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Couple New Iterative Method with Pade Approximation to Solve the Nonlinear Wave-Like Equations with Variable Coefficients

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Abstract

In this paper, some wave-like equation was solved by using a new modified iterative method as an infinite force but there is difficulty in calculating the infinite series, so the cut power series will be approximated using round the pillow. Where the PADE approximation approaches a severed energy series as a ratio between polynomials to reduce calculations and shortening of the power chain while maintaining the required accuracy. Through these methods the successful use of approximate the solut.

Keywords: wave-like equations, nonlinear PDE, Pade approximation, new iterative method.

1. Introduction

Mathematical model plays an important role in described natural and engineering problems. Many important problems are frequently modeled through linear and nonlinear differential equations. Due to this importance, researchers have been interesting to find efficient and accurate solutions to their problems. One of these problems is the nonlinear wave-like equations with variable coefficients.

However, it is difficult to obtain the exact solutions for many problems in real-life. There are many techniques and methods that have been used to solve differential equations, some of this method is used to find the exact solution for a specific problem such as separation of variables and the integral transforms (Laplace transform, Fourier transform and it.), other methods are used to find a numerical solution such as finite element method, finite difference method and it., other methods have been used to provide an analytical approximation solution for linear and nonlinear differential problems, such as the variational iteration method (VIM), homotopy perturbation method (HPM) [1-3], combine LA-transform with decomposition method [4] and Adomian decomposition method (ADM) [5-9]. Many classes of differential equations have been solved using this decomposition methods. The analytical methods provide the solution as an infinite power series, so the infinite power series must be truncated at a finite term to get numerical solution.

Consider the wave-like equation.

$$U\_tt = \sum_{i,j=1}^{n} F\_1ij(X,U,t)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}$$

$$\left(U_{xi}, U_{xj}\right) + \sum_{i=1}^{n} G_{1i}(X,U,t)\frac{\partial^p}{\partial x_i^p} G_{2i}\left(U_{xi}\right) + H(X,t,U) + S(X,t) \quad (1)$$

By the initial conditions

$$U(X,0) = a_0(X), \ U_t(X,0) = a_1(X), \quad (2)$$

Where $X = (x_1, x_2, ..., x_n)$ and F_{1ij}, G_{1i} are nonlinear functions of X, t, U, F_{2ij} and G_{2i} are non-linear functions of derivatives of x_i , x_j while *H* and *S* are non-linear functions, *k, m* and *p* are integers.

1. Modified New Iterative Method

In 2006, a new iterative method (NIM) was proposed by Gejji and Jafari to solve the linear and non-linear differential equations [10]. In [11] Alaa modified new iterative method as following:

Rewrite a general initial value problem (IVP) as:

$$L(\psi) + R(u) + N(u) = \phi \quad (3)$$

With the initial condition:

$$\frac{\partial^r \psi}{\partial t^r}\Big|_{t=0} = f_r(X) \ , \ r = 0, 1, 2..., n-1 \quad (4)$$

Where $\psi(X,t)$ is an unknown function with X is a variable with one or more dimensions, a linear operator $L(.) = \frac{\partial^n(.)}{\partial t^n}$ is a partial derivative with respect to t with n=1, 2, … ,while $R(.)$ denoted the remained of linear operator, the nonlinear operator of IVP (3) was denoted by $N(.)$ and finally the inhomogeneous part $\phi(X,t)$ is a known function.

The unknown function u can be expressed as the power series:

$$\psi = \sum_{k=0}^{\infty} \psi_k \tag{5}$$

The Laplace transform (w.r.t. to t) can be taken on both sides of equation (3) to get:

$$L\{L(\psi)\} + L\{R(\psi) + N(\psi)\} = L\{\phi\} \tag{6}$$

Now, let $M = R + N$ and applied the properties of Laplace transform, the equation (6) becomes:

$$s^n L\{\psi\} - \sum_{r=0}^{n-1} s^{n-r-1} \frac{\partial^r \psi}{\partial t^r}\Big|_{t=0} + L\{H(\psi)\} = L\{\phi\} \tag{7}$$

From (4) we have:

$$s^n L\{\psi\} - \sum_{r=0}^{n-1} s^{n-r-1} f_r + L\{H(\psi)\} = L\{\phi\} \tag{8}$$

and hence:

$$L\{\psi\} = \sum_{r=0}^{n-1} s^{-r-1} f_r + \frac{1}{s^n} L\{\phi\} - \frac{1}{s^n} L\{H(\psi)\} \tag{9}$$

The inverse Laplace transform have been taken on both sides of equation (9), to get:

$$u = \sum_{r=0}^{n-1} \frac{t^r}{r!} f_r + L^{-1}\left\{\frac{1}{s^n} L\{\phi\}\right\} - L^{-1}\left\{\frac{1}{s^n} L\{H(\psi)\}\right\} \tag{10}$$

Then (10) and (1) are equivalent, i.e., they have the same solution.

Now, (5) have been Substituted in (10) then we have:

$$\sum_{k=0}^{\infty} \psi_k(X,t) = f - L^{-1}\left\{\frac{1}{s^n} L\left\{H\left(\sum_{k=0}^{\infty} \psi_k(X,t)\right)\right\}\right\} \tag{11}$$

Where

$$f = \sum_{r=0}^{n-1} \frac{t^r}{r!} f_r + L^{-1}\left\{\frac{1}{s^n} L\{\phi\}\right\} \tag{12}$$

Then H in (11) can be written as:

$$L^{-1}\left\{\frac{1}{s^n} L\left\{H\left(\sum_{k=0}^{\infty} \psi_k\right)\right\}\right\} = L^{-1}\left\{\frac{1}{s^n} L\left\{H(\psi_0) + \sum_{k=1}^{\infty}\left[H\left(\sum_{r=0}^{k} \psi_r\right) - H\left(\sum_{r=0}^{k-1} \psi_r\right)\right]\right\}\right\} \tag{13}$$

Substituting (13) in (11), we have:

$$\sum_{k=0}^{\infty} \psi_k = f - L^{-1}\left\{\frac{1}{s^n} L\left\{H(\psi_0) + \sum_{k=1}^{\infty}\left[H\left(\sum_{r=0}^{k} \psi_r\right) - H\left(\sum_{r=0}^{k-1} \psi_r\right)\right]\right\}\right\} \tag{14}$$

Finally, the recurrence relation was defined as:

$$\{\psi_0 = f \qquad \psi_1 = - L^{-1}\left\{\frac{1}{s^n} L\{H(\psi_0)\}\right\}$$

$$\psi_{k+1} = - L^{-1}\left\{\frac{1}{s^n} L\{H(\psi_0 + \ldots + \psi_k) - H(\psi_0 + \ldots + \psi_{k-1})\}\right\}, \quad k = 1, 2, \ldots \tag{15}$$

3. Pade Approximation

Many types of problems, in various sciences as engineering and physical and it., have been successfully solved by Pade approximation [12-15]. These problems of applied sciences have been solved as an infinite power series, but it is not possible to compute the infinite series. So, the power series must be truncated at a certain term. The next step is to approximate the truncated power series by constructing a rational function matches the truncated power series as far as possible which is called Pade approximant [16]. Pade approximant approximates a truncated power series as ratio of two polynomials. The coefficients of polynomials in the numerator and denominator can be computed by matching the terms with the coefficients of truncated power series [17]. The approximation of a function using the Pade approximation is often better than the approximation by the truncated power series, because the convergence interval is greater.

In many areas and fields, the rational series solutions have been approximated by Pade approximation [14,18]. This approximation shows best performance over series approximations. Therefore, the Pade approximations are better than approximation by polynomials in numerical results.

If the series expansion of a function $\psi(t)$ is as following:

$$\psi(t) = \psi_0 + \psi_1 t + \psi_2 t^2 + \psi_3 t^3 + \psi_4 t^4 + \psi_5 t^5 + \ldots = \sum_{k=0}^{\infty} \psi_k t^k \tag{16}$$

The symbol $[n/m]_{\psi(t)}$ refers to the Pade approximant for the function $\psi(t)$ of order $[n, m]$ which is defined by:

$$[n, m]_{(t)} = \frac{P_n(t)}{Q_m(t)} = \frac{p_0 + p_1 t + \ldots + p_n t^n}{1 + q_1 t + \ldots + q_m t^m} \tag{17}$$

where the denominator and numerator have no common factors and $q_0 = 1$. The denominator and numerator in (17) are constructed so that $\psi(t)$ and $[n/m]_{\psi(t)}$ and their derivatives agree at $t = 0$ up to $n+m$. That is

$$\psi(t) - [n, m]_{\psi(t)} = O\left(t^{n+m+1}\right) \tag{18}$$

From (18), we have

$$\psi(t) \sum_{k=0}^{m} q_k t^k - \sum_{k=0}^{n} p_k t^k = O\left(t^{n+m+1}\right) \tag{19}$$

From (19), we get the following systems with $q_0 = 1$:

$$\begin{cases} \psi_{n+1} + \psi_n q_1 + \ldots + \psi_{n-m+1} q_m = 0 \\ \psi_{n+2} + \psi_{n+1} q_1 + \ldots + \psi_{n-m+2} q_m = 0 \\ \vdots \\ \psi_{n+m} + \psi_{n+m-1} q_1 + \ldots + \psi_n q_m = 0 \end{cases} \tag{20}$$

and

$$\begin{cases} p_0 = \psi_0 \\ p_1 = \psi_1 + \psi_0 q_1 \\ \vdots \\ p_n = \psi_n + \psi_{n-1} q_1 + \ldots + \psi_0 q_n \end{cases} \tag{21}$$

Firstly, from (20) all the coefficients q_i, $1 \le i \le m$ are calculated. Then, the coefficient p_i, $0 \le i \le n$ are determined from (21).

Note that, if the degree for numerator and denominator in (17) are equal or when the numerator has degree one higher than the denominator then the error (18) is smallest for a fixed value of $n+m+1$.

4. Numerical Application

In this section, the Pade approximation method has been used to approximate the analytical solutions, obtained by modified new iterative method, of some cases of wave-like equation, illustrations have been given for comparison between the approximate solution and the exact solution to indicate the accuracy of this method.

4.1 Case 1:

Consider the following nonlinear wave-like equation:

$$u_{tt} - \frac{\partial^2}{\partial x \, \partial y}\left(u_{xx} u_{yy}\right) + \frac{\partial^2}{\partial x \, \partial y}\left(xy u_x u_y\right) + u = 0$$

with initial conditions

$$u(x, y, 0) = e^{xy}, \quad u_t(x, y, 0) = e^{xy}$$

Then $L(u) = \frac{\partial^2 u}{\partial t^2}$, i.e. n=2,

$R(u) = u,\ N(u) =- \frac{\partial^2}{\partial x\,\partial y}\left(u_{xx}u_{yy}\right) + \frac{\partial^2}{\partial x\,\partial y}\left(xyu_x u_y\right)$ then

$$H(u) = u - \frac{\partial^2}{\partial x\,\partial y}\left(u_{xx}u_{yy}\right) + \frac{\partial^2}{\partial x\,\partial y}\left(xyu_x u_y\right)$$

$\phi = 0, f_0 = e^{xy}$ *and* $f_1 = e^{xy}$.

By (15) and (12) we have:

$$u_0 = f = \sum_{i=0}^{n-1} \frac{t^i}{i!} f_i + L^{-1}\left\{\frac{1}{s^2} L\{\phi\}\right\} = f_0 + t f_1 = e^{xy} + t\, e^{xy}$$

$$u_1 =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0\right)\right\}\right\} =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(e^{xy} + t\, e^{xy}\right)\right\}\right\} =- \left(\frac{1}{6}\right)e^{xy}\left(t^3 + 3\, t^2\right)$$

$$u_2 =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0 + u_1\right) - H\left(u_0\right)\right\}\right\} = \left(\frac{1}{120}\right)e^{xy}\left(t^5 + 4\, t^4\right)$$

$$u_3 =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0 + u_1 + u_2\right) - H\left(u_0 + u_1\right)\right\}\right\} =- \left(\frac{1}{5040}\right)e^{xy}\left(t^7 + 7\, t^6\right)$$

Then from (5), we have:

$$u(x, y, t) = e^{xy}\left(1 + t - \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} - \frac{t^6}{720} - \frac{t^7}{5040} + \dots\right)$$

The series can be approximated by $[4, 3]_{u(t)}$ as following:

$$[4, 3]_{u(t)} = \frac{p_0 + p_1 t + \dots + p_4 t^4}{1 + q_1 t + \dots + q_3 t^3} = a_0 + a_1 t + \dots + a_7 t^7$$

where

$a_0 = 1,\ a_1 = 1,\ a_2 =- \frac{1}{2},\ a_3 =- \frac{1}{6},\ a_4 = \frac{1}{24},\ a_5 = \frac{1}{120},\ a_6 =- \frac{1}{720},\ a_7 =- \frac{1}{5040}$

From (20) and (21) we get:

$$\{a_5 + a_4 q_1 + a_3 q_2 + a_2 q_3 = 0\ a_6 + a_5 q_1 + a_4 q_2 + a_3 q_3 = 0\ a_7 + a_6 q_1 + a_5 q_2 + a_4 q_3 = 0 \qquad (22)$$

and

$$p_2 = a_2 + a_1 q_1 + a_0 q_2 \qquad p_3 = a_3 + a_2 q_1 + a_1 q_2 + a_0 q_3\ p_4 = a_4 + a_3 q_1 + a_2 q_2 + a_1 q_3 \qquad (23)$$

By solving the system (22) we get:

$$\{q_0 = 1 \qquad q_1 =- 0.146 \qquad q_2 = 0.0345 \qquad q_3 =- 0.007 \qquad (24)$$

and substitute (24) in (23) we have:

$$\{p_0 = 1 \qquad p_1 = 0.85393 \qquad p_2 =- 0.6 \qquad p_3 =- 0.066 \qquad p_4 = 0.041747 \qquad (25)$$

Then we obtain:

$$[4, 3]_{u(t)} = \frac{0.041747\, t^4 - 0.066\, t^3 - 0.6\, t^2 + 0.85393\, t + 1}{-0.007\, t^3 + 0.0345\, t^2 - 0.146\, t + 1}$$

Then we have:

$$u(x, y, t) \cong e^{xy} \frac{0.041747\, t^4 - 0.066\, t^3 - 0.6\, t^2 + 0.85393\, t + 1}{-0.007\, t^3 + 0.03451\, t^2 - 0.146\, t + 1}$$

In the following figure, a comparison was made between the approximate solution and the exact solution in different values for *x* and *y* and *t*=[0 ,1], where the exact solution is: $u(x, y, t) = e^{x\,y}(\sin sin\,(t) \; + \cos cos\,(t)\,)$.

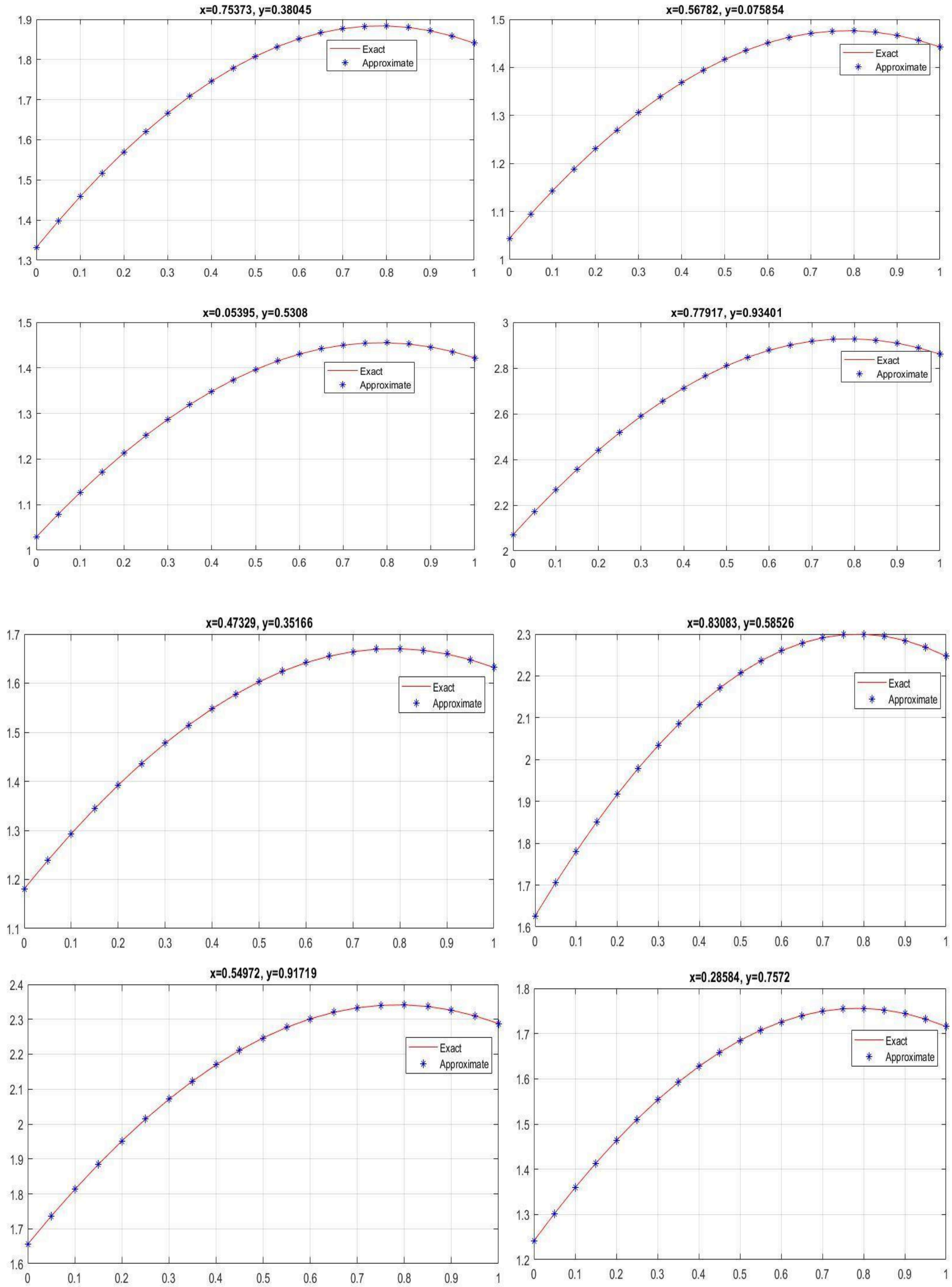

Figure 1: Illustrate case1 with *t*=0:1 and different values of *x* and *y*

4.2 Case 2:

Consider the following nonlinear wave-like equation:

$$u_{tt} - u^2 \frac{\partial^2}{\partial x^2}\left(u_x u_{xx} u_{xxx}\right) + u_x^2 \frac{\partial^2}{\partial x^2}\left(u_{xx}^3\right) - u = 0, \quad 0 < x < 1,\ t > 0$$

with initial conditions

$$u(x,0) = e^x, \quad u_t(x,0) = e^x$$

Then $L(u) = \frac{\partial^2 u}{\partial t^2}$, i.e. n=2,

$R(u) =- u,\ N(u) =- u^2 \frac{\partial^2}{\partial x^2}\left(u_x u_{xx} u_{xxx}\right) + u_x^2 \frac{\partial^2}{\partial x \partial y}\left(u_{xx}^3\right)$ then

$H(u) =- u - u^2 \frac{\partial^2}{\partial x^2}\left(u_x u_{xx} u_{xxx}\right) + u_x^2 \frac{\partial^2}{\partial x \partial y}\left(u_{xx}^3\right)$

$\phi = 0, f_0 = e^x\ and\ f_1 = e^x.$

By (15) and (12) we have:

$u_0 = f = \sum_{i=0}^{n-1} \frac{t^i}{i!} f_i + L^{-1}\left\{\frac{1}{s^2} L\{\phi\}\right\} = f_0 + t f_1 = e^x + t e^x$

$u_1 =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0\right)\right\}\right\} =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(e^x + t e^x\right)\right\}\right\} = \left(\frac{1}{6}\right) e^x \left(t^3 + 3 t^2\right)$

$u_2 =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0 + u_1\right) - H\left(u_0\right)\right\}\right\} = \left(\frac{1}{120}\right) e^x \left(t^5 + 4 t^4\right)$

$u_3 =- L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0 + u_1 + u_2\right) - H\left(u_0 + u_1\right)\right\}\right\} = \left(\frac{1}{5040}\right) e^x \left(t^7 + 7 t^6\right)$

Then from (5), we have:

$$u(x,t) = e^x\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + \dots\right)$$

The series can be approximated by $[4,3]_{u(t)}$ as following:

$$[4,3]_{u(t)} = \frac{p_0 + p_1 t + \dots + p_4 t^4}{1 + q_1 t + \dots + q_3 t^3} = a_0 + a_1 t + \dots + a_7 t^7$$

where

$a_0 = 1,\ a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{6},\ a_4 = \frac{1}{24},\ a_5 = \frac{1}{120},\ a_6 = \frac{1}{720},\ a_7 = \frac{1}{5040}$

From (20) and (21) we get:

$$\{a_5 + a_4 q_1 + a_3 q_2 + a_2 q_3 = 0\ a_6 + a_5 q_1 + a_4 q_2 + a_3 q_3 = 0\ a_7 + a_6 q_1 + a_5 q_2 + a_4 q_3 = 0 \tag{26}$$

and

$$\{\ p_0 = a_0 \quad p_1 = a_1 + a_0 q_1 \quad p_2 = a_2 + a_1 q_1 + a_0 q_2 \quad p_3 = a_3 + a_2 q_1 + a_1 q_2 + a_0 q_3\ p_4 = a_4 + a_3 q_1 + a_2 q_2 + a_1 q_3 \tag{27}$$

By solving the system (26) we get:

$$\{q_0 = 1 \quad q_1 =- \frac{3}{7} \quad q_2 = \frac{1}{14} \quad q_3 =- \frac{1}{210} \tag{28}$$

and substitute (28) in (27) we have:

$$\{p_0 = 1 \quad p_1 = \frac{4}{7} \quad p_2 = \frac{1}{7} \quad p_3 = \frac{2}{105}\ p_4 = \frac{1}{840} \tag{29}$$

Then we obtain:

$$[4,3]_{u(t)} = \frac{1+\frac{4}{7}t+\frac{1}{7}t^{2}+\frac{2}{105}t^{3}+\frac{1}{840}t^{4}}{1-\frac{3}{7}t+\frac{1}{14}t^{2}-\frac{1}{210}t^{3}}$$

Which can be simplified as following:

$$[4,3]_{u(t)} =- \frac{t^{4}+16\,t^{3}+120\,t^{2}+840\,t+840}{4\,t^{3}-60\,t^{2}+360\,t-840}$$

Then we have:

$$u(x,t)\cong -\,e^{x}\,\frac{t^{4}+16\,t^{3}+120\,t^{2}+840\,t+840}{4\,t^{3}-60\,t^{2}+360\,t-840}$$

In the following figure, a comparison was made between the approximate solution and the exact solution in different values for *x* and *t*=[0 ,1], where the exact solution is: $u(x,t) = e^{x+t}$.

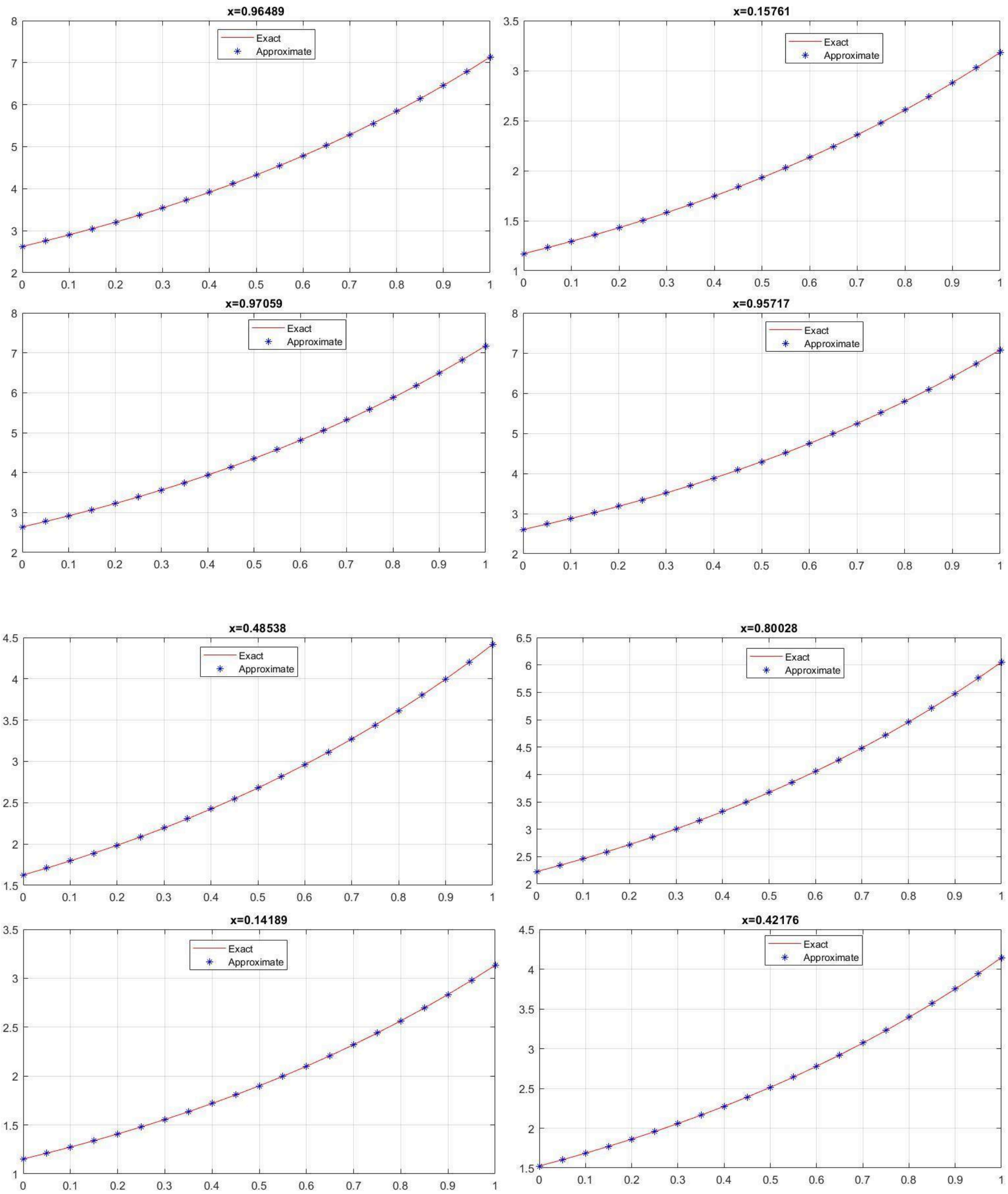

Figure 2: Illustrate case 2 with *t*=0:1 and different values of *x*

4.3 Case 3:

Consider the following nonlinear wave-like equation:

$$u_{tt} - \frac{\partial}{\partial x}\left(u_x u_{xx}\right) + x^2 u_{xx}^2 + u = 0, \quad 0 < x < 1, \ t > 0$$

By initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = x^2$$

Then $L(u) = \frac{\partial^2 u}{\partial t^2}$, i.e. n=2

$R(u) = u,\ N(u) = -\frac{\partial}{\partial x}\left(u_x u_{xx}\right) + x^2 u_{xx}^2$ then

$H(u) = u - \frac{\partial}{\partial x}\left(u_x u_{xx}\right) + x^2 u_{xx}^2$

$\phi = 0, f_0 = 0$ *and* $f_1 = x^2$.

By (15) and (12) we have:

$u_0 = f = \sum_{i=0}^{n-1} \frac{t^i}{i!} f_i + L^{-1}\left\{\frac{1}{s^2} L\{\phi\}\right\} = f_0 + t f_1 = t x^2$

$u_1 = -L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0\right)\right\}\right\} = -L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(e^x + t e^x\right)\right\}\right\} = -\frac{1}{6} x^2 t^3$

$u_2 = -L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0 + u_1\right) - H\left(u_0\right)\right\}\right\} = \frac{1}{120} x^2 t^5$

$u_3 = -L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0 + u_1 + u_2\right) - H\left(u_0 + u_1\right)\right\}\right\} = -\frac{1}{5040} x^2 t^7$

$u_4 = -L^{-1}\left\{\frac{1}{s^2} L\left\{H\left(u_0 + u_1 + u_2 + u_3\right) - H\left(u_0 + u_1 + u_3\right)\right\}\right\} = \frac{1}{3628800} x^2 t^9$

Then from (5), we have:

$$u(x, y, t) = x^2\left(t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{3628800} + \ldots\right)$$

The series can be approximated by $[4, 3]_{u(t)}$ as following:

$$[4, 3]_{u(t)} = \frac{p_0 + p_1 t + \ldots + p_5 t^5}{1 + q_1 t + \ldots + q_4 t^4} = a_0 + a_1 t + \ldots + a_9 t^9$$

where

$a_0 = a_2 = a_4 = a_6 = a_8 = 0,\ a_1 = 1,\ a_3 = -\frac{1}{6},\ a_5 = \frac{1}{120},\ a_7 = -\frac{1}{5040},\ a_9 = \frac{1}{3628800}$

From (20) and (21) we get:

$$\{a_6 + a_5 q_1 + a_4 q_2 + a_3 q_3 + a_2 q_4 = 0\ a_7 + a_6 q_1 + a_5 q_2 + a_4 q_3 + a_3 q_4 = 0\ a_8 + a_7 q_1 + a_6 q_2 + a_5 q_3 + a_4 q_4 = 0\ a_9 + a_8 q_1 + a_7 q_2 + a_6 q_3 + a_5 q_4 = 0 \quad (30)$$

and

$$\{p_0 = a_0 \quad p_1 = a_1 + a_0 q_1 \quad p_2 = a_2 + a_1 q_1 + a_0 q_2 \quad p_3 = a_3 + a_2 q_1 + a_1 q_2 + a_0 q_3 \quad p_4 = a_4 + a_3 q_1 + a_2 q_2 + a_1 q_3 + a_0 q_4\ p_5 = a_5 + a_4 q_1 + a_3 q_2 + a_2 q_3 + a_1 q_4 \quad (31)$$

By solving the system (30) we get:

$\{q_0 = 1 \quad q_1 = 0 \quad q_2 = 0.032828 \quad q_3 = 0 \quad q_4 = 0.00045$

and substitute (32) in (30) we have:

$$\{p_0 = 0 \qquad p_1 = 1 \qquad p_2 = 0$$
$$p_3 =- \ 0.13384 \ p_4 = 0 \qquad p_5 = 0.0033 \qquad (33)$$

Then we obtain:

$$[5,\ 4]_{u(t)} = \frac{t- 0.13384t^3+0.0033\, t^5}{1+0.032828\, t^2+0.00045\, t^4}$$

Then we have:

$$u(x,t) \cong x^2\ \frac{t- 0.13384t^3+0.0033\, t^5}{1+0.032828\, t^2+0.00045\, t^4}$$

In the following figure, a comparison was made between the approximate solution and the exact solution in different values for x and t=[0 ,1], where the exact solution is: $u(x,t) = x^2 sin(t)$.

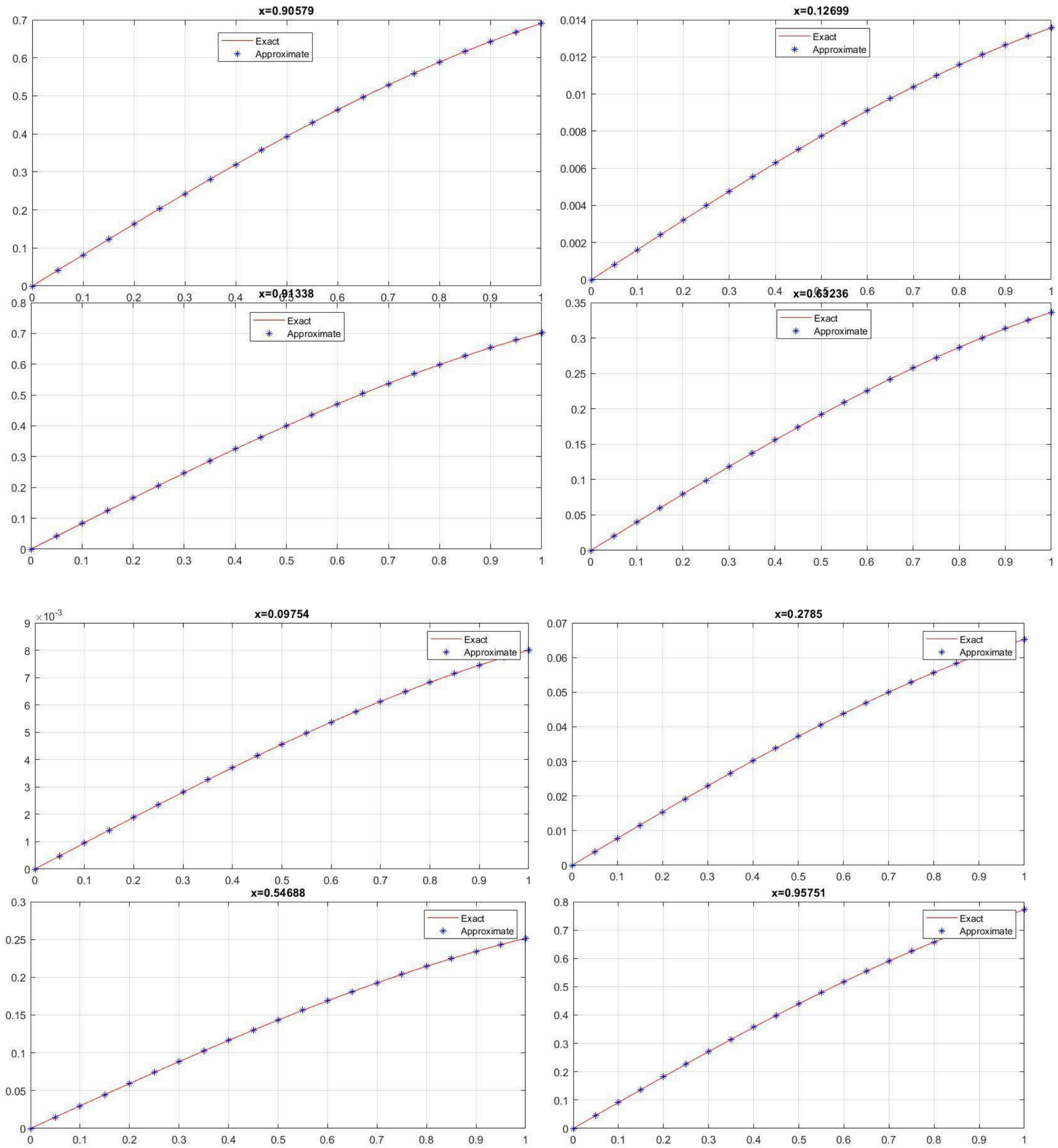

Figure 3: Illustrate case 3 with t=0:1 and different values of x

5. Conclusion

In Figures 1, 2 and 3, the solutions obtained by Pade approximation, of the non-linear wave-like equations with variable coefficients are illustrated numerically by taking different values of variables to compare these solutions with the exact solutions. These cases are shown the successful use of solution approximation by Pade approximation compared to power series expansion.

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