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# A MOMENT INEQUALITY TO CLASS USED BETTER THAN AGED IN CONVEX ORDERING UPPER TAIL (UBACT) OF LIFE DISTRIBUTIONS AND IT'S APPLICATIONS

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## ABSTRACT

The class of life distributions used better than aged in convex order upper tail ordering (UBACT) is introduced. A Moment inequality to this class (UBACT) of life distribution is given. In addition testing exponentiality versus (UBACT) class of life distribution based on a moment inequality is presented. Simulation such as critical values, Pitman's asymptotic efficiency and the power of test are discussed. Medical applications are given at the end of the paper.

## Key Words

(UBACT) class of life distribution; Moment inequalities; Testing of hypothesis; Pitman's efficiency; Monte carlo Method and the power of the test.

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## 1 INTRODUCTION

Suppose we want to purchase a used item such as a Radio, Tv, Computer, etc with unknown age. It would be of interest to have some criteria which can be used to compare the future performance of the purchased item under the buyer's assumption and it's performance under it's actual age. Many ageing concepts are introduced as a criteria for comparison such as used better than aged (UBA)class of life distributions, used better than aged in convex ordering ( UBAC) of life distribution and used better than aged convex ordering upper tail (UBACT) of life distribution.

For a nonnegative continuous random variable $X$, let $F(x) = P(X \le t)$ be a cumulative distribution function , survival function $\overline{F} = 1 - F$ and a finite mean $\mu = E(X) = \int_0^\infty \overline{F}(u)du$, let $X_t$ be the random residual lifetime at age t with survival function $\overline{F}_t = \frac{\overline{F}(t+x)}{\overline{F}(t)}, x, t \ge 0$. Now consider The following definitions for (UBA), (UBAC) and( UBAC) classes of life distributions

**Definition(1.1)**. The distribution function $F$ is said to be (UBA) if for all $x, t \ge 0$

$$\overline{F}(x+t) \ge \overline{F}(x)e^{-\gamma t}, \tag{1}$$

where $\gamma$ is called the asymptotic decay of $X$.

**Definition(1.2)**. The distribution function $F$ is said to be (UBAC) if for all $x, t \ge 0$

$$\int_t^\infty \overline{F}(x+u)du \ge \overline{F}(x)\int_t^\infty e^{-\gamma u}du, \tag{2}$$

or

$$\nu(x+t) \ge \frac{1}{\gamma}\overline{F}(x)e^{-\gamma t}, \tag{3}$$

where $\nu(x+t) = \int_{x+t}^\infty \overline{F}(u)du$ .

**Definition(1.3)**. $F$ is said to be (UBACT) if for all $x, t \ge 0$

$$\int_t^\infty \nu(x+u)du \ge \frac{1}{\gamma^2}\overline{F}(x)\int_t^\infty e^{-\gamma u}du, \tag{4}$$

or

$$\Gamma(x+t) \ge \frac{1}{\gamma^2}\overline{F}(x)e^{-\gamma t}, \tag{5}$$

where $\Gamma(x+t) = \int_{x+t}^\infty \nu(u)du$ .

The motivation of proposed (UBACT ) class of life distributions is included the famous classes of life distribution like decreasing mean residual life (DMRL), (UBA) and (UBAC) classes of life distributions, (see Willmot and Cai(2000), Al-Nachawati and Alwasel(1997), Ahmed (2004), Abu-Youssef, S. E and Bakr, M. E (2014). Thus we have

$$DMRL \subset UBA \subset UBAC \subset UBACT$$

Moreover the proposed class (UBACT) of life distribution is simple, gives more efficient for common alternatives and introduces a good power.

In this paper we introduce a new definition of (UBACT) class of life distributions based on a moment inequality in section 2. Testing this class based on a moment inequality is given in section 3. Finally, medical applications are presented for proposed test in section

4.

## 2 MOMENT INEQUALITIES

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The following theorem gives moment inequalities for (UBACT) class of life distributions.

**Theorem 2.1.** let $F$ be (UBACT) class of life distributions such that for some integers $r, s \geq 0$, $\mu_{(r+s+4)} = E(X^{r+s+4}) \leq \infty$ , then

$$\frac{\gamma^{s+3}\mu_{(r+s+4)}}{(r+s+4)!} \geq \frac{\mu_{(r+1)}}{(r+1)!}, r \geq 0 \quad . \qquad (1)$$

**Proof.** Since $F$ is UBACT, then

$$\gamma^2 \Gamma(x+t) \geq \overline{F}(x) e^{-\gamma t}, \qquad (2)$$

Multiplying both sides by $x^r t^s$ , $r, s \geq 0$, and integrating over $(0, \infty)$, w.r.t. $x, t$ , then

$$\gamma^2 \int_0^\infty \int_0^\infty x^r t^s \Gamma(x+t) dt dx \geq \int_0^\infty \int_0^\infty x^r t^s \overline{F}(x) e^{-t\gamma} dx dt \qquad (3)$$

By taking $x + t = u$ , the left hand side of (2.3) is

$$\gamma^2 \int_0^\infty \int_0^u (u-t)^r t^s \Gamma(u) dt du = \gamma^2 \int_0^\infty \int_0^u u^{r+s+1} \Gamma(u) (1 - \frac{t}{u})^r (\frac{t}{u})^s dt du$$

$$= \gamma^2 B(r+1, s+1) \int_0^\infty u^{r+s+1} \Gamma(u) du, \qquad (4)$$

where $B(r+1, s+1) = \int_0^1 (1-u)^r (u)^s du$ .

But

$$\int_0^\infty u^{r+s+1} \Gamma(u) du = \frac{1}{2} \int_0^\infty u^{r+s+1} E(X-u)^2 I(X > u) du$$

$$= E(\int_0^X u^{r+s+1} (X-u)^2 du = \frac{\mu_{(r+s+4)}}{(r+s+2)(r+s+3)(r+s+4)}. \qquad (5)$$

Then ( 2.4 ) becomes as the following:

$$\gamma \int_0^\infty \int_0^\infty x^r t^s v(x+t) dt dx = \gamma^2 B(r+1, s+1) \frac{\mu_{(r+s+4)}}{(r+s+2)(r+s+3)(r+s+4)}. \qquad (6)$$

The right hand side of (2.3) is equal to

$$\int_0^\infty t^s e^{-t\gamma} dt \int_0^\infty x^r \overline{F}(x) dx = \frac{\Gamma(s+1)\mu_{(r+1)}}{\gamma^{s+1}(r+1)!}. \qquad (7)$$

By using (2.6) and (2.7) in (2.3), (2.1) is obtained.

**Corollary 2.1.** let $r = s = 0$, then $\gamma^3 \mu_{(4)} \geq 24 \mu_{(1)}$

**Corollary 2.2.** let $r = 0$, then $\gamma^{s+3} \mu_{(s+4)} \geq (s+4)! \mu_{(1)}$

**Corollary 2.3.** let $s = 0$, then $\gamma^3 \mu_{(r+4)} \geq (r+4)(r+3)(r+2) \mu_{(r+1)}$

## 3 TESTING OF HYPOTHESES

The test depends on random sample $X_1, X_2, \ldots\ldots, X_n$ from a population with distribution function $F$ . We want to test that :

$$H_0 : \overline{F} \text{ is exponential}$$

against

$$H_1 : \overline{F} \text{ is UBACT and isn't exponential.}$$

Using theorem (2.1), we may use the following $\delta_{M_T}$ as a measure of departure from $H_0$ :

$$\delta_{M_T}(r) = \frac{\gamma^{r+3}\mu_{(2r+4)}}{(2r+4)!} - \frac{\mu_{(r+1)}}{(r+1)!}. \qquad (1)$$

Note that under $H_0 : \delta_{M_T} = 0$, while under $M_1 : \delta_{M_T} > 0$. $\hat{\delta}_{M_T}$ is the estimator of $\delta_{U_T}$ .let $X_1, X_2, \ldots, X_n$ be a random sample from $F$, $\hat{\gamma} = \frac{n}{\sum X_i}$ is the estimator of $\gamma$ and $\mu$ is estimated by $\overline{X}$, where $\overline{X} = \frac{1}{n}\sum X_i$ is the sample mean . Then $\hat{\delta}_{U_T}$ is given by using (3.1) as

$$\hat{\delta}_{M_T} = \frac{1}{n}\sum_i \left\{ \gamma^{r+3} \frac{X_i^{2r+4}}{(2r+4)!} - \frac{X_i^{r+1}}{(r+1)!} \right\}. \qquad (2)$$

to make the test statistic scale invariant, we use

$$\Delta_{M_T} = \frac{\hat{\delta}_{U_T}}{\mu^{2r+4}}.$$

which is estimated by

$$\hat{\Delta}_{M_T} = \frac{\hat{\delta}_{M_T}}{\overline{X}^{2r+4}}. \qquad (3)$$

Setting $\phi(X_1) = \frac{\gamma^{r+3} X_1^{2r+4}}{(2r+4)!} - \frac{X_1^{r+1}}{(r+1)!}$, then $\hat{\Delta}_{M_T}$ in (3.3) is a U-statistic, cf. Lee (1990). The following theorem summarizes the large sample properties of $\hat{\Delta}_{M_T}$ .

**Theorem 3.1.** As $n \to \infty, \sqrt{n}(\hat{\Delta}_{U_T} - \Delta_{M_T})$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = var[\gamma^{r+3} \frac{X_1^{2r+4}}{(2r+4)!} - \frac{X_1^{r+1}}{(r+1)!}]. \qquad (4)$$

Under $H_0 : \Delta_{M_T} = 0$ and variance $\sigma_0^2$ is given by

$$\sigma_0^2(r) = \frac{(4r+8)!}{((2r+4)!)^2} + \frac{(2r+2)!}{((r+1)!)^2} - \frac{2(3r+5)!}{(2r+4)!(r+1)!}. \qquad (5)$$

**Proof:** Since

$$\sigma^2 = var[\phi(X_1)]$$

, where

$$\phi(X_1) = \frac{\gamma^{r+3} X_1^{2r+4}}{(2r+4)!} - \frac{X_1^{r+1}}{(r+1)!}.$$

Then (3.4) follows. Under $H_0 : \Delta_{U_T} = E(\phi(X_1)) = 0$ and

$$\sigma_0^2(r) = E[\frac{X_1^{2r+4}}{(2r+4)!} - \frac{X_1^{r+1}}{(r+1)!}]^2. \qquad (6)$$

Hence (3.5) follows. The Theorem is proved. When $r = 0$,

$$\delta_{M_T}(0) = \frac{\gamma^3 \mu_{(4)}}{24} - \mu, \qquad (7)$$

in this case $\sigma_0^2 = 62$ and the test statistic

$$\hat{\delta}_{M_T}(0) = \frac{1}{24n} \sum_i \{X_i^4 - 24X_i\} \qquad (8)$$

and

$$\hat{\Delta}_{M_T} = \frac{\hat{\delta}_{M_T}}{\overline{X}^4}, \qquad (9)$$

When $r = 1$,

$$\delta_{M_T}(1) = \frac{\gamma^4 \mu_{(6)}}{6!} - \frac{\mu_2}{2!}, \qquad (10)$$

in this case $\sigma_0^2 = 865$ and the test statistic

$$\hat{\delta}_{M_T}(1) = \frac{1}{720n} \sum_i \{X_i^6 - 360X_i^2\} \qquad (11)$$

and

$$\hat{\Delta}_{M_T} = \frac{\hat{\delta}_{M_T}}{\overline{X}^6}, \qquad (12)$$

When $r = 2$,

$$\delta_{M_T}(2) = \frac{\gamma^5 \mu_{(8)}}{8!} - \frac{\mu_3}{3!}, \qquad (13)$$

in this case $\sigma_0^2 = 12560$ and the test statistic

$$\hat{\delta}_{M_T}(2) = \frac{1}{40320n} \sum_i \{X_i^8 - 6720X_i^3\} \qquad (14)$$

and

$$\hat{\Delta}_{M_T} = \frac{\hat{\delta}_{M_T}}{\overline{X}^8}, \qquad (15)$$

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{M_n}\sigma_0$ and reject $H_0$ if this exceeds the normal variate value $Z_{1-\alpha}$. To illustrate the test, we calculate, via Monte Carlo Method, the empirical critical points of $\hat{\Delta}_{M_T}$ in (3.9) for sample sizes 5(5)50. Tables (4.1) gives the percentile points for 1%, 5%, 10%, 90%, 95%, 99% .0 The calculations are based on 10000 simulated samples of sizes $n$ =5(5)50.

**Table (3.1)** Critical Values of $\hat{\Delta}_{M_T}$ in(3.9)

To compare our tests of my proposed class of life distribution with tests of other classes of life distribution on basis of

Pitman's asymptotic efficiency (PAE), we calculate Pitman's asymptotic efficiency (PAE) for UBACT and other classes of life distributions.

Here we choose $K^*$ presented by Hollander and Prochan (1975) for (DMRL), $\hat{\delta}_2$ presented by Ahmad (2004) for (UBAE) class and $\hat{\Delta}_{U-t}$ presented by Abu-Youssef et al (2014) for (UBACT) class of life distribution based on U-Statistics.

PAE of $\hat{\Delta}_{M_T}$ is given by :

$$PAE(\Delta_{M_T}(\theta)) = \left\{ \frac{d}{d\theta} \Delta_{M_T}(\theta)|_{\theta\to\theta_0} \right\} / \sigma_0. \quad (16)$$

Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

$$llll\,(i) Linear failure rate family : \overline{F}_\theta = e^{-x-\frac{\theta x^2}{2}}, x > 0, \theta > 0 (ii) Makeham family : \overline{F}_{2\theta} = e^{-x-\theta(x+e^{-x}-1)}, x > 0, \theta > 0$$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makham families. The PAE's of these alternatives of our procedure are, respectively:

$$PAE(\Delta_{M_T}, LFR) = -\frac{1}{2\sigma_0}(3r^2 + 17r + 18), \quad r \ge 0 \qquad (17)$$

$$PAE(\Delta_{M_T}, Makeham) = -\frac{1}{\sigma_0}[(r+3) - \frac{1}{2^{2r+5}} + \frac{1}{2^{r+1}}] \qquad (18)$$

**Table (3.2)**

$$|l|c|c|c|c| Distribution K^* \hat{\delta}_2 \hat{\Delta}_{U_T} \hat{\Delta}_{M_T} F_1 Linear failure 0.81 0.63 0.748 1.143 rate F_2 Makeham 0.29 0.385 0.248 0.32$$

From Table (3.2), the test statistic $\hat{\Delta}_{M_T}$ is more efficient than $K^*$ and $\hat{\Delta}_{U_T}$ for linear failure rate family and Makeham family. But the test statistic $\hat{\Delta}_{M_T}$ is more efficient than $\hat{\delta}_2$ for linear failure rate family only .

**Note that:** Since $\hat{\Delta}_{M_T}$ defines a class with parameter $r$ of test statistic, we choose $r$ that maximizes the PAE of that alternatives. If we take $r = 0$ then our test will have more efficiency than others. Finally, the power of the test statistics $\hat{\Delta}_{M_T}$ is considered for 95% percentiles in Table 4.3 for three of the most commonly used alternatives [see Hollander and Proschan (1975)], they are

$$lllll\,(i) Linear failure rate : \overline{F}_\theta = e^{-x-\frac{\theta x^2}{2}}, x > 0, \theta > 0 (ii) Makeham : \overline{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}, x \ge 0, \theta > 0 (iii) Weibull : \overline{F}_\theta = e^{-}$$

These distributions are reduced to exponential distribution for appropriate values of $\theta$ .

**Table 4.3** Power Estimate of $\hat{\Delta}_{M_T}$

## 4 APPLYING THE TEST

### Example 1

The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt Attia A. F. The ordered life times (in days) are:

10, 14, 14, 14, 14, 14, 15, 17, 18, 20, 20, 20, 20, 20, 23, 23, 24, 26, 30, 30, 31, 40, 49, 51, 52, 60, 61, 67, 71, 74, 75, 87, 96, 105, 107, 107, 107, 116, 150

Using equation (3.9), the value of test statistics, based on the above data is $\hat{\Delta}_{M_T} = 3.78044$. the critical value at

$\alpha = 0.05$ is 2.11575, then we reject $H_0$ at the significance level $\alpha = 0.05$.Therefore the data has UBACT Property.

## Example 2

Consider real data representing 40 patients suffering from blood cancer.We use the data as given in Abu-Youssef (2009). The ordered life times (in day) are:

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852.

The value of test statistics, based on the above data is $\hat{\Delta}_{M_T} = 3.7808$. the critical value at $\alpha = 0.05$ is 2.08966 This value leads to the rejecting of $H_0$ at the significance level $\alpha = 0.05$. Therefore the data has UBACT Property.

## Example 3

In an experiment at Florida state university to study the effect of methyl mercury poisoning on the life lengths of fish goldfish were subjected to various dosages of methyl mercury (Kochar (1985)). At one dosage level the ordered times to death in week are:

6, 6.143, 7.286, 8.714, 9.429, 9.857, 10.143, 11.571, 11.714, 11.714

The value of test statistics, based on the above data is $\hat{\Delta}_{M_T} = 3.7801$. the critical value at $\alpha = 0.05$ is 4.13668. Then $H_0$ at the significance level $\alpha = 0.05$ is accepted. Therefore the data hasn't UBACT Property.

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