DOI: https://doi.org/10.24297/jam.v20i.8969

Coefficient Bounds for a New Subclasses of Bi-Univalent Functions Associated with Horadam Polynomials

Najah Ali Jiben Al-Ziadi

Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya-Iraq

Abstract:

In this work we present and investigate three new subclasses of the function class Σ of bi-univalent functions in the open unit disk Δ defined by means of the Horadam polynomials. Furthermore, for functions in each of the subclasses introduced here, we obtain upper bounds for the initial coefficients $|a_2|$ and $|a_3|$. Also, we debate Fekete-Szegö inequality for functions belongs to these subclasses.

Keywords: Bi-univalent functions, Coefficient bounds, Fekete-Szegö inequality, Holomorphic function, Horadam polynomials.

2010 AMS Mathematics Subject Classification: 30C45, 30C50.

Introduction

Symbolized by $\mathcal{A}$ the function class of the shape:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are holomorphic in the open unit disk $\Delta = \{ z : z \in \mathbb{C} \; and \; |z| < 1\}$ and normalized under the conditions indicated by $f(0) = f'(0) - 1 = 0$. Furthermore, symbolized by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in U.

The Koebe One-Quarter Theorem [4] shows that the image of Δ includes a disk of radius ¼ under each function f from $\mathcal{S}$. Thereby each univalent function of this kind has an inverse f^{-1} which fulfills

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(\, f^{-1}(w)) = w \quad \left(|w| < r_0\,(f);\; r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

The function $f \in \mathcal{A}$ is considered bi-univalent in Δ if together f^{-1} and f are univalent in Δ. Indicated by the Taylor-Maclaurin series expansion (1), the class of all bi-univalent functions in Δ can be symbolized by Σ . In the year 2010, Srivastava et al. [10] refreshed the study of various classes of bi-univalent functions. Moreover, many penmans explored bounds for different subclasses of bi-univalent functions (see, for example [3,5,6,11]). The coefficient estimate problem involving the bound of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\}, \mathbb{N} = \{1,2,3, \ldots\}$) is still an open problem.

For two functions $\mathcal{D}$ and $\mathcal{Y}$, holomorphic in the open unit disk Δ, we say that the function $\mathcal{D}(w)$ is subordinate to $\mathcal{Y}(w)$ in Δ, and write

$$\mathcal{D}(w) \prec \mathcal{Y}(w) \qquad (w \in \Delta),$$

if there exists a Schwarz function $\mathcal{T}(w)$, holomorphic in Δ, with

$$\mathcal{T}(0) = 0 \; and \; |\mathcal{T}(w)| < 1 \quad (w \in \Delta),$$

such that

$$\mathcal{D}(w) = \mathcal{Y}(\mathcal{T}(w)) \qquad (w \in \Delta).$$

In special, if the function $\mathcal{Y}$ is univalent in Δ, the above subordination is equivalent to

$$\mathcal{D}(0) = \mathcal{Y}(0) \text{ and } \mathcal{D}(\Delta) \subset \mathcal{Y}(\Delta).$$

The following recurrence relation gives the Horadam polynomials $h_n(x)$ (see (8))

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (x \in \mathbb{R}, \ n \in \mathbb{N}\backslash\{1,2\}, \mathbb{N} = \{1,2,3,\dots\}), \tag{3}$$

with $h_1(x) = k, h_2(x) = bx$ and $h_3(x) = pbx^2 + kq$ where k, b, p *and* q are some real constants. The characteristic equation of repetition relationship (3) is $t^2 - pxt - q = 0$. There are two real roots of this equation

$$\alpha_1 = \frac{px + \sqrt{p^2x^2 + 4q}}{2} \quad \text{and} \quad \alpha_2 = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

The generating function of the Horadam polynomials $h_n(x)$ is indicated by

$$\Omega(x,z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{k + (b - kp)xz}{1 - pxz - qz^2} . \tag{4}$$

It should be noted that for specific values of $\mathrm{k}, \mathrm{b}, \mathrm{p}$ and q, the Horadam polynomial $h_n(x)$ leads to different polynomials, among those, we list a few cases here (see, [7 , 8], for more details) :

a) If $k = b = p = q = 1$, then we get the Fibonacci polynomials $F_n(x)$.

b) If $k = 2$ and $b = p = q = 1$, then we have the Lucas polynomials $L_n(x)$.

c) If $k = q = 1$ and $b = p = 2$, then we attain the Pell polynomials $P_n(x)$.

d) If $k = b = p = 2$ and $q = 1$, then we have the Pell-Lucas polynomials $Q_n(x)$.

e) If $k = b = 1, p = 2$ and $q = -1$, then we obtain the Chebyshev polynomials $T_n(x)$ of the first kind.

f) If $k = 1, b = p = 2$ and $q = -1$, then we attain the Chebyshev polynomials $U_n(x)$ of the second kind.

Coefficient bounds and Fekete–Szegö inequality for the class $\mathcal{K}_\Sigma(\beta, x)$

Definition 1 A function $f \in \Sigma$ is said to be in the class $\mathcal{K}_\Sigma(\beta, x)$ for $0 \le \beta \le 1$ and $x \in \mathbb{R}$, if the following conditions of subordination are satisfied:

$$(1-\beta)f'(z) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Omega(x,z) + 1 - k \tag{5}$$

and

$$(1-\beta)g'(w) + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \Omega(x,w) + 1 - k, \tag{6}$$

where the function $g = f^{-1}$ is indicated by (2) and k is real constant.

Remark 1

For $\beta = 0$, the class $\mathcal{K}_\Sigma(\beta, x)$ shortens to the class Σ' presented and investigated by Alamoush [2].

For $\beta = 1$, the class $\mathcal{K}_\Sigma(\beta, x)$ shortens to the class $\mathcal{K}_\Sigma(x)$ presented and investigated by Abirami et al. [1].

Theorem 1 Let the function $f \in \Sigma$ indicated by (1) be in the class $\mathcal{K}_\Sigma(\beta, x)$. Then

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(3-\beta)b - 4p]bx^2 - 4kq|}} \tag{7}$$

and

$$|a_3| \leq \frac{b^2x^2}{4} + \frac{|bx|}{3(\beta+1)}, \tag{8}$$

and for some $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|bx|}{3(\beta+1)} \quad if \\ |\mu - 1| \leq \dfrac{|[(3-\beta)b - 4p]bx^2 - 4kq|}{3(\beta+1)b^2x^2} \\ \dfrac{|bx|^3|\mu - 1|}{|[(3-\beta)b - 4p]bx^2 - 4kq|} \quad if \\ |\mu - 1| \geq \dfrac{|[(3-\beta)b - 4p]bx^2 - 4kq|}{3(\beta+1)b^2x^2}. \end{cases} \tag{9}$$

Proof. Let $f \in \mathcal{K}_\Sigma(\beta, x)$, $0 \leq \beta \leq 1$ and $x \in \mathbb{R}$. Then there are two holomorphic function $v, u: \Delta \to \Delta$ indicated by

$$v(z) = t_1z + t_2z^2 + t_3z^3 + \cdots \qquad (z \in \Delta)$$

and

$$u(w) = s_1w + s_2w^2 + s_3w^3 + \cdots \qquad (w \in \Delta),$$

with $v(0) = u(0) = 0, |v(z)| < 1 \; and \; |u(w)| < 1, \; z, w \in \Delta$, such that

$$(1-\beta)f'(z) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Omega(x, v(z)) + 1 - k$$

and

$$(1-\beta)g'(w) + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \Omega\big(x, u(w)\big) + 1 - k.$$

Or, in equivalent way,

$$(1-\beta)f'(z) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + h_1(x) - k + h_2(x)v(z) + h_3(x)[v(z)]^2 + \cdots \tag{10}$$

and

$$(1-\beta)g'(w) + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + h_1(x) - k + h_2(x)u(w) + h_3(x)[u(w)]^2 + \cdots. \tag{11}$$

From (10) and (11), we attain

$$(1-\beta)f'(z) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + h_2(x)t_1z + [h_2(x)t_2 + h_3(x)t_1^2]z^2 + \cdots \tag{12}$$

and

$$(1-\beta)g'(w) + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + h_2(x)s_1w + [h_2(x)s_2 + h_3(x)s_1^2]w^2 + \cdots. \tag{13}$$

Notice that if

$$|v(z)| = |t_1z + t_2z^2 + t_3z^3 + \cdots| < 1 \qquad (z \in \Delta)$$

and

$$|u(w)| = |s_1w + s_2w^2 + s_3w^3 + \cdots| < 1 \qquad (w \in \Delta),$$

then

$$|t_i| \leq 1 \; and \; |s_i| \leq 1 \quad (\, i \in \mathbb{N}\,).$$

It follows from (12) and (13) that

$$2a_2 = h_2(x)t_1, \tag{14}$$

$$3(1+\beta)a_3 - 4\beta a_2^2 = h_2(x)t_2 + h_3(x)t_1^2, \tag{15}$$

$$-2a_2 = h_2(x)s_1 \tag{16}$$

and

$$-3(1+\beta)a_3 + 2(\beta+3)a_2^2 = h_2(x)s_2 + h_3(x)s_1^2. \tag{17}$$

From (14) and (16), we find that

$$t_1 = -s_1 \tag{18}$$

and

$$8a_2^2 = [h_2(x)]^2(t_1^2 + s_1^2). \tag{19}$$

If we add (15) to (17), we get

$$(6-2\beta)a_2^2 = h_2(x)(t_2+s_2) + h_3(x)(t_1^2+s_1^2). \tag{20}$$

By using (19) in equation (20), we have

$$\left[(6-2\beta) - \frac{8h_3(x)}{[h_2(x)]^2}\right]a_2^2 = h_2(x)(t_2+s_2)\,, \tag{21}$$

which yields

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(3-\beta)b - 4p]bx^2 - 4kq|}}.$$

Next, if we deduct (17) from (15), we get

$$6(\beta+1)(a_3 - a_2^2) = h_2(x)(t_2 - s_2) + h_3(x)(t_1^2 - s_1^2). \tag{22}$$

In view of (18) and (19), equation (22) becomes

$$a_3 = \frac{[h_2(x)]^2(t_1^2+s_1^2)}{8} + \frac{h_2(x)(t_2-s_2)}{6(\beta+1)}.$$

Now, with the help of equation (3), we deduce that

$$|a_3| \leq \frac{b^2x^2}{4} + \frac{|bx|}{3(\beta+1)}.$$

Finally, by using (21) and (22) for some $\mu \in \mathbb{R}$, we get

$$a_3 - \mu a_2^2 = \frac{h_2(x)(t_2-s_2)}{6(\beta+1)} + \frac{[h_2(x)]^3(1-\mu)(t_2+s_2)}{(6-2\beta)[h_2(x)]^2 - 8h_3(x)}$$

$$= \frac{h_2(x)}{2}\left[\left(\Psi(\mu,x) + \frac{1}{3(\beta+1)}\right)t_2 + \left(\Psi(\mu,x) - \frac{1}{3(\beta+1)}\right)s_2\right],$$

where

$$\Psi(\mu,x) = \frac{[h_2(x)]^2(1-\mu)}{(3-\beta)[h_2(x)]^2 - 4h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|h_2(x)|}{3(\beta+1)} & if \quad 0 \leq |\Psi(\mu,x)| \leq \dfrac{1}{3(\beta+1)} \\ |h_2(x)||\Psi(\mu,x)| & if \quad |\Psi(\mu,x)| \geq \dfrac{1}{3(\beta+1)} \end{cases}$$

and with respect to (3), it evidently completes the proof of the theorem (1).

Remark 2 If we put $\beta = 0$ in Theorem (1), we get the outcomes which were indicated by Alamoush [2]. In addition, if we put $\beta = 1$ in Theorem (1), we get the outcomes which were indicated by Abirami et al. [1].

Coefficient bounds and Fekete–Szegö inequality for the class $\mathcal{W}_\Sigma(\alpha, x)$

Definition 2 A function $f \in \Sigma$ is said to be in the class $\mathcal{W}_\Sigma(\alpha,x)$for $0 \leq \alpha \leq 1$ and $x \in \mathbb{R}$, if the following conditions of subordination are satisfied:

$$\frac{zf'(z) + (2\alpha^2 - \alpha)z^2 f''(z)}{4(\alpha - \alpha^2)z + (2\alpha^2 - \alpha)zf'(z) + (2\alpha^2 - 3\alpha + 1)f(z)} \prec \Omega(x,z) + 1 - k \tag{23}$$

and

$$\frac{wg'(w) + (2\alpha^2 - \alpha)w^2 g''(w)}{4(\alpha - \alpha^2)w + (2\alpha^2 - \alpha)wg'(w) + (2\alpha^2 - 3\alpha + 1)g(w)} \prec \Omega(x,w) + 1 - k, \tag{24}$$

where the function $g = f^{-1}$ is indicated by (2) and k is real constant.

Remark 3 For $\alpha = 0$, the class $\mathcal{W}_\Sigma(\alpha,x)$ shortens to the class $\mathcal{W}_\Sigma(x)$ introduced and investigated by Srivastava et al. [9].

Theorem 2 Let the function $f \in \Sigma$ indicated by (1) be in the class $\mathcal{W}_\Sigma(\alpha,x)$. Then

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1)b - (1 + 3\alpha - 2\alpha^2)^2 p]bx^2 - (1 + 3\alpha - 2\alpha^2)^2 kq|}} \tag{25}$$

and

$$|a_3| \leq \frac{b^2x^2}{(1 + 3\alpha - 2\alpha^2)^2} + \frac{|bx|}{2(2\alpha^2 + 1)}, \tag{26}$$

and for some $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|bx|}{2(2\alpha^2+1)} \quad if \\ |\mu - 1| \leq \dfrac{\left|\left[\left(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1\right)b - \left(1 + 3\alpha - 2\alpha^2\right)^2 p\right]bx^2 - \left(1 + 3\alpha - 2\alpha^2\right)^2 kq\right|}{2(2\alpha^2+1)b^2x^2} \\ \dfrac{|bx|^3|\mu - 1|}{|[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1)b - (1 + 3\alpha - 2\alpha^2)^2 p]bx^2 - (1 + 3\alpha - 2\alpha^2)^2 kq|} \quad if \\ |\mu - 1| \geq \dfrac{\left|\left[\left(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1\right)b - \left(1 + 3\alpha - 2\alpha^2\right)^2 p\right]bx^2 - \left(1 + 3\alpha - 2\alpha^2\right)^2 kq\right|}{2(2\alpha^2+1)b^2x^2}. \end{cases} \tag{27}$$

Proof. Let $f \in \mathcal{W}_\Sigma(\alpha,x)$, $0 \leq \alpha \leq 1$ and $x \in \mathbb{R}$. Then there are two holomorphic function $v, u: \Delta \to \Delta$ indicated by

$$v(z) = t_1 z + t_2 z^2 + t_3 z^3 + \cdots \qquad (z \in \Delta)$$

and

$$u(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots \qquad (w \in \Delta),$$

with $v(0) = u(0) = 0, |v(z)| < 1 \; and \; |u(w)| < 1, \; z,w \in \Delta$, such that

$$\frac{zf'(z) + (2\alpha^2 - \alpha)z^2 f''(z)}{4(\alpha - \alpha^2)z + (2\alpha^2 - \alpha)zf'(z) + (2\alpha^2 - 3\alpha + 1)f(z)} \prec \Omega(x, v(z)) + 1 - k$$

and

$$\frac{wg'(w)+(2\alpha^2-\alpha)w^2g''(w)}{4(\alpha-\alpha^2)w+(2\alpha^2-\alpha)wg'(w)+(2\alpha^2-3\alpha+1)g(w)} \prec \Omega\big(x,u(w)\big)+1-k\,.$$

Or, in equivalent way,

$$\frac{zf'(z)+(2\alpha^2-\alpha)z^2f''(z)}{4(\alpha-\alpha^2)z+(2\alpha^2-\alpha)zf'(z)+(2\alpha^2-3\alpha+1)f(z)}$$

$$=1+\mathrm{h}_1(\mathrm{x})-\mathrm{k}+\mathrm{h}_2(\mathrm{x})\mathrm{v}(\mathrm{z})+\mathrm{h}_3(\mathrm{x})[\mathrm{v}(\mathrm{z})]^2+\cdots \tag{28}$$

and

$$\frac{wg'(w)+(2\alpha^2-\alpha)w^2g''(w)}{4(\alpha-\alpha^2)w+(2\alpha^2-\alpha)wg'(w)+(2\alpha^2-3\alpha+1)g(w)}$$

$$=1+h_1(x)-k+h_2(x)u(w)+h_3(x)[u(w)]^2+\cdots. \tag{29}$$

From the equations (28) and (29), we attain

$$\frac{zf'(z)+(2\alpha^2-\alpha)z^2f''(z)}{4(\alpha-\alpha^2)z+(2\alpha^2-\alpha)zf'(z)+(2\alpha^2-3\alpha+1)f(z)}$$

$$=1+h_2(x)t_1z+[h_2(x)t_2+h_3(x)t_1^2]z^2+\cdots \tag{30}$$

and

$$\frac{wg'(w)+(2\alpha^2-\alpha)w^2g''(w)}{4(\alpha-\alpha^2)w+(2\alpha^2-\alpha)wg'(w)+(2\alpha^2-3\alpha+1)g(w)}$$

$$=1+h_2(x)s_1w+[h_2(x)s_2+h_3(x)s_1^2]w^2+\cdots. \tag{31}$$

Notice that if

$$|v(z)|=|t_1z+t_2z^2+t_3z^3+\cdots|<1 \qquad (z\in\Delta)$$

and

$$|u(w)|=|s_1w+s_2w^2+s_3w^3+\cdots|<1 \qquad (w\in\Delta),$$

then

$$|t_i|\le 1 \; and \; |s_i|\le 1 \quad (\, i\in\mathbb{N}\,).$$

It follows from (30) and (31) that

$$(1+3\alpha-2\alpha^2)a_2=h_2(x)t_1, \tag{32}$$

$$(12\alpha^4-28\alpha^3+11\alpha^2+2\alpha-1)a_2^2+(4\alpha^2+2)a_3=h_2(x)t_2+h_3(x)t_1^2, \tag{33}$$

$$-(1+3\alpha-2\alpha^2)a_2=h_2(x)s_1 \tag{34}$$

and

$$(12\alpha^4-28\alpha^3+19\alpha^2+2\alpha+3)a_2^2-(4\alpha^2+2)a_3=h_2(x)s_2+h_3(x)s_1^2. \tag{35}$$

From (32) and (34), we find that

$$t_1=-s_1 \tag{36}$$

and

$$2(1+3\alpha-2\alpha^2)^2a_2^2=[h_2(x)]^2(t_1^2+s_1^2). \tag{37}$$

If we add (33) to (35), we get

$$(24\alpha^4-56\alpha^3+30\alpha^2+4\alpha+2)a_2^2=h_2(x)(t_2+s_2)+h_3(x)(t_1^2+s_1^2). \tag{38}$$

By using (37) in equation (38), we have

$$\left[(24\alpha^4-56\alpha^3+30\alpha^2+4\alpha+2)-\frac{2(1+3\alpha-2\alpha^2)^2h_3(x)}{[h_2(x)]^2}\right]a_2^2=h_2(x)(t_2+s_2), \tag{39}$$

which yields

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1)b - (1 + 3\alpha - 2\alpha^2)^2 p]bx^2 - (1 + 3\alpha - 2\alpha^2)^2 kq|}}.$$

Next, if we deduct (35) from (33), we obtain

$$4(2\alpha^2 + 1)(a_3 - a_2^2) = h_2(x)(t_2 - s_2) + h_3(x)(t_1^2 - s_1^2). \tag{40}$$

In view of (36) and (37), equation (40) becomes

$$a_3 = \frac{[h_2(x)]^2(t_1^2 + s_1^2)}{2(1 + 3\alpha - 2\alpha^2)^2} + \frac{h_2(x)(t_2 - s_2)}{4(2\alpha^2 + 1)}.$$

Now, with the help of equation (3), we deduce that

$$|a_3| \leq \frac{b^2x^2}{(1 + 3\alpha - 2\alpha^2)^2} + \frac{|bx|}{2(2\alpha^2 + 1)}.$$

Finally, by using (39) and (40) for some $\mu \in \mathbb{R}$, we get

$$a_3 - \mu a_2^2 = \frac{h_2(x)(t_2 - s_2)}{4(2\alpha^2 + 1)} + \frac{[h_2(x)]^3(1 - \mu)(t_2 + s_2)}{(24\alpha^4 - 56\alpha^3 + 30\alpha^2 + 4\alpha + 2)[h_2(x)]^2 - 2(1 + 3\alpha - 2\alpha^2)^2 h_3(x)}$$

$$= \frac{h_2(x)}{2}\left[\left(\Psi(\mu, x) + \frac{1}{2(2\alpha^2+1)}\right)t_2 + \left(\Psi(\mu, x) - \frac{1}{2(2\alpha^2+1)}\right)s_2\right],$$

where

$$\Psi(\mu, x) = \frac{[h_2(x)]^2(1 - \mu)}{(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1)[h_2(x)]^2 - (1 + 3\alpha - 2\alpha^2)^2 h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|h_2(x)|}{2(2\alpha^2 + 1)} & if \quad 0 \leq |\Psi(\mu, x)| \leq \dfrac{1}{2(2\alpha^2 + 1)} \\ |h_2(x)||\Psi(\mu, x)| & if \quad |\Psi(\mu, x)| \geq \dfrac{1}{2(2\alpha^2 + 1)} \end{cases}$$

and with respect to (3), it evidently completes the proof of the theorem (2).

Remark 4 If we put $\alpha = 0$ in Theorem (2), we get the outcomes which were indicated by Srivastava et al. [9].

Coefficient bounds and Fekete–Szegö inequality for the class $\mathcal{N}_{\Sigma}(\alpha, \gamma, x)$

Definition 3 A function $f \in \Sigma$ is said to be in the class $\mathcal{N}_{\Sigma}(\alpha, \gamma, x)$ for $0 \leq \alpha \leq 1, \gamma \in \mathbb{C}\backslash\{0\}$ and $x \in \mathbb{R}$, if the following conditions of subordination are satisfied:

$$1 + \frac{1}{\gamma}\left[\frac{\alpha z^3 f'''(z) + (1 + 2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)} - 1\right] \prec \Omega(x, z) + 1 - k \tag{41}$$

and

$$1 + \frac{1}{\gamma}\left[\frac{\alpha w^3 g'''(w) + (1 + 2\alpha)w^2 g''(w) + wg'(w)}{\alpha w^2 g''(w) + wg'(w)} - 1\right] \prec \Omega(x, w) + 1 - k, \tag{42}$$

where the function $g = f^{-1}$ is indicated by (2) and k is real constant.

Theorem 3 Let the function $f \in \Sigma$ indicated by (1) be in the class $\mathcal{N}_{\Sigma}(\alpha, \gamma, x)$. Then

$$|a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|[\gamma(2 + 4\alpha - 4\alpha^2)b - 4(1 + \alpha)^2 p]bx^2 - 4(1 + \alpha)^2 kq|}} \tag{43}$$

and

$$|a_3| \leq \frac{|\gamma|^2 b^2 x^2}{4(1+\alpha)^2} + \frac{|\gamma||bx|}{6(1+2\alpha)}, \tag{44}$$

and for some $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\gamma||bx|}{6(1+2\alpha)} \quad if \\ |\mu - 1| \leq \frac{|[\gamma(1+2\alpha-2\alpha^2)b - 2(1+\alpha)^2 p]bx^2 - 2(1+\alpha)^2 kq|}{3|\gamma|(1+2\alpha)b^2x^2} \\ \frac{|\gamma|^2|bx|^3|\mu-1|}{|[\gamma(2+4\alpha-4\alpha^2)b - 4(1+\alpha)^2 p]bx^2 - 4(1+\alpha)^2 kq|} \quad if \\ |\mu - 1| \geq \frac{|[\gamma(1+2\alpha-2\alpha^2)b - 2(1+\alpha)^2 p]bx^2 - 2(1+\alpha)^2 kq|}{3|\gamma|(1+2\alpha)b^2x^2}. \end{cases} \tag{45}$$

Proof. Let $f \in \mathcal{N}_\Sigma(\alpha,\gamma,x)$, $0 \leq \alpha \leq 1, \gamma \in \mathbb{C}\backslash\{0\}$ and $x \in \mathbb{R}$. Then there are two holomorphic function $v, u: \Delta \to \Delta$ indicated by

$$v(z) = t_1 z + t_2 z^2 + t_3 z^3 + \cdots \qquad (z \in \Delta)$$

and

$$u(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots \qquad (w \in \Delta),$$

with $v(0) = u(0) = 0, |v(z)| < 1$ *and* $|u(w)| < 1,\ z, w \in \Delta$, such that

$$1 + \frac{1}{\gamma}\left[\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)} - 1\right] \prec \Omega(x, v(z)) + 1 - k$$

and

$$1 + \frac{1}{\gamma}\left[\frac{\alpha w^3 g'''(w) + (1+2\alpha)w^2 g''(w) + wg'(w)}{\alpha w^2 g''(w) + wg'(w)} - 1\right] \prec \Omega\big(x, u(w)\big) + 1 - k.$$

Or, in equivalent way,

$$1 + \frac{1}{\gamma}\left[\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)} - 1\right]$$
$$= 1 + h_1(x) - k + h_2(x)v(z) + h_3(x)[v(z)]^2 + \cdots \tag{46}$$

and

$$1 + \frac{1}{\gamma}\left[\frac{\alpha w^3 g'''(w) + (1+2\alpha)w^2 g''(w) + wg'(w)}{\alpha w^2 g''(w) + wg'(w)} - 1\right]$$
$$= 1 + h_1(x) - k + h_2(x)u(w) + h_3(x)[u(w)]^2 + \cdots. \tag{47}$$

From (46) and (47), we get

$$1 + \frac{1}{\gamma}\left[\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)} - 1\right]$$
$$= 1 + h_2(x)t_1 z + [h_2(x)t_2 + h_3(x)t_1^2]z^2 + \cdots \tag{48}$$

and

$$1 + \frac{1}{\gamma}\left[\frac{\alpha w^3 g'''(w) + (1+2\alpha)w^2 g''(w) + wg'(w)}{\alpha w^2 g''(w) + wg'(w)} - 1\right]$$
$$= 1 + h_2(x)s_1 w + [h_2(x)s_2 + h_3(x)s_1^2]w^2 + \cdots. \tag{49}$$

Notice that if

$$|v(z)| = |t_1 z + t_2 z^2 + t_3 z^3 + \cdots| < 1 \qquad (z \in \Delta)$$

and

$$|u(w)| = |s_1 w + s_2 w^2 + s_3 w^3 + \cdots| < 1 \qquad (w \in \Delta),$$

then

$$|t_i| \leq 1 \; and \; |s_i| \leq 1 \quad (\, i \in \mathbb{N}\,).$$

It follows from (48) and (49) that

$$\frac{2(1+\alpha)}{\gamma} a_2 = h_2(x) t_1, \tag{50}$$

$$\frac{6(1+2\alpha)}{\gamma} a_3 - \frac{4(1+\alpha)^2}{\gamma} a_2^2 = h_2(x) t_2 + h_3(x) t_1^2, \tag{51}$$

$$-\frac{2(1+\alpha)}{\gamma} a_2 = h_2(x) s_1 \tag{52}$$

and

$$\frac{6(1+2\alpha)}{\gamma}(2a_2^2 - a_3) - \frac{4(1+\alpha)^2}{\gamma} a_2^2 = h_2(x) s_2 + h_3(x) s_1^2. \tag{53}$$

From (50) and (52), we find that

$$t_1 = -s_1 \tag{54}$$

and

$$\frac{8(1+\alpha)^2}{\gamma^2} a_2^2 = [h_2(x)]^2 (t_1^2 + s_1^2). \tag{55}$$

If we add (51) to (53), we get

$$\frac{(4 + 8\alpha - 8\alpha^2)}{\gamma} a_2^2 = h_2(x)(t_2 + s_2) + h_3(x)(t_1^2 + s_1^2). \tag{56}$$

By using (55) in equation (56), we have

$$\left[\frac{(4 + 8\alpha - 8\alpha^2)}{\gamma} - \frac{8(1+\alpha)^2 h_3(x)}{\gamma^2 [h_2(x)]^2}\right] a_2^2 = h_2(x)(t_2 + s_2), \tag{57}$$

which yields

$$|a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|[\gamma(2 + 4\alpha - 4\alpha^2)b - 4(1+\alpha)^2 p]bx^2 - 4(1+\alpha)^2 kq|}}.$$

Next, if we deduct (53) from (51), we get

$$\frac{12(1+2\alpha)}{\gamma}(a_3 - a_2^2) = h_2(x)(t_2 - s_2) + h_3(x)(t_1^2 - s_1^2). \tag{58}$$

In view of (54) and (55), equation (58) becomes

$$a_3 = \frac{\gamma^2 [h_2(x)]^2 (t_1^2 + s_1^2)}{8(1+\alpha)^2} + \frac{\gamma h_2(x)(t_2 - s_2)}{12(1+2\alpha)}.$$

Now, with the help of equation (3), we conclude that

$$|a_3| \leq \frac{|\gamma|^2 b^2 x^2}{4(1+\alpha)^2} + \frac{|\gamma||bx|}{6(1+2\alpha)}.$$

Finally, by using (57) and (58) for some $\mu \in \mathbb{R}$, we get

$$a_3 - \mu a_2^2 = \frac{\gamma h_2(x)(t_2 - s_2)}{12(1+2\alpha)} + \frac{\gamma^2 [h_2(x)]^3 (1 - \mu)(t_2 + s_2)}{\gamma(4 + 8\alpha - 8\alpha^2)[h_2(x)]^2 - 8(1+\alpha)^2 h_3(x)}$$

$$= \frac{\gamma h_2(x)}{2}\left[\left(\Psi(\mu,x) + \frac{1}{6(1+2\alpha)}\right)t_2 + \left(\Psi(\mu,x) - \frac{1}{6(1+2\alpha)}\right)s_2\right],$$

where

$$\Psi(\mu,x) = \frac{\gamma[h_2(x)]^2(1-\mu)}{\gamma(2+4\alpha-4\alpha^2)[h_2(x)]^2 - 4(1+\alpha)^2 h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{|\gamma||h_2(x)|}{6(1+2\alpha)} & if \quad 0 \le |\Psi(\mu,x)| \le \dfrac{1}{6(1+2\alpha)} \\ |\gamma||h_2(x)||\Psi(\mu,x)| & if \ |\Psi(\mu,x)| \ge \dfrac{1}{6(1+2\alpha)} \end{cases}$$

and with respect to (3), it evidently completes the proof of the theorem (3).

References

1. Abirami C, Magesh N, Yamini J. Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials. Abstract and Applied Analysis. 2020; Article ID 7391058, 8 pages.
2. Alamoush AG. On a subclass of bi-univalent functions associated to Horadam polynomials. Int. J. Open Problems Complex Analysis. 2020; 12(1): 58-65.
3. Bulut S. A new comprehensive subclass of analytic bi-close-to-convex function. Turk. J. Math. 2019; 43: 1414-1424.
4. Duren PL. Univalent Functions. Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
5. Frasin BA, Aouf MK. New subclasses of bi-univalent functions. Appl. Math. Lett. 2011; 24: 1569-1573.
6. Güney HÖ, Murugusundaramoorthy G, Sokól. Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers. Acta Univ. Sapient. Math. 2018; 10(1):70-84.
7. Horadam AF, Mahon JM. Pell and Pell-Lucas polynomials. Fibonacci Quart. 1985; 23: 7-20.
8. Horzum T, Kocer EG. On some properties of Horadam polynomials. Int. Math. Forum. 2009; 4: 1243-1252.
9. Srivastava HM, Altinkaya Ş, Yalçin S. Certain subclasses of bi-univalent functions associated with the Horadam polynomials. Iran. J. Sci. Technol. Trans. A Sci. 2019; 43: 1873-1879.
10. Srivastava HM, Mishra AK, Gochhayat P. Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. 2010; 23: 1188–1192.
11. Wanas AK, Yalçin S. Horadam polynomials and their applications to new family of bi-univalent functions with respect to symmetric conjugate points. Proyecciones Journal of Math. 2021; 40(1): 107-116.