphi_{\lambda}(M)$=0 ,where$\varphi_{\lambda} \in \mathrm{Hom}(M,R)$.

Proof:

By the same way of the second side of proof of proposition (4.11) by using T(M) is fully cancellation instead of T(M) is max-fully cancellation .

Now ,we end this section by the following proposition

Proposition(4.13):

Let M be a cancellation R-module .T(M) is max-fully cancellation ideal ,if M is fully cancellation module .

Proof:-

By the steps of the first side of proof of proposition (4.11) and we take M is fully cancellation instead of M is max-fully cancellation module.

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