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W-Power N-Binormal Operator on Hilbert Space

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Abstract: In this paper we present a new class of operators on Hilbert space called w-power n-binormal operator. We study this operator and give some properties of it.

Keywords: Normal operator, Binormal operator, Hilbert space.

Introduction: Consider B(H) be the algebra of all bounded linear operators on Hilbert space H. An operator S called normal if$S^*S = SS^*$ .In [2] Campbell, Stephen ,L. introduce the class binormal of operator which is defined as $S^*SSS^* = SS^*S^*S$ ..In[4] Panayappan, S. and Sivamani give a new class of operators called n-binormal and it is defined as $S^*S^nS^nS^* = S^nS^*S^*S^n$. In this paper we defied a new class of operators on Hilbert space as $(S^w)^*S^nS^n(S^w)^* = S^n(S^w)^*(S^w)^*S^n$called w-power n-binormal operator and study some properties of it.

Main Result

Definition 1.1 Let S be bounded operator. S is called w-power n-binormal operator if and only if $(S^w)^*S^nS^n(S^w)^* = S^n(S^w)^*(S^w)^*S^n$,where w,n are nonnegative integer .

Example 1.2 : Let S be a weighted shift operator of non- zero weights $\{\beta_r\}_{r=0}^{\infty}$ .then S is w-power n-binormal operator if and only if $(\overline{\beta_{r-1}} \dots \overline{\beta_{r-w}})(\beta_{r-w} \dots \beta_{r-w+n-1})(\beta_{r-w+n} \dots \beta_{r-w+2n-1})(\overline{\beta_{r-w+2n-1}} \dots \overline{\beta_{r-2w+2n}})z_{r-2w+2n} = (\beta_r \dots \beta_{r+n-1})(\overline{\beta_{r+n-1}} \dots \overline{\beta_{r+n-w}})(\overline{\beta_{r+n-w-1}} \dots \overline{\beta_{r+n-2w}})(\beta_{r+2n-2w} \dots \beta_{r+2n-2w-1})\, z_{r+2n-2w}$

Proof: Suppose $\{z_r\}_{r=0}^{\infty}$ be orthogonal basis of H. Hence $Sz_r = \beta_r\, z_{r+1}$,

$S^*z_r = \overline{\beta_{r-1}}\, z_{r-1}$, $S^nz_r = (\beta_r \dots \beta_{r+n-1})z_{r+n}$ and

$(S^w)^*z_r = (S^*)^w z_r = (\overline{\beta_{r-1}} \dots \overline{\beta_{r-w}})z_{r-w}$

$$(S^w)^*S^nS^n(S^w)^*z_r = (S^w)^*S^n\,(\overline{\beta_{r-1}} \dots \overline{\beta_{r-w}})\; S^n z_{r-w}$$

$$= (S^w)^*S^n\,(\overline{\beta_{r-1}} \dots \overline{\beta_{r-w}})(\beta_{r-w} \dots \beta_{r-w+n-1})z_{r-w+n}$$

$$= (\overline{\beta_{r-1}} \dots \overline{\beta_{r-w}})(\beta_{r-w} \dots \beta_{r-w+n-1})(\beta_{r-w+n} \dots \beta_{r-w+2n-1})(\overline{\beta_{r-w+2n-1}} \dots \overline{\beta_{r-2w+2n}})z_{r-2w+2n}$$

$$S^n(S^w)^*(S^w)^*S^nz_r = S^n(S^w)^*\,(\beta_r \dots \beta_{r+n-1})(S^w)^*\, z_{r+n}$$

$$= S^n(S^w)^*\,(\beta_r \dots \beta_{r+n-1})(\overline{\beta_{r+n-1}} \dots \overline{\beta_{r+n-w}})\, z_{r+n-w}$$

$$= S^n\,(\beta_r \dots \beta_{r+n-1})(\overline{\beta_{r+n-1}} \dots \overline{\beta_{r+n-w}})(\overline{\beta_{r+n-w-1}} \dots \overline{\beta_{r+n-2w}})\, z_{r+n-2w}$$

$$= (\beta_r \dots \beta_{r+n-1})(\overline{\beta_{r+n-1}} \dots \overline{\beta_{r+n-w}})(\overline{\beta_{r+n-w-1}} \dots \overline{\beta_{r+n-2w}})(\beta_{r+2n-2w} \dots \beta_{r+2n-2w-1})\, z_{r+2n-2w}$$

Hence, S is w-power n-binormal operator if and only if

$$(\overline{\beta_{r-1}} \dots \overline{\beta_{r-w}})(\beta_{r-w} \dots \beta_{r-w+n-1})(\beta_{r-w+n} \dots \beta_{r-w+2n-1})(\overline{\beta_{r-w+2n-1}} \dots \overline{\beta_{r-2w+2n}})z_{r-2w+2n}$$
$$= (\beta_r \dots \beta_{r+n-1})(\overline{\beta_{r+n-1}} \dots \overline{\beta_{r+n-w}})(\overline{\beta_{r+n-w-1}} \dots \overline{\beta_{r+n-2w}})(\beta_{r+2n-2w} \dots \beta_{r+2n-2w-1})\, z_{r+2n-2w}$$

Proposition 1.3 Suppose S be abounded operator on H, then it is be w-power n-binormal operator if and only if S is a n-power w-binormal operator .

Proof: Let S be w-power n-binormal operator then $(S^w)^*S^nS^n(S^w)^* = S^n(S^w)^*(S^w)^*S^n$. Therefore, we need to prove that S is a n-power w-binormal operator.

$(S^n)^* S^w S^w (S^n)^* = [[(S^n)^* S^w S^w (S^n)^*]^*]^*$

$= [[S^w (S^n)^*]^* \ [(S^n)^* S^w]^*]^*$

$= [\ [(S^n)^* S^w]\ ^*]^* \ [[S^w (S^n)^*]^*]^*$

$= [(S^w)^* \ S^n]^* \ [S^n \ (S^w)^*]^*$

$= [\ S^n (S^w)^* \ (S^w)^* \ S^n]^*$, since S w-power n-binormal operator

$= [(S^w)^* S^n S^n (S^w)^*]^*$

$= [S^n (S^w)^*]^* \ [(S^w)^* \ S^n]^*$

$= S^w (S^n)^* (S^n)^* S^w$

Thus, S is a n-power w-binormal operator. The convers is similarly.

Definition 1.4 [3]: If A, B are bounded operator on Hilbert space H. Then A, B are unitary equivalent if there is an isomorphism $U: H \to H$ such that $B = UAU^*$.

Proposition 1.5 If S is w-power n-binormal operator**,**

1.- then S^* is w-power n-binormal operator.

2.- If S^{-1} exist then, S^{-1} is w-power n-binormal operator.

3.- If $T \in B(H)$ is unitary equivalent to S then T is w-power n-binormal operator.

Proof

1. Since S is w-power n-binormal operator**,** then $(S^w)^* S^n S^n (S^w)^* = S^n (S^w)^* (S^w)^* S^n$ **.**

$((S^*)^w)^* (S^*)^n (S^*)^n ((S^*)^w)^* = S^w (S^n)^* (S^n)^* S^w$ By above proposition we have,

$= (S^*)^w S^n S^n (S^*)^w$

$= (S^*)^n S^w S^w (S^*)^n$

$= (S^*)^n ((S^*)^*)^w ((S^*)^*)^w (S^*)^n$

$= (S^*)^n ((S^*)^w)^* ((S^*)^w)^* (S^*)^n$

Hence**,** S^* is w-power n-binormal operator.

2. Consider $((S^{-1})^w)^* (S^{-1})^n (S^{-1})^n ((S^{-1})^w)^*$

$= ((S^*)^w)^{-1} (S^n)^{-1} (S^n)^{-1} \ ((S^*)^w)^{-1}$

$= [S^n \ (S^*)^w]^{-1} \ [(S^*)^w \ S^n]^{-1}$

$= \ [(S^*)^w S^n \ S^n \ (S^*)^w]^{-1}$, by above proposition

$= [S^n (S^w)^* (S^w)^* S^n]^{-1}$

$= [S^n \ (S^*)^w]^{-1} \ [\ S^n (S^*)^w]^{-1}$

$= (S^n)^{-1} ((S^*)^w)^{-1} \ ((S^*)^w)^{-1} (S^n)^{-1}$

$= (S^{-1})^n ((S^{-1})^w)^* ((S^{-1})^w)^* (S^{-1})^n$

Hence, S^{-1} is w-power n-binormal operator.

3. Since T is unitary equivalent to S then $T = USU^*$,therefore $(USU^*)^n = US^nU^*$

$(T^w)^* T^n T^n (T^w)^* = ((USU^*)^w)^* (USU^*)^n (USU^*)^n ((USU^*)^w)^*$

$= (US^wU^*)^* (US^nU^*)(US^nU^*)(US^wU^*)^*$

$= (U(S^w)^*U^*) \ \ (US^nU^*)(US^nU^*)(U(S^{w*})U^*)$

$= U(S^w)^* S^n S^n (S^w)^* U^*$

$$= US^nU^*U(S^w)^*U^*U(S^w)^*U^*US^nU^*$$

$$= (US^nU^*)(US^wU^*)^*(US^wU^*)^*(US^nU^*)$$

$$= (USU^*)^n((USU^*)^w)^*((USU^*)^w)^*(USU^*)^n = T^n(T^w)^*(T^w)^*T^n$$

Hence T is w-power n-binormal operator

Proposition 1.6 Let S be a bounded operator. If S is w-power n-binormal operator then S^{nw} is binormal operator.

Proof: Suppose that S is w-power n-binormal operator then $(S^w)^*S^nS^n(S^w)^* = S^n(S^w)^*(S^w)^*S^n$ it is clear that $(S^m)^* = (S^*)^m$ for each nonnegative integer m**.**

$$(S^{nw})^*S^{nw}S^{nw}(S^{nw})^* = ((S^w)^n)^*S^{nw}S^{nw}((S^w)^n)^*$$

$$= \underbrace{(S^wS^w\dots S^w)^*}_{n-times}\underbrace{(S^nS^n\dots S^n)}_{m-times}\underbrace{(S^nS^n\dots S^n)}_{m-times}\underbrace{(S^wS^w\dots S^w)^*}_{n-times}$$

$$= \underbrace{(S^w)^*(S^w)^*\dots(S^w)^*}_{n-times}\underbrace{(S^nS^n\dots S^n)}_{m-times}\underbrace{(S^nS^n\dots S^n)}_{m-times}\underbrace{(S^w)^*(S^w)^*\dots(S^w)^*}_{n-times}$$

$$= (S^w)^*(S^w)^*\dots S^n(S^w)^*S^nS^n\dots S^n.S^nS^n\dots(S^w)^*S^n.(S^w)^*(S^w)^*\dots(S^w)^*$$

$$\vdots$$

$$= \underbrace{(S^nS^n\dots S^n)}_{m-times}\underbrace{(S^w)^*(S^w)^*\dots(S^w)^*}_{n-times}\underbrace{(S^w)^*(S^w)^*\dots(S^w)^*}_{n-times}\underbrace{(S^nS^n\dots S^n)}_{m-times}$$

$$= \underbrace{(S^nS^n\dots S^n)}_{m-times}\underbrace{(S^wS^w\dots S^w)^*}_{n-times}\underbrace{(S^wS^w\dots S^w)^*}_{n-times}\underbrace{(S^nS^n\dots S^n)}_{m-times}$$

$$= S^{nw}((S^w)^n)^*((S^w)^n)^*S^{nw}$$

$$= S^{nw}(S^{nw})^*(S^{nw})^*S^{nw}.$$

Hence, S^{nw} is binormal operator.

Theorem 1.7 The set of all w-power n-binormal operators on H is a closed subset of B(H) under scalar multiplication.

Proof: Let

$$W(H) = \{S \in B(H): S \text{ is w} - \text{power n} - \text{binormal operator on H for some nonnegative integer w}\}$$

Let $S \in W(H)$ then we have S is w-power n-binormal operator and thus $(S^w)^*S^nS^n(S^w)^* = S^n(S^w)^*(S^w)^*S^n$.

Let γ be a scalar, hence

$$((\gamma S)^w)^*(\gamma S)^n(\gamma S)^n((\gamma S)^w)^* = (\bar{\gamma})^w(S^w)^*\gamma^nS^n\ \gamma^nS^n\ (\bar{\gamma})^w(S^w)^*$$

$$= (\bar{\gamma})^w\gamma^n\ \gamma^n(\bar{\gamma})^w\ (S^w)^*\ S^n\ S^n\ (S^w)^*$$

$$= (\bar{\gamma})^w\gamma^n\ \gamma^n(\bar{\gamma})^w\ S^n(S^w)^*\ (S^w)^*\ S^n$$

$$= \gamma^nS^n\ (\bar{\gamma})^w(S^w)^*\ \ (\bar{\gamma})^w(S^w)^*\ \gamma^nS^n$$

$$= (\gamma S)^n((\gamma S)^w)^*((\gamma S)^w)^*(\gamma S)^n$$

Thus $\gamma S \in W(H)$,

Let S_k be a sequence in $W(H)$ and converge to S, then we can get that

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$$= \|(S^w)^*S^nS^n(S^w)^* - (S_k^w)^*S_k^nS_k^n(S_k^w)^* + S_k^n(S_k^w)^*(S_k^w)^*S_k^n - S^n(S^w)^*(S^w)^*S^n\|$$

$$\leq \|(S^w)^*S^nS^n(S^w)^* - (S_k^w)^*S_k^nS_k^n(S_k^w)^*\| + \|S_k^n(S_k^w)^*(S_k^w)^*S_k^n - S^n(S^w)^*(S^w)^*S^n\| \to 0 \text{ as } k \to \infty.$$

Hence, $(S^w)^*S^nS^n(S^w)^* = S^n(S^w)^*(S^w)^*S^n$ therefore $S \in W(H)$.

Then, $W(H)$ is closed subset.

Theorem 1.8: If R and S are w-power n-binormal operators on H ,and let S commute with R then (SR) is w-power n-binormal operator on H**.**

Proof:

$$((SR^w))^*(SR)^n(SR)^n((SR)^w)^* = (R^w)^*(S^w)^*(R)^n(S)^n(R)^n(S)^n(R^w)^*(S^w)^*$$
$$= (R^w)^*(S^w)^*(S)^n(S)^n(R)^n(R)^n(S^w)^*(R^w)^*$$
$$= (R^w)^*(S^w)^*(S)^n(S)^n(R)^n(S^w)^*(R)^n(R^w)^*$$
$$= (R^w)^*(S)^n(S^w)^*(S)^n(S^w)^*(R)^n(R)^n(R^w)^*$$
$$= (S)^n(R^w)^*(S^w)^*(S)^n(S^w)^*(R)^n(R)^n(R^w)^*$$
$$= (S)^n(R^w)^*(S^w)^*(S)^n(R)^n(S^w)^*(R)^n(R^w)^*$$
$$= (S)^n(R^w)^*(S^w)^*(R)^n(S)^n(S^w)^*(R)^n(R^w)^*$$
$$= (S)^n(R^w)^*(R)^n(S^w)^*(S)^n(S^w)^*(R)^n(R^w)^*$$
$$= (S)^n(R)^n(R^w)^*(S^w)^*(S)^n(S^w)^*(R)^n(R^w)^*$$
$$= (S)^n(R)^n(R^w)^*(S^w)^*(S^w)^*(S)^n(R)^n(R^w)^*$$
$$= (S)^n(R)^n(R^w)^*(S^w)^*(S^w)^*(R^w)^*(S)^n(R)^n$$
$$= (S)^n(R)^n(R^w)^*(S^w)^*(R^w)^*(S^w)^* \quad (R)^n(S)^n$$
$$= (SR)^n((SR^w))^*((SR)^w)^*(SR)^n$$

Theorem 1.9: Let $S_1, S_2, \dots, S_k$, are w-power n-binormal operators on H. Then the direct sum $(S_1 \oplus S_2 \oplus \dots \oplus S_k)$ is w-power n-binormal operator on H**.**

Proof: Since every operator of $S_1, S_2, \dots, S_k$ is w-power n-binormal, then

$({S_l}^w)^*{S_l}^n{S_l}^n({S_l}^w)^* = {S_l}^n({S_l}^w)^*({S_l}^w)^*{S_l}^n$ for all $l = 1,1, \dots, k$

$$((S_1 \oplus S_2 \oplus \dots \oplus S_k)^w)^*(S_1 \oplus S_2 \oplus \dots \oplus S_k)^n(S_1 \oplus S_2 \oplus \dots \oplus S_k)^n((S_1 \oplus S_2 \oplus \dots \oplus S_k)^w)^*$$
$$= ({S_1}^w \oplus {S_2}^w \oplus \dots \oplus {S_k}^w)^*({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)({S_1}^w \oplus {S_2}^w \oplus \dots \oplus {S_k}^w)^*$$
$$= [({S_1}^w)^* \oplus ({S_2}^w)^* \oplus \dots \oplus ({S_k}^w)^*]({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)[({S_1}^w)^* \oplus ({S_2}^w)^* \oplus \dots \oplus ({S_k}^w)^*]$$
$$= ({S_1}^w)^*{S_1}^n{S_1}^n({S_1}^w)^* \oplus ({S_2}^w)^*{S_2}^n{S_2}^n({S_2}^w)^* \oplus \dots \oplus ({S_k}^w)^*{S_k}^n{S_k}^n({S_k}^w)^*$$
$$= {S_1}^n({S_1}^w)^*({S_1}^w)^*{S_1}^n \oplus {S_2}^n({S_2}^w)^*({S_2}^w)^*{S_2}^n \oplus \dots \oplus {S_k}^n({S_k}^w)^*({S_k}^w)^*{S_k}^n$$
$$= ({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)\,[({S_1}^w)^* \oplus ({S_2}^w)^* \oplus \dots \oplus ({S_k}^w)^*][({S_1}^w)^* \oplus ({S_2}^w)^* \oplus \dots \oplus ({S_k}^w)^*]({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)$$
$$= ({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)({S_1}^w \oplus {S_2}^w \oplus \dots \oplus {S_k}^w)^*({S_1}^w \oplus {S_2}^w \oplus \dots \oplus {S_k}^w)^*({S_1}^n \oplus {S_2}^n \oplus \dots \oplus {S_k}^n)$$
$$= (S_1 \oplus S_2 \oplus \dots \oplus S_k)^n((S_1 \oplus S_2 \oplus \dots \oplus S_k)^w)^*((S_1 \oplus S_2 \oplus \dots \oplus S_k)^w)^*(S_1 \oplus S_2 \oplus \dots \oplus S_k)^n$$

Thus, $(S_1 \oplus S_2 \oplus \dots \oplus S_k)$ is w-power n-binormal operator on H.

Theorem 1.10: Let $S_1, S_2, \dots, S_k$, are w-power n-binormal operators on H. Then the tenser product $(S_1 \otimes S_2 \otimes \dots \otimes S_k)$ is w-power n-binormal operator on H**.**

Proof: Since every operator of $S_1, S_2, \dots, S_k$ is w-power n-binormal, then

$({S_l}^w)^*{S_l}^n{S_l}^n({S_l}^w)^* = {S_l}^n({S_l}^w)^*({S_l}^w)^*{S_l}^n$ for all $l = 1,1, \dots, k$

$$((S_1 \otimes S_2 \otimes \dots \otimes S_k)^w)^*(S_1 \otimes S_2 \otimes \dots \otimes S_k)^n(S_1 \otimes S_2 \otimes \dots \otimes S_k)^n((S_1 \otimes S_2 \otimes \dots \otimes S_k)^w)^*(\boldsymbol{x_1} \otimes \boldsymbol{x_2} \otimes \dots \otimes \boldsymbol{x_k})$$
$$= ({S_1}^w \otimes {S_2}^w \otimes \dots \otimes {S_k}^w)^*({S_1}^n \otimes {S_2}^n \otimes \dots \otimes {S_k}^n)({S_1}^n \otimes {S_2}^n \otimes \dots \otimes {S_k}^n)({S_1}^w \otimes {S_2}^w \otimes \dots \otimes {S_k}^w)^*(\boldsymbol{x_1} \otimes \boldsymbol{x_2} \otimes \dots \otimes \boldsymbol{x_k})$$

$= [(S_1{}^w)^*\otimes(S_2{}^w)^*\otimes \dots \otimes(S_k{}^w)^*](S_1{}^n\otimes S_2{}^n\otimes \dots \otimes S_k{}^n)(S_1{}^n\otimes S_2{}^n\otimes \dots \otimes S_k{}^n)[(S_1{}^w)^*\otimes(S_2{}^w)^*\otimes \dots \otimes(S_k{}^w)^*](\boldsymbol{x_1}\otimes\boldsymbol{x_2}\otimes \dots \otimes\boldsymbol{x_k})$

$= (S_1{}^w)^*S_1{}^nS_1{}^n(S_1{}^w)^*\boldsymbol{x_1}\otimes(S_2{}^w)^*S_2{}^nS_2{}^n(S_2{}^w)^*\boldsymbol{x_2}\otimes \dots \otimes(S_k{}^w)^*S_k{}^nS_k{}^n(S_k{}^w)^*\boldsymbol{x_k}$

$= S_1{}^n(S_1{}^w)^*(S_1{}^w)^*S_1{}^n\boldsymbol{x_1}\otimes S_2{}^n(S_2{}^w)^*(S_2{}^w)^*S_2{}^n\boldsymbol{x_2}\otimes \dots \otimes S_k{}^n(S_k{}^w)^*(S_k{}^w)^*S_k{}^n\boldsymbol{x_k}$

$= (S_1{}^n\otimes S_2{}^n\otimes \dots \otimes S_k{}^n) [(S_1{}^w)^*\otimes(S_2{}^w)^*\otimes \dots \otimes(S_k{}^w)^*][(S_1{}^w)^*\otimes(S_2{}^w)^*\otimes \dots \otimes(S_k{}^w)^*](S_1{}^n\otimes S_2{}^n\otimes \dots \otimes S_k{}^n)(\boldsymbol{x_1}\otimes\boldsymbol{x_2}\otimes \dots \otimes\boldsymbol{x_k})$

$= (S_1{}^n\otimes S_2{}^n\otimes \dots \otimes S_k{}^n)(S_1{}^w\otimes S_2{}^w\otimes \dots \otimes S_k{}^w)^*(S_1{}^w\otimes S_2{}^w\otimes \dots \otimes S_k{}^w)^*(S_1{}^n\otimes S_2{}^n\otimes \dots \otimes S_k{}^n)(\boldsymbol{x_1}\otimes\boldsymbol{x_2}\otimes \dots \otimes\boldsymbol{x_k})$

$= (S_1\otimes S_2\otimes \dots \otimes S_k)^n((S_1\otimes S_2\otimes \dots \otimes S_k)^w)^*((S_1\otimes S_2\otimes \dots \otimes S_k)^w)^*(S_1\otimes S_2\otimes \dots \otimes S_k)^n(\boldsymbol{x_1}\otimes\boldsymbol{x_2}\otimes \dots \otimes\boldsymbol{x_k})$

Thus, $(S_1\otimes S_2\otimes \dots \otimes S_k)$ is w-power n-binormal operator on H.

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