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**Some Properties of Two Integral Operators of a New Class of Univalent Functions Defined by a Linear Operator**

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**Abstract:**

The main object of this paper is to introduce and investigate an differential operator $D^{k+1+\eta}_{\delta z^{\eta}}\vartheta(z)$ of holomorphic function, and we determine conditions on the order ρ of the functions in the class $N(\rho)$ such that the integral operators will be in this class .

**Keywords:** Holomorphic function, differential operator, integral operators ,class $N(\rho)$ .

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**Introduction**

Let $A$ be the set of all holomorphic functions of the form

$$\vartheta(z) = z + \sum_{j=2}^{\infty} a_j z^j \qquad 1.1$$

Defined in the open unit disk $U = \{z\in C: |z| < 1\}$ and normalized by the condition $\vartheta(0) = \vartheta'(0) - 1 = 0$.Moreover,by S ,we shall denote the class of all functions in A, which are univalent in $U$ .A function $\vartheta(z)$ belonging to S is said to be starlike of order ρ $(0\leq\rho < 1)$ if it satisfies

$$R_e\left(\frac{z\vartheta'(z)}{\vartheta(z)}\right) > \rho, (z\in U)$$

We denote the set of these starlike functions of order ρ which lie in $U$ by the symbol $S^*(\rho)$ which is a subclass of A .Also a function $\vartheta(z)\in A$ will be contained in the class $R(\rho)$ if and only if

$$R_e\left(\vartheta'(z)\right) > \rho, (z\in U)$$

the subclass $N(\rho)$ of A containing a functions $\vartheta(z)$ so that it fulfills the condition

$$R_e\left(\frac{z\vartheta''(z)}{\vartheta'(z)}\right) < \rho \ ,(\rho > 1 , z\in U)$$

where many researchers presented the class $N(\rho)$ in their research ,and we mention from these sources [1], [8].

The integral operator

$$H_{\gamma}^{\alpha_j\beta_j}\left(\vartheta_1.\vartheta_2.\cdots.\vartheta_l\right) = \int_0^z \gamma t^{\gamma-1}\prod_{j=1}^{l}\left(\left(\vartheta_j(t)\right)^{\alpha_j}\left(\frac{\vartheta_j(t)}{t}\right)^{\beta_j dt}\right)^{\frac{1}{\gamma}} \qquad 1.2$$

Introduced and studied by Frasin [2],Narayan and Panigrahi [7],when $\gamma = 1$ , $\alpha_j = 0$ and $\beta_j = \frac{1}{S_j}(S_j > 0)$ for all $j = 1, 2,. ..., l$, we have the integral operator

$$F(z) = \int_0^z \prod_{j=1}^{n}\left(\frac{\vartheta_j(t)}{t}\right)^{\frac{1}{S_j}} dt \ , \qquad 1.3$$

Recall that Ularu and et.al.[6] introduced and studied the following integral

$$B(z) = \int_0^z \left(te^{\vartheta_j(t)}\right)^{\alpha_j} dt \qquad 1.4$$

Now, we define the differential operator as

$$D^{0+0}_{\delta z^0}\vartheta(z) = \vartheta(z)$$

$$D^{0+\eta}_{\delta z^\eta}\vartheta(z) = \frac{\Gamma(2)}{\Gamma(2-\eta)}z + \sum_{j=1}^{\infty}\frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}a_j z^j = D^{\eta}_{\delta z^\eta}\vartheta(z) \quad 1.5$$

$$D^{1+\eta}_{\delta z^\eta}\vartheta(z) = D_\delta(D^{\eta}_{\delta z^\eta}\vartheta(z)) = (1-\delta)D^{\eta}_{\delta z^\eta}\vartheta(z) + \delta z(D^{\eta}_{\delta z^\eta}\vartheta(z))'$$

$$= (1-\delta\eta)\frac{\Gamma(2)}{\Gamma(2-\eta)}z + \sum_{j=1}^{\infty}\frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}(1-\delta+\delta(j-\eta))a_j z^j;$$

$$\vdots$$

$$D^{k+\eta}_{\delta z^\eta}\vartheta(z) = D_\delta(D^{k-1+\eta}_{\delta z^\eta}\vartheta(z)) = (1-\delta)(D^{k-1+\eta}_{\delta z^\eta}) + \delta z(D^{k-1+\eta}_{\delta z^\eta})' \quad 1.6$$

$$= (1-\delta\eta)^k\frac{\Gamma(2)}{\Gamma(2-\eta)}z + \sum_{j=1}^{\infty}\frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}(1-\delta+\delta(j-\eta))^k a_j z^j, \; k.\eta \in N \text{ and } \delta \geq 0;$$

$$D^{k+1+\eta}_{\delta z^\eta}\vartheta(z) = D_\delta(D^{k+\eta}_{\delta z^\eta}\vartheta(z)) = (1-\delta)(D^{k+\eta}_{\delta z^\eta}) + \delta z(D^{k+\eta}_{\delta z^\eta})' \quad 1.7$$

$$= (1-\delta\eta)^{k+1}\frac{\Gamma(2)}{\Gamma(2-\eta)}z + \sum_{j=1}^{\infty}\frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}(1-\delta+\delta(j-\eta))^{k+1} a_j z^j.$$

To define a new family say $Q(\delta, \eta, k, \gamma, \rho)$ which includes various new subfamilies .

**Definition1.** for $0 \leq \rho < 1$, $\vartheta \in A$ give by 1.1 and $\gamma \geq 0$, a function $\vartheta$ is said to be in the family $Q(\delta, \eta, k, \gamma, \rho)$ if it satisfies the following condition

$$\left| \frac{D^{k+1+\eta}_{\delta z^\eta}\vartheta(z)}{z}\left(\frac{z}{D^{k+\eta}_{\delta z^\eta}\vartheta(z)}\right)^{\gamma} - 1 \right| < 1-\rho, \; (z \in U), \quad 1.8$$

where $k, \eta \in N$, $\delta \geq 0$.

The family $Q(\delta, \eta, k, \gamma, \rho)$ includes various new subfamilies of holomorphic functions .we observe that:

(1) set $\eta = 0, \delta = 1, k = 0$ in 1.5 and $\gamma = 1$,then the family $Q(1, 0, 0, 1, \rho)$ reduces to the class $Q(1, \rho) \equiv S^*(\rho)$ [see,3] .

(2) set $\delta = 1, \eta = 0, k = 0$ in 1.5 and $\gamma = 0$ ,then the family $Q(1, 0, 0, 0, \rho)$ reduces to the class $Q(0, \rho) \equiv R(\rho)$ .

(3) set $\delta = 1, \eta = 0, k = 0$ in 1.5,then the family $Q(1, 0, 0, \gamma, \rho)$ reduces to the class $Q(\gamma, \rho)$[see,6].

(4) putting $\delta = 1, \eta = 0, k = 0$ in 1.5 and $\gamma = 2$,then the family $Q(1, 0, 0, 2, \rho)$ reduces to the class $Q(2, \rho) \equiv Q(\rho)$ [see,4].

In this research ,we determined the conditions on the order $\rho$ to ensure that the two integral operators will be in $N(\rho)$ . Many results and conclusions are obtained.

**Lemma1.1**[5] (Schwarz lemma). Let $\vartheta(z)$ be holomorphic ,with $\vartheta(0) = 0$ .Then

$$|\vartheta(z)| \leq |z|, \; |z| < 1$$

and the equality satisfied if $\vartheta(z) = \lambda z$ , for $\lambda \in C$ (where $|\lambda|=1$ ).

**2-Main Results**

**Theorem 2.1.** If the functions $\vartheta_j(z)$ in $A$, for $j = 1, 2, ..., l$ be in $Q(\delta, \eta, k, \gamma, \rho)$ ,$A_j \geq 1, \delta \geq 0, 0 \leq \rho < 1$ , $z \in U$ then the integral operator

$$F(z) = \int_0^z \prod_{j=1}^{l}\left(\frac{D^{k+\eta}_{\delta z^\eta}\vartheta_j(t)}{t}\right)^{\frac{1}{S_j}} dt,$$

be in $N(\sigma)$ , where

$$\sigma = \sum_{j=1}^{l} \frac{1}{\delta|S_j|}\{(2-\rho)A_j^{\gamma-1}+1\}+1$$

and $\sum_{j=1}^{l} \frac{1}{\delta|S_j|}\left\{(2-\rho)A_j^{\gamma-1}+1\right\}+1>0,\ S_j\in\mathbb{C}\setminus\{0\}$ ,for all $j=1.\dots.l$ .

**Proof.** Let

$$F(z)=\int_0^z \prod_{j=1}^{l}\left(\frac{D_{\delta z^\eta}^{k+\eta}\vartheta_j(t)}{t}\right)^{\frac{1}{S_j}} dt$$

Calculate the derivative of two sides, we have

$$F_l'(z)=\left(\frac{D_{\delta z^\eta}^{k+\eta}\vartheta_1(z)}{z}\right)^{\frac{1}{S_1}}\left(\frac{D_{\delta z^\eta}^{k+\eta}\vartheta_2(z)}{z}\right)^{\frac{1}{S_2}}\dots\dots\dots\left(\frac{D_{\delta z^\eta}^{k+\eta}\vartheta_l(z)}{z}\right)^{\frac{1}{S_l}}$$

Take the derivative of both sides, we get

$$\frac{F''(z)}{F'(z)}=\frac{1}{S_1}\frac{1}{\left(\frac{D_{\delta z^\eta}^{k+\eta}\vartheta_1(z)}{z}\right)}\left(\frac{z(D_{\delta z^\eta}^{k+\eta}\vartheta_1(t))'-D_{\delta z^\eta}^{k+\eta}\vartheta_1(z)}{z^2}\right)+\dots+\frac{1}{S_l}\frac{1}{\left(\frac{D_{\delta z^\eta}^{k+\eta}\vartheta_l(z)}{z}\right)}\left(\frac{z(D_{\delta z^\eta}^{k+\eta}\vartheta_l(t))'-D_{\delta z^\eta}^{k+\eta}\vartheta_l(z)}{z^2}\right)$$

Multiply both sides by z

$$\frac{zF''(z)}{F'(z)}=\frac{1}{S_1}\left(\frac{z^3(D_{\delta z^\eta}^{k+\eta}\vartheta_1(t))'-z^2D_{\delta z^\eta}^{k+\eta}\vartheta_1(z)}{z^2}\right)\frac{1}{D_{\delta z^\eta}^{k+\eta}\vartheta_1(z)}+\dots+\frac{1}{S_l}\left(\frac{z^3(D_{\delta z^\eta}^{k+\eta}\vartheta_l(t))'-z^2D_{\delta z^\eta}^{k+\eta}\vartheta_l(z)}{z^2}\right)\frac{1}{D_{\delta z^\eta}^{k+\eta}\vartheta_l(z)}$$

$$\frac{zF''(z)}{F'(z)}=\frac{1}{S_1}\left(\frac{z(D_{\delta z^\eta}^{k+\eta}\vartheta_1(t))'}{D_{\delta z^\eta}^{k+\eta}\vartheta_1(z)}-1\right)+\dots+\frac{1}{S_l}\left(\frac{(D_{\delta z^\eta}^{k+\eta}\vartheta_l(t))'}{D_{\delta z^\eta}^{k+\eta}\vartheta_l(z)}-1\right)$$

Thus, we get

$$R_e\left\{\frac{zF''(z)}{F'(z)}+1\right\}=R_e\left(\sum_{j=1}^{l}\frac{1}{S_j}\left(\frac{(D_{\delta z^\eta}^{k+\eta}\vartheta_j(t))'}{D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}-1\right)+1\right)$$

Since $z(D_{\delta z^\eta}^{k+\eta}\vartheta_j(t))'=\frac{D_{\delta z^\eta}^{k+1+\eta}\vartheta_j(z)-(1-\delta)D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}{\delta}$

So, we have

$$R_e\left\{\frac{zF''(z)}{F'(z)}+1\right\}=R_e\left(\sum_{j=1}^{l}\frac{1}{S_j}\left(\frac{D_{\delta z^\eta}^{k+1+\eta}\vartheta_j(z)-(1-\delta)D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}{\delta D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}-1\right)+1\right)$$

$$=R_e\left(\sum_{j=1}^{l}\frac{1}{\delta S_j}\left\{\frac{D_{\delta z^\eta}^{k+1+\eta}\vartheta_j(z)}{D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}-1\right\}+1\right)$$

$$<\left(\sum_{j=1}^{l}\frac{1}{\delta|S_j|}\left\{\left|\frac{D_{\delta z^\eta}^{k+1+\eta}\vartheta_j(z)}{D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}\right|+1\right\}+1\right)$$

$$<\left(\sum_{j=1}^{l}\frac{1}{\delta|S_j|}\left\{\left|\frac{D_{\delta z^\eta}^{k+1+\eta}\vartheta_j(z)}{z}\left(\frac{z}{D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}\right)^{\gamma}\right|\left|\left(\frac{D_{\delta z^\eta}^{k+\eta}\vartheta_j(z)}{z}\right)^{\gamma-1}\right|+1\right\}+1\right)$$

Since $\vartheta_j(z)\in Q(\delta,\eta,k,\gamma,\rho)$ and $|\vartheta_j(z)|\le A_j$ ,applying Schwarz Lemma ,we obtain

$$R_e\left\{\frac{zF''(z)}{F'(z)}+1\right\}<\left(\sum_{j=1}^{l}\frac{1}{\delta|S_j|}\left\{\left|\frac{D_{\delta z^{\eta}}^{k+1+\eta}\vartheta_j(z)}{z}\left(\frac{z}{D_{\delta z^{\eta}}^{k+\eta}\vartheta_j(z)}\right)^{\gamma}\right|\left[A_j\right]^{\gamma-1}+1\right\}+1\right)$$

$$<\left(\sum_{i=1}^{l}\frac{1}{\delta|S_j|}\left\{\left|\frac{D_{\delta z^{\eta}}^{k+1+\eta}\vartheta_j(z)}{z}\left(\frac{z}{D_{\delta z^{\eta}}^{k+\eta}\vartheta_j(z)}\right)^{\gamma}-1+1\right|\left[A_j\right]^{\gamma-1}+1\right\}+1\right)$$

$$<\left(\sum_{j=1}^{l}\frac{1}{\delta|S_j|}\left\{\left\{\left|\frac{D_{\delta z^{\eta}}^{k+1+\eta}\vartheta_j(z)}{z}\left(\frac{z}{D_{\delta z^{\eta}}^{k+\eta}\vartheta_j(z)}\right)^{\gamma}-1\right|+1\right\}\left[A_j\right]^{\gamma-1}+1\right\}+1\right)$$

$$<\left(\sum_{j=1}^{l}\frac{1}{\delta|S_j|}\left\{\{1-\rho+1\}\left[A_j\right]^{\gamma-1}+1\right\}+1\right)$$

$$<\left(\sum_{j=1}^{l}\frac{1}{\delta|S_j|}\left\{(2-\rho)\left[A_j\right]^{\gamma-1}+1\right\}+1\right)=\sigma$$

Therefore $F(z)$ will be in $N(\sigma)$ . The proof is complete.

If (1) $\gamma = 1$ (2) $\delta = 1$ (3) $\delta = 1$ , $\gamma = 1$ in the above theorem , we have the following Corollaries .

**Corollary 2.1.** If the functions $\vartheta_j(z)$ in $A$,for $j = 1, 2,. ..., l$ be in $S^*(\rho)$,$0\leq\rho < 1$, $A_j\geq 1$ , $z$ in $\in U$ then $F_l(z)$ is in $N(\sigma)$ , where

$$\sigma = \sum_{j=1}^{l}\frac{1}{\delta|S_j|}\{(2-\rho)+1\}+1 ,$$

and $\sum_{j=1}^{l}\frac{1}{\delta|S_j|}(2-\rho)+1>0$, $S_j\in\mathbb{C}\backslash\{0\}$ ,for all $j = 1, ..., l$ .

**Corollary 2.2**[6, Theorem 2.1]**.** If the functions $\vartheta_j(z)$ in $A$,for $j = 1, 2, ..., l$ be in $Q(\gamma,\rho)$ ,$A_j\geq 1$, $\delta\geq 0$, $0\leq\rho < 1$ , $z$ in $U$ then $F_l(z)$ is in $N(\sigma)$ , where

$$\sigma = \sum_{j=1}^{l}\frac{1}{|S_j|}\{(2-\rho)A_j^{\gamma-1}+1\}+1$$

and $\sum_{j=1}^{l}\frac{1}{|S_j|}(2-\rho)A_j^{\gamma-1}+1>0$, $S_j\in\mathbb{C}\backslash\{0\}$ ,for all $j = 1. .... l$ .

**Corollary 2.3**[6, Corollary 2.2]**.** If the functions $\vartheta_j(z)$ in $A$,for $j = 1, 2, ..., l$ be in $S^*(\rho)$ ,$A_j\geq 1$, $\delta\geq 0$, $0\leq\rho < 1$ , $z$ in $U$ then I.O. defined by 2.1 is in $N(\sigma)$ , where

$$\sigma = \sum_{j=1}^{l}\frac{1}{|S_j|}\{(2-\rho)+1\}+1$$

And $\sum_{j=1}^{l}\frac{1}{|S_j|}\{(2-\rho)+1\}+1>0$, $S_j\in\mathbb{C}\backslash\{0\}$ ,for all $j = 1. .... l$ .

**Theorem 2.2.** For $j = 1, 2, ..., l$ the functions $\vartheta_j(z)\in A$, be in $Q(\delta,\eta,k,\gamma,\rho)$ ,$A_j\geq 1$

,$\delta \geq 0, 0 \leq \rho < 1$ , $z$ in $U$ then the integral operator

$$B(z) = \int_0^z \left( te^{D_{\delta z^\eta}^{k+\eta} \vartheta_j(t)} \right)^{\alpha_j} dt$$

be in $N(\tau)$ , where

$$\tau = \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ \{2 - \rho\} \left(A_j\right)^\gamma + (1 - \delta) A_j + \delta \right\} + 1$$

and $\sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ \{2 - \rho\} \left(A_j\right)^\gamma + (1 - \delta) A_j + \delta \right\} > 0,\ \alpha_j \in \mathbb{C} \backslash \{0\}$ ,for all $j = 1, ..., l$ .

**Proof.** Let

$$B(z) = \int_0^z \left( te^{D_{\delta z^\eta}^{k+\eta} \vartheta_j(t)} \right)^{\alpha_j} dt$$

Calculate the derivative of two sides, we have

$$F_l'(z) = \left(ze^{D_{\delta z^\eta}^{k+\eta} \vartheta_1(z)}\right)^{\alpha_1} \left(ze^{D_{\delta z^\eta}^{k+\eta} \vartheta_2(z)}\right)^{\alpha_2} \ldots\ldots\ldots \left(ze^{D_{\delta z^\eta}^{k+\eta} \vartheta_l(z)}\right)^{\alpha_l}$$

Take the derivative of both sides, we get

$$\frac{F''(z)}{F'(z)} = \alpha_1 \frac{1}{\left(ze^{D_{\delta z^\eta}^{k+\eta} \vartheta_1(z)}\right)} \left( ze^{D_{\delta z^\eta}^{k+\eta} \vartheta_1(z)} \left( D_{\delta z^\eta}^{k+\eta} \vartheta_1(z) \right)' + e^{D_{\delta z^\eta}^{k+\eta} \vartheta_1(z)} \right)$$

$$+ \ldots + \alpha_l \frac{1}{\left(ze^{D_{\delta z^\eta}^{k+\eta} \vartheta_l(z)}\right)} \left( ze^{D_{\delta z^\eta}^{k+\eta} \vartheta_l(z)} \left( D_{\delta z^\eta}^{k+\eta} \vartheta_l(z) \right)' + e^{D_{\delta z^\eta}^{k+\eta} \vartheta_l(z)} \right)$$

This will lead to the following result after multiplying by z

$$\frac{z(F''(z))}{F'(z)} = \left( \alpha_1 \left[ z(D_{\delta z^\eta}^{k+\eta} \vartheta_1(z))' + 1 \right] + \ldots + \alpha_l \left[ z(D_{\delta z^\eta}^{k+\eta} \vartheta_l(z))' + 1 \right] \right)$$

Hence, we get

$$R_e\left\{ \frac{zF''(z)}{F'(z)} + 1 \right\} = R_e \left( \sum_{j=1}^{l} \alpha_i \left( z(D_{\delta z^\eta}^{k+\eta} \vartheta_j(z))' + 1 \right) + 1 \right)$$

Since $z(D_{\delta z^\eta}^{k+\eta} \vartheta_j(t))' = \frac{D_{\delta z^\eta}^{k+1+\eta} \vartheta_j(z) - (1-\delta) D_{\delta z^\eta}^{k+\eta} \vartheta_j(z)}{\delta}$

So, we have

$$R_e\left\{ \frac{zF''(z)}{F'(z)} + 1 \right\} = R_e \left( \sum_{j=1}^{l} \frac{\alpha_j}{\delta} \left( D_{\delta z^\eta}^{k+1+\eta} \vartheta_j(z) - (1 - \delta) D_{\delta z^\eta}^{k+\eta} \vartheta_j(z) + \delta \right) + 1 \right)$$

$$= R_e \left( \sum_{j=1}^{l} \frac{\alpha_j}{\delta} \left\{ D_{\delta z^\eta}^{k+\eta} \vartheta_j(z) \left( \frac{D_{\delta z^\eta}^{k+1+\eta} \vartheta_j(z)}{D_{\delta z^\eta}^{k+\eta} \vartheta_j(z)} - (1 - \delta) \right) + \delta \right\} + 1 \right)$$

$$< \left( \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ \left| D_{\delta z^\eta}^{k+\eta} \vartheta_j(z) \right| \left( \left| \frac{D_{\delta z^\eta}^{k+1+\eta} \vartheta_j(z)}{D_{\delta z^\eta}^{k+\eta} \vartheta_j(z)} \left( \frac{z}{D_{\delta z^\eta}^{k+\eta} \vartheta_j(z)} \right)^\gamma \left( \frac{D_{\delta z^\eta}^{k+\eta} \vartheta_j(z)}{z} \right)^{\gamma - 1} \right| + (1 - \delta) \right) + \delta \right\} + 1 \right)$$

$$< \left( \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ \left| D_{\delta z^{\eta}}^{k+\eta} \vartheta_j(z) \right| \left( \left[ \left\{ \left| \frac{D_{\delta z^{\eta}}^{k+1+\eta} \vartheta_j(z)}{D_{\delta z^{\eta}}^{k+\eta} \vartheta_j(z)} \left( \frac{z}{D_{\delta z^{\eta}}^{k+\eta} \vartheta_j(z)} \right)^{\gamma} - 1 \right| + 1 \right\} \left| \left( \frac{D_{\delta z^{\eta}}^{k+\eta} \vartheta_j(z)}{z} \right)^{\gamma-1} \right| \right] + (1-\delta) \right) + \delta \right\} + 1 \right)$$

$$< \left( \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ A_j \left( \left[ \left\{ \left| \frac{D_{\delta z^{\eta}}^{k+1+\eta} \vartheta_j(z)}{D_{\delta z^{\eta}}^{k+\eta} \vartheta_j(z)} \left( \frac{z}{D_{\delta z^{\eta}}^{k+\eta} \vartheta_j(z)} \right)^{\gamma} - 1 \right| + 1 \right\} (A_j)^{\gamma-1} \right] + (1-\delta) \right) + \delta \right\} + 1 \right)$$

$$< \left( \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ A_j \left( \left[ \{1 - \rho + 1\} (A_j)^{\gamma-1} \right] + (1-\delta) \right) + \delta \right\} + 1 \right)$$

$$< \left( \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ A_j \left( \left[ \{2 - \rho\} (A_j)^{\gamma-1} \right] + (1-\delta) \right) + \delta \right\} + 1 \right)$$

$$< \left( \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ \{2 - \rho\} (A_j)^{\gamma} + (1-\delta) A_j + \delta \right\} + 1 \right) = \tau$$

Therefore $B(z)$ is in $N(\tau)$.

The proof is complete.

If (1) $\gamma = 0$ (2) $\gamma = 1$ (3) $\delta = 1$(4) $\delta = 1$, $\gamma = 0$ (5) $\delta = 1$, $\gamma = 1$ in the above theorem. Then, we obtained the following results.

**Corollary 2.4.** Let the functions $\vartheta_j(z)$ in $A$,for$j = 1, 2, ..., l$ be in $R(\rho)$ ,$A_j \geq 1$ ,$\delta \geq 0$, $0 \leq \rho < 1$ , $z$ in $U$ then $B(z)$ is in $N(\tau)$ , where

$$\tau = \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ \{2 - \rho\} + (1-\delta) A_j + \delta \right\} + 1$$

and $\sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ \{2 - \rho\} + (1-\delta) A_j + \delta \right\} > 0$, $\alpha_j \in \mathbb{C} \setminus \{0\}$ ,for all $j = 1, ..., l$ .

**Corollary 2.5.** Let the functions $\vartheta_j(z)$ in $A$,for $j = 1, 2, ..., l$ be in $S^*(\rho)$ ,$A_j \geq 1$ ,$\delta \geq 0$, $0 \leq \rho < 1$ , $z$ in $U$ then $B(z)$is in $N(\tau)$ , where

$$\tau = \sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ (\{2 - \rho\} + (1-\delta)) A_j + \delta \right\} + 1$$

and $\sum_{j=1}^{l} \frac{|\alpha_j|}{\delta} \left\{ (\{2 - \rho\} + (1-\delta)) A_j + \delta \right\} > 0$, $\alpha_j \in \mathbb{C} \setminus \{0\}$ ,for all $j = 1, ..., l$ .

**Corollary 2.6.** Let the functions $\vartheta_j(z)$ in $A$,for$j = 1, 2, ..., l$ be in $Q(\gamma, \rho)$ ,$A_j \geq 1$ ,$\delta \geq 0$, $0 \leq \rho < 1$ , $z$ in $U$ then $B(z)$is in $N(\tau)$ , where

$$\tau = \sum_{j=1}^{l} |\alpha_j| \left\{ \{2 - \rho\} (A_j)^{\gamma} + 1 \right\} + 1$$

and $\sum_{j=1}^{l} |\alpha_j| \left\{ \{2 - \rho\} (A_j)^{\gamma} + 1 \right\} > 0$, $\alpha_j \in \mathbb{C} \setminus \{0\}$ ,for all $j = 1, ..., l$ .

**Corollary 2.7.** Let the functions $\vartheta_j(z)$ in $A$,for $j = 1, 2, ..., l$ be in $R(\rho)$ ,$A_j \geq 1$ ,$\delta \geq 0$, $0 \leq \rho < 1$ , $z$ in $U$ then $B(z)$ is in $N(\tau)$ , where

$$\tau = \sum_{j=1}^{l} |\alpha_j| \{3 - \rho\} + 1$$

and $\sum_{j=1}^{l}\left|\alpha_j\right|\{3 - \rho\} > 0,\ \alpha_j \in \mathbb{C}\backslash\{0\}$ ,for all $j = 1, ..., l$ .

**Corollary 2.8.** Let the functions $\vartheta_j(z)$ in $A$,for $j = 1, 2, ..., l$ be in $S^*(\rho)$ ,$A_j \geq 1$ ,$\delta \geq 0, 0 \leq \rho < 1$ , $z$ in $U$ then $B(z)$is in $N(\tau)$ , where

$$\tau = \sum_{j=1}^{l}\left|\alpha_j\right|\left\{\{2 - \rho\}A_j + 1\right\} + 1$$

and $\sum_{j=1}^{l}\left|\alpha_j\right|\left\{\{2 - \rho\}A_j + 1\right\} > 0,\ \alpha_j \in \mathbb{C}\backslash\{0\}$ ,for all $j = 1, ..., l$ .

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