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Using Proposed Approach to Solve nonlinear Partial Differential Equations

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Abstract:

The purpose of this research is to employ a new method to solve nonlinear differential equations to obtain precise analytical solutions and overcome computation challenges without the need to discretize the domain or assume the presence of a small parameter, where the method demonstrated a quick and highly accurate solving nonlinear partial differential equations with initial conditions, in compared to existing methods. The phases of the proposed method are straightforward to implement, highly precise, and quickly converge to the correct result.

Keywords: suggested approach, converging analysis, nonlinear differential equations.

Introduction

A large number of significant ones were described using partial differential equations. Real-world patterns such as pollution, heat, waves, contamination, and response pattern. [1-5]. That is why coming up with a solution is crucial. Differential equations can be solved using a variety of techniques. Many writers have concentrated on solving non-linear PDEs using various techniques during the past few years, including HAM [5], VIM [6,7], DTM [8], ADM [9-10] and coupled method [11]. Recently,

modifications of effective techniques are now more widely employed to solve various partial differential equation types. In order to solve linear PDEs with variable coefficients in 2001, ADM was utilized by Wazwaz [14] to provide analytical answers to the nonlinear parabolic problem with time-varying and space-varying physical variables. ADM was utilized by Soufyane and Boulmaf [15]. The numerical answer to the regularized long-wave equation with non-constant coefficients was achieved by Achouri and Omrani [16] by the utilization of ADM. Additionally, a variety of numerical techniques are employed to discover the numerical solution of nonlinear PDEs like [17-18]. In this article, we'll introduce a new approach to solve nonlinear PDEs and obtain the exact analytical solution by ease steps.

Description of proposed approach

In this section, a suggested method for resolving equations using partial differentials are described. Take into account the following form of the non-linear partial differential equation:

$$L_t u(X,t) + L_X u(X,t) + L_{Xt} u(X,t) + R(u(X,t)) + N(u(X,t)) = g(X,t),\ X = (x_1, x_2, ...) \qquad (1).$$

where L_t represents the differential of t with the highest order., $L_X$represents The differential of x_i with the highest possible order , where i can be any number from 1 to n, and so on. $g(X,t)$ is an inhomogeneous or forcing term, and L_{Xt} is the highest order mix differential between x_i and t. R represents the linear terms of less derivatives that remain, N is an analytic nonlinear term. Partial solutions are solutions for u(X,t) that are acquired from the equations of the operators $L_X u$, $L_t u$ and $L_{Xt} u$. These are the solutions that are referred to as "partial." It has been established in the past that these several partial answers are interchangeable, and that all of them eventually lead to the same correct answer. However, the choice of the operators $L_X u$, $L_t u$ and $L_{Xt} u$ that should be utilized in order to solve the problem is mostly dependent on two grounds:

(i) In order to reduce the amount of effort required for computation, the operator with the lowest order should be used.

(ii) In order to quicken the process of evaluating the many components of the solution, the operator with the lowest order that is chosen should have the most well-known conditions.

Assuming that the operator L_t satisfies the requirements of both of the selection bases, we will now set

$$L_t(X,t) = g(X,t) - L_X u(X,t) - L_{Xt} u(X,t) - R(u(X,t)) - N(u(X,t)) \qquad (2)$$

Applying L_t^{-1} , so (2) gives

$u(x, y) = \emptyset_0 + L_t^{-1} g(X, t) - L_t^{-1} L_X u(X, t) - L_t^{-1} L_y u(X, t) - L_t^{-1} R(u(X, t)) - L_t^{-1} N(u(X, t))$
(3)

where $\emptyset_0 = u(X, 0),\ if\ L = \frac{\partial}{\partial t}$, $\emptyset_0 = u(X, 0) + t u_t(X, 0),\ if\ L = \frac{\partial^2}{\partial t^2}$, and so on

The solution u is given through an endless succession with the form:

$u(X, t) = \sum_{n=0}^{\infty} u_n(X, t)$ (4), such that

$u_0 = \emptyset_0 + L_t^{-1} g(X, t)$

$u_{n+1} = - L_t^{-1} L_X u_n(X, t) - L_t^{-1} L_y u_n(X, t) - L_t^{-1} R\left(u_n(X, t)\right) - L_t^{-1} N_n$

Regarding the phrase "nonlinear," $N(u)$ is

$N(u) = \sum_{k=0}^{\infty} N_k$ (5)

Where N_k can be calculated by

$N_k = \frac{1}{k!}\left[\frac{\partial^k N(u)}{\partial t^k}\right]_{t=0}$, $where\ \frac{\partial^k}{\partial t^k} u = k!\, u_k,\ \ k = 0, 1, 2, ...$ (6)

This method is very easy particularly if compare with Adomain decomposition method precisely in finding the nonlinear part the following section clears that . we will named this approach Fatema method.

Application

Example 1: Take into consideration the nonlinear differential equation of the second order, which is:

$u_t + u^2 u_x = 0$, (7)

With IC: $u(x, 0) = 2x$, $t > 0$, rewrite the eq.(7) as follow $u_t = - u^2 u_x$

It's clear that $X = x$, $R(u) = 0$, $N(u) = - u^2 u_x$ and $g(x, t) = 0$

$L_t(u) = - N(u)$ (8), such that $L_t = \frac{\partial}{\partial t}$

We take the $\left(L_t^{-1}\right)$ in both directions of eq.(8)

$u(x, t) = u(x, 0) - L_t^{-1}(N(u))$

$u_0 = 2x, \quad u_{(n+1)}(x, t) = - L_t^{-1}\left(N_n\right) \Rightarrow$

$u_1 = - L_t^{-1}\left(N_0\right)$

Now, calculate the nonlinear part as follow:

$N_0 = N(u) = u_0^2 u_{0x} = 8x^2$

$u_1 = - L_t^{-1}\left[N_0\right] = - L_t^{-1}\left[8x^2\right]$

$u_1 = - 8x^2 t$,

$u_2 = - L_t^{-1}\left[N_1\right]$

$N_1 = \left[\frac{\partial}{\partial t} N(u)\right]_{t=0} = \frac{\partial}{\partial t}\left(u^2 u_x\right) = \left(u^2 u_{xt} + 2u u_t u_x\right)$

$N_1 = \left(u_0^2 u_{1x} + 2u_0 u_1 u_{0x}\right)$

$N_1 = -128x^3 t$

$u_2 = -L_t^{-1}\left[-128x^3 t\right]$

$u_2 = 64x^3 t^2$, with the same procedure we get

$u_3 = -640x^4 t^3$, and so on.

The solution to equation 7 in the form of a series is provided by

$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \ldots$

$u = 2x - 8x^2 t + 64x^3 t^2 - 640x^4 t^3 + \ldots$

From above we observe the solution u is

$u = \{2x\ t = 0\ \frac{1}{4t}\left(\sqrt{1 + 16xt} - 1\right)\ t > 0$

Example 2: We will solve the fourth-order nonlinear differential equation in this example while considering the mixed derivative. Consider the problem to be as follows:

$u_{xt} - 6uu_{xx} - 6\left(u_x\right)^2 + u_{xxxx} + 3u_{yy} = 0\,,\qquad (9)$

Upon initial stipulation, $u_x(x, y, 0) = -\frac{1}{2}csc^2\left(\frac{1}{2}(x + y)\right) coth\left(\frac{1}{2}(x + y)\right)$ *, rewrite the eq.(9) as follow*

$u_{xt} = 6uu_{xx} + 6\left(u_x\right)^2 - u_{xxxx} - 3u_{yy}$

It's clear that $X = (x, y),\ R(u) = -u_{xxxx} - 3u_{yy},\ N(u) = 6uu_{xx} + 6\left(u_x\right)^2$ *and* $g(x, y, t) = 0$

$L_t(u_x) = N(u) - u_{xxxx} - 3u_{yy}\qquad (10),\quad$ *such that* $L_t = \frac{\partial}{\partial t}$

We take the $\left(L_t^{-1}\right)$ *to both sides of eq.(10)*

$u_x(x, y, t) = u_x(x, y, 0) + L_t^{-1}\left(N(u) - u_{xxxx} - 3u_{yy}\right)$

$u_{0x} = \frac{-1}{2}csch^2\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}(x + y)\right) \Rightarrow$

$u_0 = \frac{1}{2}csch^2\left(\frac{1}{2}(x + y)\right)$

$u_{(n+1)x}(x, y, t) = L_t^{-1}\left(N_n - u_{nxxxx} - 3u_{nyy}\right) \Rightarrow$

$u_{(n+1)}(x, y, t) = L_x^{-1}\left[L_t^{-1}\left(N_n - u_{nxxxx} - 3u_{nyy}\right)\right], L_x = \frac{\partial}{\partial x}$

$u_1 = L_x^{-1}\left[L_t^{-1}\left(N_0 - u_{0xxxx} - 3u_{0yy}\right)\right]$

$u_{0xx} = u_{0yy} = \frac{1}{4}csch^4\left(\frac{1}{2}(x + y)\right) + \frac{1}{2}csch^2\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(x + y)\right)$

$u_{0xxx} = -csch^4\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}(x + y)\right) - \frac{1}{2}csch^2\left(\frac{1}{2}(x + y)\right)coth^3\left(\frac{1}{2}(x + y)\right)$

$u_{0xxxx} = \frac{1}{2}csch^6\left(\frac{1}{2}(x + y)\right) + \frac{11}{4}csch^4\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(x + y)\right) + \frac{1}{2}csch^2\left(\frac{1}{2}(x + y)\right)coth^4\left(\frac{1}{2}(x + y)\right)$

Now, calculate the nonlinear part as follow:

$N_0 = N(u) = 6uu_{xx} + 6\left(u_x\right)^2 = 6u_0 u_{0xx} + 6\left(u_{0x}\right)^2$

$N_0 = \frac{3}{4}csch^6\left(\frac{1}{2}(x + y)\right) + 3csch^4\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(x + y)\right)$

$u_1 = L_x^{-1}\left[t\left[\frac{3}{4}csch^6\left(\frac{1}{2}(x + y)\right) + 3csch^4\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(x + y)\right) - \frac{1}{2}csch^6\left(\frac{1}{2}(x + y)\right) - \frac{11}{4}csch^4\left(\frac{1}{2}\ (x + y)\right)coth^2\left(\frac{1}{2}\ (\right.\right.\right.$

$u_1 = L_x^{-1}\left[t\left[\frac{1}{4}csch^6\left(\frac{1}{2}(x + y)\right) + \frac{1}{4}csch^4\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(x + y)\right) - \frac{1}{2}csch^2\left(\frac{1}{2}(x + y)\right)coth^4\left(\frac{1}{2}\ (x + y)\right) - \frac{3}{4}csch^4\left(\frac{1}{2}\ (x\right.\right.\right.$

$u_1 = L_x^{-1}\left[\left[-\ csch^4\left(\frac{1}{2}(x + y)\right) - 2csch^2\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(x + y)\right)\right]t\right]$

$u_1 = tL_x^{-1}\left[d\left(2csch^2\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}(x + y)\right)\right)\right], \qquad (11)$

after integrating eq. (11) w.r.t. x we get

$\Rightarrow u_1 = 2tcsch^2\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}(x + y)\right)$

$u_{1x} = t\left[-\ csch^4\left(\frac{1}{2}(x + y)\right) - 2csch^2\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(x + y)\right)\right]$

$u_{1xx} = \frac{t}{2}\left[4csch^4\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}(x + y)\right) + 4csch^4\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}(x + y)\right) + 4csch^2\left(\frac{1}{2}(x + y)\right)coth^3\left(\frac{1}{2}(x + y)\right)\right]$

$u_{1xxx} = 2t\left(-\ csch^6\left(\frac{1}{2}(x + y)\right) - 4csch^4\left(\frac{1}{2}\ (x + y)\right)coth^2\left(\frac{1}{2}\ (x + y)\right)\right) + t\left(-\ 3csch^4\left(\frac{1}{2}(x + y)\right)coth^2\left(\frac{1}{2}(\ x + y)\right) - 2csch^2\right.$

$u_{1xxxx} = 17tcsch^6\left(\frac{1}{2}\ (x + y)\right)coth\left(\frac{1}{2}\ (x + y)\right) + 26tcsch^4\left(\frac{1}{2}(x + y)\right)coth^3\left(\frac{1}{2}\ (x + y)\right) + 2tcsch^2\left(\frac{1}{2}\ (x + y)\right)coth^5\left(\frac{1}{2}\ (x +\right.$

$u_2 = L_x^{-1}\left[L_t^{-1}\left(N_1 - u_{1xxxx} - 3u_{1yy}\right)\right]$

$N_1 = \left[\frac{\partial}{\partial t}N(u)\right]_{t=0} = \frac{\partial}{\partial t}\left(6uu_{xx} + 6\left(u_x\right)^2\right) = 6\left(uu_{txx} + u_t u_{xx} + 2u_x u_{tx}\right)$

$N_1 = 6\left(u_0 u_{1xx} + u_1 u_{0xx} + 2u_{0x}u_{1x}\right)$

$N_1 = 21tcsch^6\left(\frac{1}{2}\ (x + y)\right)coth\left(\frac{1}{2}\ (x + y)\right) + 24tcsch^4\left(\frac{1}{2}\ (x + y)\right)coth^3\left(\frac{1}{2}\ (x + y)\right)$

$u_2 = L_x^{-1}\left[L_t^{-1}\left[21tcsch^6\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}(x + y)\right) + 24tcsch^4\left(\frac{1}{2}(x + y)\right)coth^3\left(\frac{1}{2}(x + y)\right) - 17tcsch^6\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}\ (\right.\right.\right.$

$u_2 = \frac{t^2}{2!}L_x^{-1}\left[-\ 16csch^4\left(\frac{1}{2}(x + y)\right)coth\left(\frac{1}{2}\ (x + y)\right) - 8csch^2\left(\frac{1}{2}(x + y)\right)coth^3\left(\frac{1}{2}(x + y)\right)\right]$

$u_2 = \frac{t^2}{2!}L_x^{-1}\left[-\ 8csch^4\left(\frac{1}{2}\ (x + y)\right)coth\left(\frac{1}{2}(\ x + y)\right) - 8\left(csch^4\left(\frac{1}{2}\ (x + y)\right)coth\left(\frac{1}{2}(x + y)\right) + csch^2\left(\frac{1}{2}(x + y)\right)coth^3\left(\frac{1}{2}\ (x +\right.\right.\right.$

$u_2 = \frac{t^2}{2!}\left[4csch^4\left(\frac{1}{2}(\ x + y)\right) + 8csch^2\left(\frac{1}{2}\ (x + y)\right)coth^2\left(\frac{1}{2}\ (x + y)\right)\right]$

and so forth

$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \ldots$

$u = \frac{1}{2}csch^2\left(\frac{1}{2}(x + y)\right) + 2tcsch^2\left(\frac{1}{2}\ (x + y)\right)coth\left(\frac{1}{2}(x + y)\right) + \frac{t^2}{2!}\left(4csch^4\left(\frac{1}{2}(x + y)\right) + 8csch^2\left(\frac{1}{2}\ (x + y)\right)coth^2\left(\frac{1}{2}(x + y\right.\right.$

$u = \left[\frac{1}{2}csch^2\left(\frac{1}{2}\ (x + y - 4t)\right)\right]_{t=0} + t\left[\frac{\partial}{\partial t}\ \left(\frac{1}{2}csch^2\left(\frac{1}{2}(x + y - 4t)\right)\right)\right]_{t=0} + \frac{t^2}{2!}\left[\frac{\partial^2}{\partial t^2}\ \left(\frac{1}{2}\ csch^2\left(\frac{1}{2}(x + y - 4t)\right)\right)\right]_{t=0} + \ldots$

That comes pretty close to becoming the solution to the problem:

$$u = \frac{1}{2}csch^2\left(\frac{1}{2}(x + y - 4t)\right)$$

Conclusion

While ADM and its variations can be used to solve the same examples by employing an Adomain polynomial to arrive at exact analytical solution, the nonlinear terms in this article the suggested approach is simpler to compute because they don't involve the use of Adomain polynomial or using any unknown parameter or complex rule to calculate those boundaries. Therefore, this methodis very efficient, simple to apply, and rapidly converges to precise results.

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