DOI: https://doi.org/10.24297/jam.v20 i.9253

# Certain Families of Holomorphic and Sălăgean Type Bi-Univalent Functions Defined by (p,q)-Lucas Polynomials Involving a Modified Sigmoid Activation Function

Ali Mohammed Ramadhan[1] and Najah Ali Jiben Al-Ziadi [2]

[1] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya-Iraq

[2] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya-Iraq

[1] edu-math.post15@qu.edu.iq [2] najah.ali@qu.edu.iq

**Abstract:**

The aim of the present paper is to introduce a certain families of holomorphic and Sălăgean type bi-univalent functions by making use $(p, q)$ −Lucas polynomials involving the modified sigmoid activation function $\phi(\delta) = \frac{2}{1+e^{-\delta}}$, $\delta \geq 0$ in the open unit disk $\Delta$. For functions belonging to these subclasses, we obtain upper bounds for the second and third coefficients. Also, we debate Fekete-Szegö inequality for these families. Further, we point out several certain special cases for our results.

**Keywords:** Holomorphic function, Bi-univalent functions, Fekete-Szegö inequality, Lucas polynomials, Sălăgean operator, modified sigmoid function.

**2010 AMS Mathematics Subject Classification:** 30C45, 30C50.

## Introduction

Symbolized by $\mathcal{A}$ the functions class of the form:

$$k(s) = s + \sum_{n=2}^{\infty} d_n s^n, \tag{1}$$

which are holomorphic in the open unit disk $\Delta = \{ s : s \in \mathbb{C} \; and \; |s| < 1\}$ and normalized under the conditions indicated by $k(0) = k'(0) - 1 = 0$. Furthermore, symbolized by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\Delta$ .

For each $k \in \mathcal{S}$, the Koebe one-quarter theorem [6] states that the image of the open unit disk $\Delta$ under $k$ contains a disk of radius $1/4$. Thus, every univalent function $k$ has an inverse $k^{-1}$, which is defined by

$$k^{-1}\big(k(s)\big) = s \qquad (s \in \Delta)$$

and

$$k\big(k^{-1}(r)\big) = r \quad \left(|r| < r_0\,(k);\; r_0(k) \geq \frac{1}{4}\right),$$

where

$$k^{-1}(r) = h(r) = r - d_2 r^2 + (2d_2^2 - d_3) r^3 - (5d_2^3 - 5d_2 d_3 + d_4) r^4 + \cdots. \tag{2}$$

Let $D$ be the class of function $F$ which is holomorphic in $\Delta$ with

$$F(0) = 0 \qquad and \qquad |F(s)| < 1 \qquad ( s \in \Delta ).$$

Let $k(s)$ and $y(s)$ be holomorphic in $\Delta$ then the function $k(s)$ is said to be subordinate to $y(s)$ in $\Delta$ written by

$$k(s) \prec y(s) \qquad (s \in \Delta), \tag{3}$$

such that $k(s) = y\big(F(s)\big), (s \in \Delta)$. From the definition of the subordination, it is easy to show that the subordination (3) implies that

$$k(0) = y(0) \text{ and } k(\Delta) \subset y(\Delta). \tag{4}$$

In particular, if $y(s)$ is univalent in $\Delta$, then the subordination (3) is equivalent to the condition (4).

The function $k \in \mathcal{A}$ is considered bi-univalent in $\Delta$ if both $k$ and $k^{-1}$ are univalent in $\Delta$. Indicated by the Taylor-Maclaurin series expansion (1), the class of all bi-univalent function in $\Delta$ can be symbolized by $\Sigma$. In the year 2010, Srivastava et al. [15] refreshed the study of various classes of bi-univalent functions. Moreover, many penmaus explored bounds for different subclasses of bi-univalent function (see, for examples [1, 2, 3, 4, 5, 8, 13, 14, 16, 17]). The coefficient estimate problem involving the bound of $|d_n|$ ( $n \in \mathbb{N} - \{1,2\}, \mathbb{N} = \{1,2,3,\dots\}$) is still an open problem.

Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p,q)$ −polynomials $L_{p,q,n}(x)$, or briefly $L_n(x)$ are given by the following recurrence relation (see[9, 10]):

$$L_n(x) = p(x)L_{n-1}(x) + q(x)L_{n-2}(x) \qquad (n \in \mathbb{N} - \{1\}),$$

with

$$\begin{aligned} &L_0(x) = 2, \\ &L_1(x) = p(x), \\ &L_2(x) = p^2(x) + 2q(x), \\ &L_3(x) = p^3(x) + 3p(x)q(x), \dots \end{aligned} \tag{5}$$

The generating function of the Lucas polynomials $L_n(x)$ (see [11]) is given by:

$$G_{L_n(x)}(s) = \sum_{n=0}^{\infty} L_n(x)s^n = \frac{2 - p(x)s}{1 - p(x)s - q(x)s^2}.$$

Note that for particular values of $p$ and $q$, the $(p,q)$ −polynomial $L_n(x)$ leads to various polynomials, among those, we list few cases here (see, [11] for more details, also [2]):

1) For $p(x) = x$ and $q(x) = 1$, we obtain the Lucas polynomials $L_n(x)$.
2) For $p(x) = 2x$ and $q(x) = 1$, we attain the pell-Lucas polynomials $Q_n(x)$.
3) For $p(x)$ =1 and $q(x) = 2x$, we attain the Jacobsthal-Lucas polynomials $j_n(x)$.
4) For $p(x) = 3x$ and $q(x) = -2$, we obtain the Fermat-Lucas polynomials $f_n(x)$.
5) For $p(x) = 2x$ and $q(x) = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind.

Let $\mathcal{A}_\phi$ denoted the class of functions of the form:

$$k_\phi(s) = s + \sum_{n=2}^{\infty} \frac{2}{1+e^{-\delta}} d_n\, s^n = s + \sum_{n=2}^{\infty} \phi(\delta) d_n\, s^n, \tag{6}$$

where $\phi(\delta) = \frac{2}{1+e^{-\delta}}$ is the sigmoid activation function and $\delta \geq 0$. Also, $\mathcal{A}_1 = \mathcal{A}$ (see[7]).

We consider a differential operator $D^t$, $t \in N \cup \{0\}$, (see[12]) for $k_\phi$ belongs to $\mathcal{A}_\phi$, defined by

$$D^0 k_\phi(s) = k_\phi(s);\ D^1 k_\phi(s) = Dk_\phi(s) = sk'_\phi(s);\ D^t k_\phi(s) = D\left(D^{t-1}k_\phi(s)\right).$$

We note that

$$D^t k_\phi(s) = s + \sum_{n=2}^{\infty} \frac{2n^t}{1+e^{-\delta}} d_n\, s^n = s + \sum_{n=2}^{\infty} n^t \phi(\delta) d_n\, s^n. \tag{7}$$

In this paper, we introduce a certain families of bi-univalent functions defined through the $(p,q)$ −Lucas polynomials. Furthermore, we derive coefficient estimates and Fekete–Szegö inequality for functions defined in those classes.

**Cofficient bounds and Fekete–Szegö inequality for the class $\mathcal{W}_\Sigma(\alpha, t, \phi(\delta); x)$**

**Definition 1** A function $k \in \Sigma$ is said to be in the class $\mathcal{W}_\Sigma(\alpha, t, \phi(\delta); x)$ for $0 \leq \alpha \leq 1$ , $t \in N \cup \{0\}$ and $\phi(\delta) = \frac{2}{1+e^{-\delta}}$, $\delta \geq 0$, if the following conditions of subordination are satisfied:

$$\frac{s\left(D^t k_\phi(s)\right)' + (2\alpha^2 - \alpha)\, s^2 \left(D^t k_\phi(s)\right)''}{4(\alpha - \alpha^2)s + (2\alpha^2 - \alpha)s\left(D^t k_\phi(s)\right)' + (2\alpha^2 - 3\alpha + 1)D^t k_\phi(s)} \prec G_{L_n(x)}(s) - 1 \tag{8}$$

and

$$\frac{r\left(D^t h_\phi(r)\right)' + (2\alpha^2 - \alpha)\, r^2 \left(D^t h_\phi(r)\right)''}{4(\alpha - \alpha^2)r + (2\alpha^2 - \alpha)r\left(D^t h_\phi(r)\right)' + (2\alpha^2 - 3\alpha + 1)D^t h_\phi(r)} \prec G_{L_n(x)}(r) - 1, \tag{9}$$

where the function $h = k^{-1}$ is indicated by (2).

**Remark 1**

1) For $\alpha = 0$, the function class $\mathcal{W}_\Sigma(\alpha, t, \phi(\delta); x)$ shortens to the function class $S^*_\Sigma(\phi(\delta); x)$ presented and investigated by Swamy et al. [16].
2) For $\alpha = 0$, $t = 0$ and $\phi(\delta) = 1$, the function class $\mathcal{W}_\Sigma(\alpha, t, \phi(\delta); x)$ shortens to the function class $S(x)$ presented and investigated by Altinkaya [1].
3) For $\alpha = \frac{1}{2}$, $t = 0$ and $\phi(\delta) = 1$, the function class $\mathcal{W}_\Sigma(\alpha, t, \phi(\delta); x)$ shortens to the function class $\mathcal{W}_\Sigma(\tau = 1; x)$ presented and investigated by Altinkaya and Yalçin [3].

**Theorem 1** Let the function $k \in \Sigma$ indicated by (1) be in the class $\mathcal{W}_\Sigma(\alpha, t, \phi(\delta); x)$. Then

$$|d_2| \leq \frac{|px|\sqrt{|px|}}{2^t \phi(\delta)\sqrt{|[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1) - (1 + 3\alpha - 2\alpha^2)^2]p^2(x) - 2(1 + 3\alpha - 2\alpha^2)^2 q(x)|}} \tag{10}$$

and

$$|d_3| \leq \frac{|p(x)|}{2(2\alpha^2 + 1)3^t\phi(\delta)} + \frac{p^2(x)}{(1 + 3\alpha - 2\alpha^2)3^t\phi(\delta)}, \tag{11}$$

and for some $\mu \in \mathbb{R}$,

$$|d_3 - \mu d_2^2| \leq \begin{cases} \dfrac{|p(x)|}{2(2\alpha^2 + 1)3^t\phi(\delta)} \quad if \\ \left|1 - \mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right| \leq \dfrac{|[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1) - (1 + 3\alpha - 2\alpha^2)^2]p^2(x) - 2(1 + 3\alpha - 2\alpha^2)^2 q(x)|}{2(2\alpha^2 + 1)p^2(x)} \\ \dfrac{|p(x)|^3\left|1 - \mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right|}{|[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1) - (1 + 3\alpha - 2\alpha^2)^2]p^2(x) - 2(1 + 3\alpha - 2\alpha^2)^2 q(x)|3^t\phi(\delta)} \quad if \\ \left|1 - \mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right| \geq \dfrac{|[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1) - (1 + 3\alpha - 2\alpha^2)^2]p^2(x) - 2(1 + 3\alpha - 2\alpha^2)^2 q(x)|}{2(2\alpha^2 + 1)p^2(x)}. \end{cases} \tag{12}$$

**Proof.** Let $k \in \mathcal{W}_\Sigma(\alpha, t, \phi(\delta); x)$ be given by Taylor-Maclaurin expansion (1). Then, there are two holomorphic functions $u$ *and* $v$ such that

$$u(0) = 0, \qquad v(0) = 0,$$

$$|u(s)| = |m_1 s + m_2 s^2 + \cdots| < 1 \;,\; |v(r)| = |n_1 r + n_2 r^2 + \cdots| < 1 \qquad (\forall\, s, r \in \Delta).$$

Hence, we can write

$$\frac{s\left(D^t k_\phi(s)\right)' + (2\alpha^2 - \alpha)\, s^2 \left(D^t k_\phi(s)\right)''}{4(\alpha - \alpha^2)s + (2\alpha^2 - \alpha)s\left(D^t k_\phi(s)\right)' + (2\alpha^2 - 3\alpha + 1)D^t k_\phi(s)} = G_{L_n(x)}\big(u(s)\big) - 1$$

and

$$\frac{r\left(D^t h_\phi(r)\right)' + (2\alpha^2-\alpha)\,r^2\left(D^t h_\phi(r)\right)''}{4(\alpha-\alpha^2)r + (2\alpha^2-\alpha)r\left(D^t h_\phi(r)\right)' + (2\alpha^2-3\alpha+1)D^t h_\phi(r)} = G_{L_n(x)}\big(v(r)\big) - 1.$$

Or, equivalently,

$$\frac{s\left(D^t k_\phi(s)\right)' + (2\alpha^2-\alpha)\,s^2\left(D^t k_\phi(s)\right)''}{4(\alpha-\alpha^2)s + (2\alpha^2-\alpha)s\left(D^t k_\phi(s)\right)' + (2\alpha^2-3\alpha+1)D^t k_\phi(s)} = -1 + L_0(x) + L_1(x)u(s) + L_2(x)[u(s)]^2 + \cdots$$

and

$$\frac{r\left(D^t h_\phi(r)\right)' + (2\alpha^2-\alpha)\,r^2\left(D^t h_\phi(r)\right)''}{4(\alpha-\alpha^2)r + (2\alpha^2-\alpha)r\left(D^t h_\phi(r)\right)' + (2\alpha^2-3\alpha+1)D^t h_\phi(r)} = -1 + L_0(x) + L_1(x)v(r) + L_2(x)[v(r)]^2 + \cdots.$$

From the above equalities, we obtain

$$\frac{s\left(D^t k_\phi(s)\right)' + (2\alpha^2-\alpha)\,s^2\left(D^t k_\phi(s)\right)''}{4(\alpha-\alpha^2)s + (2\alpha^2-\alpha)s\left(D^t k_\phi(s)\right)' + (2\alpha^2-3\alpha+1)D^t k_\phi(s)} = 1 + L_1(x)m_1 s + [L_1(x)m_2 + L_2(x)m_1^2]s^2 + \cdots. \quad (13)$$

and

$$\frac{r\left(D^t h_\phi(r)\right)' + (2\alpha^2-\alpha)\,r^2\left(D^t h_\phi(r)\right)''}{4(\alpha-\alpha^2)r + (2\alpha^2-\alpha)r\left(D^t h_\phi(r)\right)' + (2\alpha^2-3\alpha+1)D^t h_\phi(r)} = 1 + L_1(x)n_1 r + [L_1(x)n_2 + L_2(x)n_1^2]r^2 + \cdots. \quad (14)$$

Additionally, it is fairly well known that

$$|m_i| \le 1 \ \ and \ \ |n_i| \le 1 \quad (\, i \in \mathbb{N}\,). \quad (15)$$

Thus upon comparing the corresponding coefficients in (13) and (14), we have

$$(1+3\alpha-2\alpha^2)2^t\phi(\delta)d_2 = L_1(x)m_1, \quad (16)$$

$$(12\alpha^4-28\alpha^3+11\alpha^2+2\alpha-1)2^{2t}\,\phi^2(\delta)d_2^2 + (4\alpha^2+2)3^t\phi(\delta)d_3 = L_1(x)m_1 + L_2(x)m_1^2, \quad (17)$$

$$-(1+3\alpha-2\alpha^2)2^t\phi(\delta)d_2 = L_1(x)n_1 \quad (18)$$

and

$$(12\alpha^4-28\alpha^3+19\alpha^2+2\alpha+3)2^{2t}\,\phi^2(\delta)d_2^2 - (4\alpha^2+2)3^t\phi(\delta)d_3 = L_1(x)n_2 + L_2(x)n_1^2. \quad (19)$$

From (16) and (18), we can easily see that

$$m_1 = -n_1 \quad (20)$$

and

$$(1+3\alpha-2\alpha^2)^2 2^{2t+1}\phi^2(\delta)d_2^2 = L_1^2(x)(m_1^2+n_1^2). \quad (21)$$

If we add (17) to (19), we get

$$(24\alpha^4-56\alpha^3+30\alpha^2+4\alpha+2)2^{2t}\,\phi^2(\delta)d_2^2 = L_1(x)(m_2+n_2) + L_2(x)(m_1^2+n_1^2). \quad (22)$$

By using (21) in equation (22), we have

$$d_2^2 = \frac{L_1^3(x)(m_2+n_2)}{[(24\alpha^4-56\alpha^3+30\alpha^2+4\alpha+2)L_1^2(x) - 2(1+3\alpha-2\alpha^2)^2L_2(x)]\,2^{2t}\,\phi^2(\delta)}, \quad (23)$$

which yields

$$|d_2| \le \frac{|px|\sqrt{|px|}}{2^t\phi(\delta)\,\sqrt{|[(12\alpha^4-28\alpha^3+15\alpha^2+2\alpha+1)-(1+3\alpha-2\alpha^2)^2]p^2(x) - 2(1+3\alpha-2\alpha^2)^2q(x)|}}.$$

By subtracting (19) from (17) and in view of (20), we obtain

$$4(2\alpha^2+1)[3^t\phi(\delta)d_3 - 2^{2t}\,\phi^2(\delta)d_2^2] = L_1(x)(m_2-n_2) + L_2(x)(m_1^2-n_1^2)$$

$$d_3 = \frac{L_1(x)(m_2 - n_2)}{4(2\alpha^2+1)3^t\phi(\delta)} + \frac{2^{2t}\,\phi(\delta)}{3^t} d_2^2. \tag{24}$$

Then, in view of (21), (24) becomes

$$d_3 = \frac{L_1(x)(m_2 - n_2)}{4(2\alpha^2+1)3^t\phi(\delta)} + \frac{L_1^2(x)(m_1^2 + n_1^2)}{2(1+3\alpha-2\alpha^2)^2 3^t\phi(\delta)}.$$

Applying (5), we deduce that

$$|d_3| \le \frac{|p(x)|}{2(2\alpha^2+1)3^t\phi(\delta)} + \frac{p^2(x)}{(1+3\alpha-2\alpha^2)3^t\phi(\delta)}.$$

From (24), for $\mu \in \mathbb{R}$, we write

$$d_3 - \mu d_2^2 = \frac{L_1(x)(m_2 - n_2)}{4(2\alpha^2+1)3^t\phi(\delta)} + \left(\frac{2^{2t}\,\phi(\delta)}{3^t} - \mu\right) d_2^2. \tag{25}$$

By substituting (23) in (25), we get

$$d_3 - \mu d_2^2 = \frac{L_1(x)(m_2 - n_2)}{4(2\alpha^2+1)3^t\phi(\delta)} + \left(\frac{2^{2t}\,\phi(\delta)}{3^t} - \mu\right)\frac{L_1^3(x)(m_2+n_2)}{[(24\alpha^4 - 56\alpha^3 + 30\alpha^2 + 4\alpha + 2)L_1^2(x) - 2(1+3\alpha-2\alpha^2)^2 L_2(x)]2^{2t}\,\phi^2(\delta)}$$

$$= \frac{L_1(x)}{2}\left[\left(\Omega(\mu,x) + \frac{1}{2(2\alpha^2+1)3^t\phi(\delta)}\right)m_2 + \left(\Omega(\mu,x) - \frac{1}{2(2\alpha^2+1)3^t\phi(\delta)}\right)n_2\right],$$

where

$$\Omega(\mu,x) = \frac{L_1^2(x)\left(\frac{2^{2t}\phi(\delta)}{3^t} - \mu\right)}{[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1)L_1^2(x) - (1+3\alpha-2\alpha^2)^2 L_2(x)]\,2^{2t}\,\phi^2(\delta)}.$$

Hence, in view of (15), we conclude that

$$|d_3 - \mu d_2^2| \le \begin{cases} \dfrac{|L_1(x)|}{2(2\alpha^2+1)3^t\phi(\delta)} & if\ \ |\Omega(\mu,x)| \le \dfrac{1}{2(2\alpha^2+1)3^t\phi(\delta)} \\ |L_1(x)||\Omega(\mu,x)| & if\ \ |\Omega(\mu,x)| \ge \dfrac{1}{2(2\alpha^2+1)3^t\phi(\delta)}, \end{cases}$$

which evidently completes the proof of Theorem 1.

**Remark 2**

1) If we put $\alpha = 0$ in Theorem 1, we get the outcomes which were indicated by Swamy et al.[16].
2) If we put $\alpha = 0$ , $t = 0$ and $\phi(\delta) = 1$ in Theorem 1, we get the outcomes which were indicated by Altinkaya [1].
3) If we put $\alpha = \frac{1}{2}$, $t = 0$ and $\phi(\delta) = 1$ in Theorem 1, we get the outcomes which were indicated by Altinkaya and Yalçin [3].

**Cofficient bounds and Fekete–Szegö inequality for the class $\mathcal{K}_\Sigma(\alpha, t, \phi(\delta); x)$**

**Definition 2** A function $k \in \Sigma$ is said to be in the class $\mathcal{K}_\Sigma(\alpha, t, \phi(\delta); x)$ for $0 \le \alpha \le 1$, $t \in N \cup \{0\}$ and $\phi(\delta) = \frac{2}{1+e^{-\delta}}$, $\delta \ge 0$, if the following conditions of subordination are satisfied:

$$(1-\alpha)\left(D^t k_\phi(s)\right)' + \alpha\left(1 + \frac{s\left(D^t k_\phi(s)\right)''}{\left(D^t k_\phi(s)\right)'}\right) \prec G_{L_n(x)}(s) - 1 \tag{26}$$

and

$$(1-\alpha)\left(D^t h_\phi(r)\right)' + \alpha\left(1 + \frac{r\left(D^t h_\phi(r)\right)''}{\left(D^t h_\phi(r)\right)'}\right) \prec G_{L_n(x)}(r) - 1, \tag{27}$$

where the function $h = k^{-1}$ is indicated by (2).

**Remark 3**

1) For $\alpha = 1$, $t = 0$ and $\phi(\delta) = 1$, the function class $\mathcal{K}_\Sigma(\alpha, t, \phi(\delta); x)$ shortens to the function class $C(x)$ presented and investigated by Altinkaya [1].
2) For $\alpha = 0$, $t = 0$ and $\phi(\delta) = 1$, the function class $\mathcal{K}_\Sigma(\alpha, t, \phi(\delta); x)$ shortens to the function class $\mathcal{W}_\Sigma(\tau = 1; x)$ presented and investigated by Altinkaya and Yalçin [3].

**Theorem 2** Let the function $k \in \Sigma$ indicated by (1) be in the class $\mathcal{K}_\Sigma(\alpha, t, \phi(\delta); x)$. Then

$$|d_2| \le \frac{|px|\sqrt{|px|}}{2^t\phi(\delta)\sqrt{|(1+\alpha)p^2(x) + 8q(x)|}} \tag{28}$$

and

$$|d_3| \le \frac{|p(x)|}{(1+\alpha)3^{t+1}\phi(\delta)} + \frac{p^2(x)}{4\,\phi(\delta)\,3^t}, \tag{29}$$

and for some $\mu \in \mathbb{R}$,

$$|d_3 - \mu d_2^2| \le \begin{cases} \dfrac{|p(x)|}{(1+\alpha)3^{t+1}\phi(\delta)} & if\ \left|1 - \mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right| \le \dfrac{|(1+\alpha)p^2(x) + 8q(x)|}{3(1+\alpha)p^2(x)} \\ \dfrac{|p(x)|^3\left|1 - \mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right|}{|(1+\alpha)p^2(x) + 8q(x)|3^t\phi(\delta)} & if\ \left|1 - \mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right| \ge \dfrac{|(1+\alpha)p^2(x) + 8q(x)|}{3(1+\alpha)p^2(x)}. \end{cases} \tag{30}$$

**Proof.** Let $k \in \mathcal{K}_\Sigma(\alpha, t, \phi(\delta); x)$ be given by Taylor-Maclaurin expansion (1). Then, there are two holomorphic functions $u\ and\ v$ such that

$$u(0) = 0, \qquad v(0) = 0,$$

$$|u(s)| = |m_1 s + m_2 s^2 + \cdots| < 1, \qquad |v(r)| = |n_1 r + n_2 r^2 + \cdots| < 1 \qquad (\forall s, r \in \Delta).$$

Hence, we can write

$$(1-\alpha)\left(D^t k_\phi(s)\right)' + \alpha\left(1 + \frac{s\left(D^t k_\phi(s)\right)''}{\left(D^t k_\phi(s)\right)'}\right) = G_{L_n(x)}\big(u(s)\big) - 1$$

and

$$(1-\alpha)\left(D^t h_\phi(r)\right)' + \alpha\left(1 + \frac{r\left(D^t h_\phi(r)\right)''}{\left(D^t h_\phi(r)\right)'}\right) = G_{L_n(x)}\big(v(r)\big) - 1.$$

Or, equivalently,

$$(1-\alpha)\left(D^t k_\phi(s)\right)' + \alpha\left(1 + \frac{s\left(D^t k_\phi(s)\right)''}{\left(D^t k_\phi(s)\right)'}\right) = -1 + L_0(x) + L_1(x)u(s) + L_2(x)[u(s)]^2 + \cdots$$

and

$$(1-\alpha)\left(D^t h_\phi(r)\right)' + \alpha\left(1 + \frac{r\left(D^t h_\phi(r)\right)''}{\left(D^t h_\phi(r)\right)'}\right) = -1 + L_0(x) + L_1(x)v(r) + L_2(x)[v(r)]^2 + \cdots.$$

From the above equalities, we obtain

$$(1-\alpha)\left(D^t k_\phi(s)\right)' + \alpha\left(1 + \frac{s\left(D^t k_\phi(s)\right)''}{\left(D^t k_\phi(s)\right)'}\right) = 1 + L_1(x)m_1 s + [L_1(x)m_2 + L_2(x)m_1^2]s^2 + \cdots \tag{31}$$

and

$$(1-\alpha)\left(D^t h_\phi(r)\right)' + \alpha\left(1+\frac{r\left(D^t h_\phi(r)\right)''}{\left(D^t h_\phi(r)\right)'}\right) = 1 + L_1(x)n_1 r + [L_1(x)n_2 + L_2(x)n_1^2]r^2 + \cdots. \tag{32}$$

Additionally, it is fairly well known that

$$|m_i| \le 1 \quad and \quad |n_i| \le 1 \quad (\, i \in \mathbb{N}\,). \tag{33}$$

Thus upon comparing the corresponding coefficients in (31) and (32), we have

$$2^{t+1}\phi(\delta)d_2 = L_1(x)m_1, \tag{34}$$

$$(1+\alpha)3^{t+1}\phi(\delta)d_3 - \alpha\, 2^{2(t+1)}\,\phi^2(\delta)d_2^2 = L_1(x)m_1 + L_2(x)m_1^2, \tag{35}$$

$$-\,2^{t+1}\phi(\delta)d_2 = L_1(x)n_1 \tag{36}$$

and

$$-\,(1+\alpha)3^{t+1}\phi(\delta)d_3 + (\alpha+3)2^{2t+1}\,\phi^2(\delta)d_2^2 = L_1(x)n_2 + L_2(x)n_1^2. \tag{37}$$

From (34) and (36), we can easily see that

$$m_1 = -n_1 \tag{38}$$

and

$$2^{2t+3}\phi^2(\delta)d_2^2 = L_1^2(x)(m_1^2+n_1^2). \tag{39}$$

If we add (35) to (37), we get

$$(3-\alpha)2^{2t+1}\,\phi^2(\delta)d_2^2 = L_1(x)(m_2+n_2) + L_2(x)(m_1^2+n_1^2). \tag{40}$$

By using (39) in equation (40), we have

$$d_2^2 = \frac{L_1^3(x)(m_2+n_2)}{[(3-\alpha)L_1^2(x) - 4L_2(x)]\,2^{2t+1}\,\phi^2(\delta)}, \tag{41}$$

which yields

$$|d_2| \le \frac{|px|\sqrt{|px|}}{2^t\phi(\delta)\,\sqrt{|(1+\alpha)p^2(x) + 8q(x)|}}.$$

By subtracting (37) from (35) and in view of (38), we obtain

$$(1+\alpha)[2\phi(\delta)3^{t+1}d_3 - \;3\phi^2(\delta)2^{2t+1}d_2^2] = L_1(x)(m_2-n_2) + L_2(x)(m_1^2-n_1^2)$$

$$d_3 = \frac{L_1(x)(m_2-n_2)}{2(1+\alpha)3^{t+1}\phi(\delta)} + \frac{2^{2t}\,\phi(\delta)}{3^t}d_2^2. \tag{42}$$

Then, in view of (39), (42) becomes

$$d_3 = \frac{L_1(x)(m_2-n_2)}{2(1+\alpha)3^{t+1}\phi(\delta)} + \frac{L_1^2(x)(m_1^2+n_1^2)}{8\,\phi(\delta)\,3^t}.$$

Applying (5), we deduce that

$$|d_3| \le \frac{|p(x)|}{(1+\alpha)3^{t+1}\phi(\delta)} + \frac{p^2(x)}{4\,\phi(\delta)\,3^t}.$$

From (42), for $\mu \in \mathbb{R}$, we write

$$d_3 - \mu d_2^2 = \frac{L_1(x)(m_2-n_2)}{2(1+\alpha)3^{t+1}\phi(\delta)} + \left(\frac{2^{2t}\,\phi(\delta)}{3^t} - \mu\right)d_2^2. \tag{43}$$

By substituting (41) in (43), we get

$$d_3 - \mu d_2^2 = \frac{L_1(x)(m_2-n_2)}{2(1+\alpha)\phi(\delta)3^{t+1}} + \left(\frac{2^{2t}\,\phi(\delta)}{3^t} - \mu\right)\frac{L_1^3(x)(m_2+n_2)}{[(3-\alpha)L_1^2(x) - 4L_2(x)]\,2^{2t+1}\,\phi^2(\delta)},$$

$$= \frac{L_1(x)}{2}\left[\left(\Omega(\mu,x) + \frac{1}{(1+\alpha)3^{t+1}\phi(\delta)}\right)m_2 + \left(\Omega(\mu,x) - \frac{1}{(1+\alpha)3^{t+1}\phi(\delta)}\right)n_2\right],$$

where

$$\Omega(\mu,x) = \frac{L_1^2(x)\left(\frac{2^{2t}\phi(\delta)}{3^t} - \mu\right)}{[(3-\alpha)L_1^2(x) - 4L_2(x)]\, 2^{2t}\,\phi^2(\delta)}.$$

Hence, in view of (33), we conclude that

$$|d_3 - \mu d_2^2| \le \begin{cases} \frac{|L_1(x)|}{(1+\alpha)3^{t+1}\phi(\delta)} & if \quad |\Omega(\mu,x)| \le \frac{1}{(1+\alpha)3^{t+1}\phi(\delta)} \\ |L_1(x)||\Omega(\mu,x)| & if \quad |\Omega(\mu,x)| \ge \frac{1}{(1+\alpha)3^{t+1}\phi(\delta)}, \end{cases}$$

which evidently completes the proof of Theorem 2.

**Remark 4**

1) If we put $\alpha = 1$, $t = 0$ and $\phi(\delta) = 1$ in Theorem 2, we get the outcomes which were indicated by Altinkaya [1].
2) If we put $\alpha = 0$, $t = 0$ and $\phi(\delta) = 1$ in Theorem 2, we get the outcomes which were indicated by Altinkaya and Yalçin [3].

**Cofficient bounds and Fekete–Szegö inequality for the class $\mathcal{T}_\Sigma(\beta, t, \phi(\delta); x)$**

**Definition 3** A function $k \in \Sigma$ is said to be in the class $\mathcal{T}_\Sigma(\beta, t, \phi(\delta); x)$ for $\beta \ge 0$, $t \in N \cup \{0\}$ and $\phi(\delta) = \frac{2}{1+e^{-\delta}}$, $\delta \ge 0$, if the following conditions of subordination are satisfied:

$$\frac{\left(s\left(D^t k_\phi(s)\right)' + \beta\left(s^2\left(D^t k_\phi(s)\right)''\right)\right)'}{\left(D^t k_\phi(s)\right)'} \prec G_{L_n(x)}(s) - 1 \tag{44}$$

and

$$\frac{\left(r\left(D^t h_\phi(r)\right)' + \beta\left(r^2\left(D^t h_\phi(r)\right)''\right)\right)'}{\left(D^t h_\phi(r)\right)'} \prec G_{L_n(x)}(r) - 1, \tag{45}$$

where the function $h = k^{-1}$ is indicated by (2).

**Remark 5** For $\beta = 0$, $t = 0$ and $\phi(\delta) = 1$, the function class $\mathcal{T}_\Sigma(\beta, t, \phi(s); x)$ shortens to the function class $C(x)$ presented and investigated by Altinkaya [1].

**Theorem 3** Let the function $k \in \Sigma$ indicated by (1) be in the class $\mathcal{T}_\Sigma(\beta, t, \phi(\delta); x)$. Then

$$|d_2| \le \frac{|px|\sqrt{|px|}}{2^t\phi(\delta)\sqrt{|2(1+3\beta+8\beta^2)p^2(x) + 8(1+2\beta)^2 q(x)|}} \tag{46}$$

and

$$|d_3| \le \frac{|p(x)|}{2(1+3\beta)3^{t+1}\phi(\delta)} + \frac{p^2(x)}{4(1+2\beta)^2\, 3^t\phi(\delta)}, \tag{47}$$

and for some $\mu \in \mathbb{R}$,

$$|d_3-\mu d_2^2| \leq \begin{cases} \dfrac{|p(x)|}{2(1+3\beta)3^{t+1}\phi(\delta)} \quad if \\ \left|1-\mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right| \leq \dfrac{|(1+3\beta+8\beta^2)p^2(x)+4(1+2\beta)^2q(x)|}{3(1+3\beta)p^2(x)} \\ \dfrac{|p(x)|^3\left|1-\mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right|}{2|(1+3\beta+8\beta^2)p^2(x)+4(1+2\beta)^2q(x)|3^t\phi(\delta)} \quad if \\ \left|1-\mu\dfrac{3^t}{2^{2t}\phi(\delta)}\right| \geq \dfrac{|(1+3\beta+8\beta^2)p^2(x)+4(1+2\beta)^2q(x)|}{3(1+3\beta)p^2(x)}. \end{cases} \tag{48}$$

**Proof.** Let $k \in \mathcal{T}_\Sigma(\beta,t,\phi(\delta);x)$ be given by Taylor-Maclaurin expansion (1). Then, there are two holomorphic functions $u$ and $v$ such that

$$u(0)=0, \qquad v(0)=0,$$

$$|u(s)|=|m_1 s+m_2 s^2+\cdots|<1, \qquad |v(r)|=|n_1 r+n_2 r^2+\cdots|<1 \qquad (\forall s,r\in\Delta).$$

Hence, we can write

$$\frac{\left(s\left(D^t k_\phi(s)\right)'+\beta\left(S^2\left(D^t k_\phi(s)\right)''\right)\right)'}{\left(D^t k_\phi(s)\right)'} = G_{L_n(x)}(u(s))-1$$

and

$$\frac{\left(r\left(D^t h_\phi(r)\right)'+\beta\left(r^2\left(D^t h_\phi(r)\right)''\right)\right)'}{\left(D^t h_\phi(r)\right)'} = G_{L_n(x)}(v(r))-1.$$

Or, equivalently,

$$\frac{\left(s\left(D^t k_\phi(s)\right)'+\beta\left(S^2\left(D^t k_\phi(s)\right)''\right)\right)'}{\left(D^t k_\phi(s)\right)'} = -1+L_0(x)+L_1(x)u(s)+L_2(x)[u(s)]^2+\cdots$$

and

$$\frac{\left(r\left(D^t h_\phi(r)\right)'+\beta\left(r^2\left(D^t h_\phi(r)\right)''\right)\right)'}{\left(D^t h_\phi(r)\right)'} = -1+L_0(x)+L_1(x)v(r)+L_2(x)[v(r)]^2+\cdots.$$

From the above equalities, we obtain

$$\frac{\left(s\left(D^t k_\phi(s)\right)'+\beta\left(S^2\left(D^t k_\phi(s)\right)''\right)\right)'}{\left(D^t k_\phi(s)\right)'} = 1+L_1(x)m_1 s+[L_1(x)m_2+L_2(x)m_1^2]s^2+\cdots \tag{49}$$

and

$$\frac{\left(r\left(D^t h_\phi(r)\right)'+\beta\left(r^2\left(D^t h_\phi(r)\right)''\right)\right)'}{\left(D^t h_\phi(r)\right)'} = 1+L_1(x)n_1 r+[L_1(x)n_2+L_2(x)n_1^2]r^2+\cdots. \tag{50}$$

Additionally, it is fairly well known that

$$|m_i|\leq 1, \qquad |n_i|\leq 1 \qquad (i\in\mathbb{N}). \tag{51}$$

Thus upon comparing the corresponding coefficients in (49) and (50), we have

$$(1+2\beta)\,2^{t+1}\phi(\delta)d_2 = L_1(x)m_1, \tag{52}$$

$$2(1+3\beta)3^{t+1}\phi(\delta)d_3-(1+2\beta)\,2^{2(t+1)}\,\phi^2(\delta)d_2^2=L_1(x)m_1+L_2(x)m_1^2, \tag{53}$$

$$-(1+2\beta)2^{t+1}\phi(\delta)d_2=L_1(x)n_1 \tag{54}$$

and

$$(2+7\beta)2^{2(t+1)}\,\phi^2(\delta)d_2^2-2(1+3\beta)3^{t+1}\phi(\delta)d_3=L_1(x)n_1+L_2(x)n_1^2. \tag{55}$$

From (52) and (54), we can easily see that

$$m_1=-n_1 \tag{56}$$

and

$$(1+2\beta)^2\,2^{2t+3}\phi^2(\delta)d_2^2=L_1^2(x)(m_1^2+n_1^2). \tag{57}$$

If we add (53) to (55), we get

$$(1+5\beta)2^{2(t+1)}\,\phi^2(\delta)d_2^2=L_1(x)(m_2+n_2)+L_2(x)(m_1^2+n_1^2). \tag{58}$$

By using (57) in equation (58), we have

$$d_2^2=\frac{L_1^3(x)(m_2+n_2)}{[(1+5\beta)L_1^2(x)-2(1+2\beta)^2L_2(x)]\,2^{2(t+1)}\,\phi^2(\delta)}, \tag{59}$$

which yields

$$|d_2|\le\frac{|px|\sqrt{|px|}}{2^t\phi(\delta)\,\sqrt{|2(1+3\beta+8\beta^2)p^2(x)+8(1+2\beta)^2q(x)|}}.$$

By subtracting (55) from (53) and in view of (56), we obtain

$$(1+3\beta)\left[4\phi(\delta)3^{t+1}d_3-\;3\phi^2(\delta)2^{2(t+1)}d_2^2\right]=L_1(x)(m_2-n_2)+L_2(x)(m_1^2-n_1^2)$$

$$d_3=\frac{L_1(x)(m_2-n_2)}{4(1+3\beta)3^{t+1}\phi(\delta)}+\frac{2^{2t}\,\phi(\delta)}{3^t}d_2^2. \tag{60}$$

Then, in view of (57), (60) becomes

$$d_3=\frac{L_1(x)(m_2-n_2)}{4(1+3\beta)3^{t+1}\phi(\delta)}+\frac{L_1^2(x)(m_1^2+n_1^2)}{8(1+2\beta)^2\,\phi(\delta)\,3^t}.$$

Applying (5), we deduce that

$$|d_3|\le\frac{|p(x)|}{2(1+3\beta)3^{t+1}\phi(\delta)}+\frac{p^2(x)}{4(1+2\beta)^2\,3^t\,\phi(\delta)}.$$

From (60), for $\mu\in\mathbb{R}$, we write

$$d_3-\mu d_2^2=\frac{L_1(x)(m_2-n_2)}{4(1+3\beta)3^{t+1}\phi(\delta)}+\left(\frac{2^{2t}\,\phi(\delta)}{3^t}-\mu\right)d_2^2.$$

$$d_3-\mu d_2^2=\frac{L_1(x)(m_2-n_2)}{4(1+3\beta)3^{t+1}\,\phi(\delta)}+\left(\frac{2^{2t}\,\phi(\delta)}{3^t}-\mu\right)\frac{L_1^3(x)(m_2+n_2)}{[(1+5\beta)L_1^2(x)-2(1+2\beta)^2L_2(x)]\,2^{2(t+1)}\,\phi^2(\delta)},$$
$$=\frac{L_1(x)}{2}\left[\left(\Omega(\mu,x)+\frac{1}{2(1+3\beta)3^{t+1}\phi(\delta)}\right)m_2+\left(\Omega(\mu,x)-\frac{1}{2(1+3\beta)3^{t+1}\phi(\delta)}\right)n_2\right],$$

where

$$\Omega(\mu,x)=\frac{L_1^2(x)\left(\frac{2^{2t}\phi(\delta)}{3^t}-\mu\right)}{[(1+5\beta)L_1^2(x)-2(1+2\beta)^2L_2(x)]\,2^{2t+1}\,\phi^2(\delta)}.$$

Hence, in view of (51), we conclude that

$$|d_3-\mu d_2^2|\le\begin{cases}\dfrac{|L_1(x)|}{2(1+3\beta)3^{t+1}\phi(\delta)} & if\quad |\Omega(\mu,x)|\le\dfrac{1}{2(1+3\beta)3^{t+1}\phi(\delta)}\\[2ex] |L_1(x)||\Omega(\mu,x)| & if\quad |\Omega(\mu,x)|\ge\dfrac{1}{2(1+3\beta)3^{t+1}\phi(\delta)},\end{cases}$$

which evidently completes the proof of Theorem 3.

**Remark 6** If we put $\beta = 0$, $t = 0$ and $\phi(\delta) = 1$ in Theorem 3, we get the outcomes which were indicated by Altinkaya [1].

**References**


1. Ş. Altinkaya, Inclusion properties of Lucas polynomials for bi-univalent functions introduced through the q-analogue of the Noor integral operator, Turkish J. Math., 43(2019), 620-629.
2. Ş. Altinkaya and S. Yalçin, On the $(p, q)$ −Lucas polynomial coefficient bounds of the bi-univalent functions class $\sigma$, Boletin dela Sociedad Matematica Mexicana, (2018), 1-9.
3. Ş. Altinkaya and S. Yalçin, $(p, q)$ −Lucas polynomials and their applications to bi-univalent functions, Proyecciones, 39(5)(2019), 1093-1105.
4. N. A. J. Al-Ziadi and A. K. Wanas, Coefficient bounds and Fekete-Szegö inequality for a certain families of bi-prestarlike functions defined by (M,N)-Lucas polynomials, Journal of Advances in Mathematics, 20(2021), 121-134.
5. S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, C. R. Acad. Paris. Ser. I., 352(6)(2014), 479-484.
6. P. L. Duren, Univalent Functions, Vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA, (1983).
7. O. A. Fadipe-Joseph, B. B. Kadir, S. E. Akinwumi and E. O. Adeniran, Polynomial bounds for a class of univalent function involving Sigmoid function, Khayyam J. Math. 4(1)(2018). 7-20.
8. B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions. Appl. Math. Lett., 24(2011), 1569-1573.
9. A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, Fibonacci Quart., 23(1985), 7-20.
10. T. Horzum and E. Gökçen Koçer, On some properties of Horadam polynomials, Int. Math. Forum, 4(2009), 1243-1252.
11. A. Lupas, A guide of Fibonacci and Lucas polynomials, Octogon Math. Mag., 7(1999), 3-12.
12. G. Ş. Sălăgean, Subclasses of univalent functions, Lecture Notesin Math., Springer, Berlin, 1013(1983), 362-372.
13. T. G. Shaba and A. K. Wanas, Coefficient bounds for a new family of bi-univalent functions associated with (U,V )-Lucas polynomials, Int. J. Nonlinear Anal. Appl. 13(1) (2022), 615-626.
14. H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, Acta Universitatis Apulensis, 41(2015), 153-164.
15. H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23(2010), 1188–1192.
16. S. R. Swamy, P. K. Mamatha, N. Magesh and J. Yamini, Certain subclasses of bi-univalent functions defined by Sălăgean operator with the $(\mathrm{p}, \mathrm{q})$ −Lucas polynomial, Advances in Mathematics: Scientific Journal, 9(8)(2020), 6017-6025.
17. S. R. Swamy, A. K. Wanas and Y. Sailaja, Some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m,n)-Lucas polynomials, Communications in Mathematics and Applications, 11(4)(2020), 563-574.