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Distributions generated by the boundary values of functions in Privalov spaces

Mejdin Saliji[1], Bedrije Bedzeti[2], Vesna Manova Erakovikj[3]

[1]Faculty of Education, Ss. Ukshin Hoti, Prizren, Kosovo

[2]Faculty of Mathematics and Natural Sciences, State University of Tetovo, Tetovo, Republic of North Macedonia.

[3]Faculty of Mathematics and Natural Sciences, Ss. Cyril and Methodius University, Skopje, Republic of North Macedonia.

mejdins@gmail.com, bedrije_a@hotmail.com, vesname@pmf.ukim.mk

Abstract

We characterise the distributions generated by the boundary values of functions from Privalov spaces.

1. Introduction

We use the following notation and preliminaries. U stands for the open unit disc in C and T is its boundary, i.e. $U = \{z \in C||z| < 1\}, T = \partial U,$ *and* Π^+ is the upper half plane, meaning $\Pi^+ = \{z \in C|Imz > 0\}$. For a function f holomorphic on a region Ω we right $f\epsilon H(\Omega)$. $L^p(\Omega)$ is the space of measurable functions on Ω such that $\int_\Omega |f(x)|^p dx < \infty$; L^p_{loc} is the space of measurable functions on Ω such that for every compact set $K \subset \Omega$ the following holds $\int_K |f(x)|^p dx < \infty$.

Privalov spaces on U and Π^+ and their properties: Privalov class, denoted with $N^p, 1 < p < \infty$, consists of all functions $f\epsilon H$(U) such that

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} (\log^+|f(re^{i\theta}|)^p d\theta < \infty.$$

.

Theorem. ([8]) The function f, holomorphic on U, belongs to N^p if and only if for every $\varepsilon > 0$ there exist $\delta > 0$ such that for every measurable set $E \subset T$, satisfying $m(E) < \delta$ the following holds

$$\int_E (\log^+|f(re^{i\theta})|)^p d\theta < \varepsilon, \qquad \text{for all } 0 \le r < 1.$$

Theorem. ([8]) The function f, holomorphic on U, belongs to N^p if and only if the subharmonic function $z \mapsto (log^+|f(z)|)^p \ (z\epsilon U)$ has a harmonic majorant.

Every function in Nevalina class, $N(U)$, because of Fatou's lemma, has a nontangentional (radial) limit on T almost everywere; every function in Privalov class, $N^p(U)$, has a nontangentional (radial) limit on T almost everywere, in both cases we denote the boundary value with $f^*(e^{i\theta}) = \lim_{r\to 1} f(e^{i\theta})$.

The class $N^p(\Pi^+)$, $p > 1$, is introduced analogously to $N^p(U)$, and is the set of all holomorphic functions on Π^+ satisfying

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} (\log (1 + |f(x + iy)|)^p dx < \infty.$$

Every $f \in N^p(\Pi^+)$ has a nontangentional limit $f^*(x)$ almost everywhere on the real axis.

Theorem. ([12]) The set L is bounded in $N^p(\Pi^+)$ if and only if

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i) There exist $C > 0$ such that

$$\int_R (\log(1+|f^*(x)|)^p dx < C$$

for all $f \in L$.

ii) For every $\varepsilon > 0$, exist $\delta > 0$ such that

$$\int_E (\log(1+|f^*(x)|)^p dx < \varepsilon$$

for all $f \in L$, and every Lebesgue measurable $E \subset R$ satisfying $m(E) < \delta$.

Distributions: $C^\infty(R^n)$ denotes the set of all complex valued functions infinitely differentiable on R^n; $C_0^\infty(R^n)$ is the subset of $C^\infty(R^n)$ which contains compactlly supported functions. Support of the function f denoted with $suppf$ is the cloasure of the set $\{x: f(x) \neq 0\}$ in R^n. $D = D(R^n)$ denotes the space $C_0^\infty(R^n)$ in which the convergence is defined in the following way: the sequence $\{\varphi_\lambda\}$, of functions $\varphi_\lambda \epsilon D$, converges to $\varphi \epsilon D$ when $\lambda \to \lambda_0$ if and only if there exist compact subset of R^n such that $supp\ \varphi_\lambda \subseteq K$ for all λ, $supp\ \varphi \subseteq K$, and for every n-tuple α of nonegative integers the sequence $\{D_x^\alpha(\varphi_\lambda(x))\}$ converges to $\{D_x^\alpha(\varphi(x))\}$ uniformly on K when $\lambda \to \lambda_0$. With $D' = D'(R^n)$ is denoted the space of all continuous, linear functionals on D, where the continuity is in the sense: from $\varphi_\lambda \to \varphi$ in D when $\lambda \to \lambda_0$ it follows that $\langle T, \varphi_\lambda \rangle \to \langle T, \varphi \rangle$ in C, when $\lambda \to \lambda_0$.

The space D' is called the space of distributions. We use the convention $\langle T, \varphi \rangle = T(\varphi)$ for the value of the functional Tacting on the function φ.

Let $\varphi \epsilon D$ and $f(x) \in L^1_{loc}(R^n)$. Then the functional T_f on D defined with

$$\langle T_f, \varphi \rangle = \int_{R^n} f(t)\varphi(t)dt, \varphi \epsilon D,$$

is an element in D' and it is called the regular distribution generated by the function f.

2. Main results

Theorem. ([5]) Sufficient and necessary condition for the measurable function $\varphi(e^{it})$ defined on T to coincide almost everywhere on T with the boundary value $f^*(e^{it})$ of some function $f(z)$ in $N(U)$, is to exist a sequence of polynomials $\{P_n(z)\}$ such that:

i. $\{P_n(e^{i\theta})\}$ converges to $\varphi(e^{i\theta})$ almost everywhere on T;

ii. $\overline{\lim_{n\to\infty}} \int_0^{2\pi} (\log^+|P_n(e^{i\theta})|)d\theta < \infty$.

Theorem 1. Let $T_{f^*} \in D'$ is generated by the boundary value $f^*(x)$ of a function $f(z)$ in $N^p(\Pi^+)$. There exist sequence of polynomials $\{P_n(z)\}$, $z \in \Pi^+$, and respectively $\{T_n\}$, $T_n \in D'$, generated by the boundary values $P_n^*(x)$ of the polynomials $P_n(z)$, i.e. $T_n = T_{P_n^*}$ such that:

i. $T_n \to T_{f^*}$ in D' when $n \to \infty$,

ii. $\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} (\log\ (1+|P_n^*(x)|))^p |\varphi(x)|\, dx < \infty$ for every $\varphi \epsilon D$.

Proof. Let the assumptions of the theorem hold. Since $f \in N^p(\Pi^+)$, one has $f \in H(\Pi^+)$ and there exist a constant $C > 0$ such that

$$\int_{-\infty}^{\infty} \log(1+|f(x+iy)|)^p\, dx \leq C \text{ for every } z = x+iy \in \Pi^+. \qquad (1)$$

Let $\{y_n\}$ be a sequence of positive real numbers satisfying $\lim_{n\to\infty} y_n = 0$. We define a sequence of complex functions $\{F_n(z)\}$ with

$$F_n(z) = f(z+iy_n).$$

The functions $F_n(z)$ are holomorphic on $\Pi^+ \cup R$. Margelijan theorem implies that for arbitrary compact subset K of $\Pi^+ \cup R$ with complement being connected, for the functions $F_n(z)$ there exist polynomials $P_n(z)$ such that $|F_n(z) - P_n(z)| < \varepsilon_n$, for all $z \in K$, where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ when $n \to \infty$.

In what follows we prove i. and ii.

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i. Let $\varphi \in D,\ supp\ \varphi = K.$ Then

$$|\langle T_n, \varphi\rangle - \langle T_{f^*}, \varphi\rangle| = \left|\int_{-\infty}^{\infty} P_n^*(x)\varphi(x)dx - \int_{-\infty}^{\infty} f^*(x)\varphi(x)dx\right|$$

$$= \left|\int_{-\infty}^{\infty} [P_n^*(x) - f^*(x)]\varphi(x)dx\right| = \left|\int_K [P_n^*(x) - f^*(x)]\varphi(x)dx\right|$$

$$\leq M(\int_K [P_n^*(x) - f^*(x)]dx \leq M\varepsilon_n^{'} m(K) \to 0$$

when $n \to \infty$.

In the previous calculations we use the notation $m(K)$ for the Lebesgue measure of the set K, $M = \max\{\varphi(x) : x \in K\}$ and $\varepsilon_n^{'} = \varepsilon_n + [f^*(x) - F_n(x)]$. It is obvious that $\varepsilon_n^{'} \to 0$ when $n \to \infty$. The Later calculation implies that $\langle T_n, \varphi\rangle \to \langle T_{f^*}, \varphi\rangle$ when $n \to \infty$ for every, but fixed, $\varphi \epsilon D$, meaning $T_n \to T_{f^*}$ weakly in D'. To prove the convergence in the strong topology it suffices to prove the same convergence for $\varphi \in B$ for an arbitrary bounded set in D. Choose $B \subset D$, arbitrary bounded set. The condition of boundnes implies that there exists a compact set K such that $supp\ \varphi \in K$, $||\varphi||_{D(K)} < M$, for every $\varphi \in B$. Note that the calculations at the beginning of the paragraph hold for every $\varphi \in B$ and the new compact set chosen for the boundness condition. Hence, $T_n \to T_{f^*}$ in D'.

(ii)

$$\int_{-\infty}^{\infty} (\log(1 + |P_n^*(x)|)^p |\varphi(x)|dx$$

$$= \int_K (\log(1 + |P_n^*(x) + F_n(x) - F_n(x)|)^p |\varphi(x)|dx$$

$$\leq \int_K (\log(1 + |P_n^*(x) - F_n(x)| + |F_n(x)|))^p |\varphi(x)|dx$$

$$\leq \int_K (\log(1 + |F_n(x)| + |P_n^*(x) - F_n(x)|))^p |\varphi(x)|dx$$

$$\leq M2^{p-1}\int_K (\log(1 + |F_n(x)|)^p dx + M2^{p-1}\int_K |P_n^*(x) - F_n(x)|)^p\, dx$$

$$\leq MC + M\varepsilon_n^p\, m(K).$$

Because $\varepsilon_n \to 0, n \to \infty$ we get $\int_R$ (Log(1 + $|P_n^*(x)|)^p |\varphi(x)|dx < C'$ meaning

$\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} (\log(1 + |P_n^*(x)|)^n |\varphi(x)|\, dx < \infty$, for all $\varphi \epsilon D$.

In the proof of ii. We use the inequalities $|a + b| \leq |a| + |b|$, $\log(1 + a + b) \leq \log(1 + a) + b,\ for\ a, b > 0$ and $(a + b)^p \leq 2^{p-1}(a^p + b^p),\ for\ p \geq 1.$

Theorem 2. Let φ_0 be a localy integrable function and $T_{\varphi_0} \in D'$ is generated by the function φ_0. Let there exist sequence of polynomials $P_n(z)$ satisfying the conditions:

i. The sequence of distributions generated by the boundary values $P_n^*(x)\ of\ P_n(z)$ converges to T_{φ_0} in $D^{'}$ when $n \to \infty$;

ii. $\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} (\log(1 + P_n(x + iy))^p\, |\varphi(x)|\, dx < C < \infty, \forall z = x + iy \in \Pi^+,\ \varphi \epsilon D.$

There exists a function $f \in H(\Pi^+)$ such that

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$$\int_K (\log(1+|f(x+iy)|)^p dx < C < \infty, \forall z = x+iy \in \Pi^+,$$

for every compact $K \subset R$, and

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x)\, dx = \langle T_{\varphi_0}, \varphi\rangle.$$

Proof. Let the assumptions of the theorem are fulfilled. In [3] it is proven that from i., i.e.

$\lim_{n\to\infty} \int_R P_n^*(x)\varphi(x)dx = \int_R \varphi_0(x)\varphi(x)dx, \varphi\epsilon D,$

implies the existence of $f \in H(\Pi^+)$ such that the sequence of polynomials converges to f, uniformly on arbitrary compact subsets of Π^+ when $n \to \infty$.

Firstly we will prove that this function f is holomorphic and satisfies the condition

$$\int_K \log(1+|f(x+iy)|)^p\ dx \le C$$

for all $z = x+iy \in \Pi^+$and arbitrary compact set $K \subset R$.

Indeed, we use the condition ii., i.e.

$\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty}(\log(1+|P_n(x+iy)|)^p\, |\varphi(x)|\, dx < C < \infty, \forall z = x+iy \in \Pi^+, \varphi\epsilon D.$

Let K be compact set. There exists $\varphi(x)\epsilon C_0^\infty(R^n)$, $\varphi(x) = 1, \forall x\epsilon K$. To obtain the last statement, it is enough to take characteristic function of the set K and to regularize it. Substitution of such φ in to ii., implies that for every $n \in N$,

$$\int_K (\log(1+|P_n(x+iy)|))^p\ dx < C < \infty, \forall z = x+iy \in \Pi^+.$$

Now,

$$\int_K \log(1+|f(x+iy)|)^p\, dx = \int_K \lim_{n\to\infty}(\log(1+|P_n(x+iy)|)^p$$

$$\le \overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} (\log(1+|P_n(x+iy)|)^p\ dx < C < \infty,$$

i.e.

$\int_K \log(1+|f(x+iy)|)^p\ dx \le C < \infty$ for arbitrary compact set $K \subset R$ and every $z = x+iy \in \Pi^+$.

It remains to be proved that $\lim_{y\to 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x)\, dx = \langle T_{\varphi_0}, \varphi\rangle$, for every $\varphi \in D$.

Let $\varphi\epsilon D$ and $supp\varphi = K \subset R$. Then

$$\lim_{y\to 0^+} \int_R (f(x+iy)\varphi(x)\, dx = \lim_{y\to 0^+} \int_R \lim_{n\to\infty} (P_n(x+iy)\varphi(x)\, dx =$$

$$= \lim_{y\to 0^+} \lim_{n\to\infty} \int_K (P_n(x+iy)\varphi(x)dx = \lim_{n\to\infty} \lim_{y\to 0^+} \int_K (P_n(x+iy)\varphi(x)dx =$$

$$= \lim_{n\to\infty} \int_K P_n^*(x+iy)\varphi(x)dx = \int_R \varphi_0(x)\varphi(x)dx = \langle T_{\varphi_0}, \varphi\rangle,$$

$for\ every\ \varphi\epsilon D.$

The previous equalities are obvious, exept the following

$$\lim_{y\to 0^+} \lim_{n\to\infty} \int_K P_n(x+iy)\varphi(x)dx = \lim_{n\to\infty} \lim_{y\to 0^+} \int_K P_n(x+iy)\varphi(x)dx \dots .(*)$$

for $z = x+iy \in \Pi^+$.

We will prove $(*)$.

To do that we consider the sequence of functions $\{g_n(y)\}$ defined by

$$g_n(y) = \int_K (P_n(x+iy)\varphi(x)dx\,, x+iy \in K_1.$$

for K_1 compact subset of Π^+ such that $z \in K_1$ for $Re(z) \in K$. Because $\{P_n(x+iy)\}$ converges to $(x+iy)$ uniformly on K_1, when $n \to \infty$, one obtains that for fixed y

$$\lim_{n\to\infty} g_n(y) = \int_K (P_n(x+iy)\varphi(x)dx = \int_K (f(x+iy)\varphi(x)dx = g(y),$$

i.e. the sequence $\{g_n(y)\}$ converges to $g(y)$ when $n \to \infty$. We will prove that this convergence is uniform on ImK_1, which will imply the statement. Indeed,

$$0 \le \sup_y |g_n(x+iy) - g(x+iy)| = \sup_y \left|\int_K \mathrm{P_n(x+iy)\varphi(x)dx} - \int_K f(x+iy)\varphi(x)dx\right|$$

$$= \sup_y \left|\int_K [P_n(x+iy) - f(x+iy)]\varphi(x)dx\right|$$

$$\le \sup_y \int_K |(P_n(x+iy) - (f(x+iy)||\varphi(x)|dx$$

$$\le M\sup_y \int_K |P_n(x+iy) - f(x+iy)|dx.$$

Since $P_n(x+iy) \to f(x+iy)$ uniformly on K_1, it follows that

$$\int_K |P_n(x+iy) - f(x+iy)|dx$$

converges to 0 uniformly on $Im(K_1)$ meaning

$$\lim_{n\to\infty} \sup_y \int_K |(P_n(x+iy) - f(x+iy)|dx = 0.$$

Finally, $\lim_{n\to\infty} \sup_y |g_n(x+iy) - g(x+iy)| = 0.$

3. Conclusion

We obtain necessary and sufficient condition for a distribution generated from an element of the Privalov class to be boundary value of analytic functions on upper half space. The boundary values are taken in the distributional sense.

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Conflicts of Interest

The authors don't have competing for any interests

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