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Golden Ratio

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Abstract

This paper introduces the unique geometric features of 1:2:$\sqrt{5}$ right triangle, which is observed to be the quintessential form of Golden Ratio (φ). The 1:2:$\sqrt{5}$ triangle, with all its peculiar geometric attributes described herein, turns out to be the real 'Golden Ratio Triangle' in every sense of the term. This special right triangle also reveals the fundamental Pi:Phi (π:φ) correlation, in terms of precise geometric ratios, with an extreme level of precision. Further, this 1:2:$\sqrt{5}$ triangle is found to have a classical geometric relationship with 3-4-5 Pythagorean triple. The perfect complementary relationship between1:2:$\sqrt{5}$ triangle and 3-4-5 triangle not only unveils several new aspects of Golden Ratio, but it also imparts the most accurate π:φ correlation, which is firmly premised upon the classical geometric principles. Moreover, this paper introduces the concept of special right triangles; those provide the generalised geometric substantiation of all Metallic Means.

Keywords: Golden Ratio, Fibonacci sequence, Pi (π), Phi(φ), Metallic Means, Golden Ratio and Pi Relationship, Golden triangle, Divine Proportion, Golden Mean, Right Triangle, 1:2:$\sqrt{5}$ Triangle, 3-4-5 Triangle, Pythagoras Theorem

Introduction

The prime objective of this work is to introduce and elaborate the characteristic geometry of 1:2:$\sqrt{5}$ right triangle, which is resplendent with Golden Ratio(φ), embedded in its every aspect. The remarkable expression of Golden Ratio in every geometric feature of 1:2:$\sqrt{5}$ triangle, including all its angles and side lengths, its 'Incenter-Excenters Orthocentric system', its Gergonne and Nagel triangles, and also the Nobbs points and the Gergonne line, various triangle centers as well as the Incircle of 1:2:$\sqrt{5}$ triangle, make this triangle the real Golden Ratio Trigon in geometry. Moreover, this special right triangle, with Golden Proportions in all its manifestations, is observed to have a classical geometric intimacy with Regular Pentagon, as described in this paper.

This special right triangle, with its catheti in ratio 1:2, not only manifests as the ultimate geometric substantiation of Golden Ratio, but it also reveals the extremely accurate Pi:Phi correlation, based upon precise geometric ratios. Although, Pi is a transcendental number, and the Golden Ratio is an algebraic number, i.e. the solution to the polynomial equation $x^2 - x - 1 = 0$, and even though the Golden Ratio is the "most irrational" of all irrational numbers, this special right angled triangle 1:2:$\sqrt{5}$, with all its distinctive geometric features, enables us to derive the precise correlation between these two important constants, viz. φ and π.

Moreover, this 1:2:$\sqrt{5}$ triangle is also found to possess a classical geometric relationship with 3-4-5 right angled triangle, which is the first of the primitive Pythagorean triples. The perfect complementary relationship between these two right triangles, with a perfect synergy between their corresponding sides and angles, conjointly reveal the most important aspects of Golden Ratio, in a phenomenal way described in this paper.

Furthermore, this paper puts forward the concept of the special right triangles those represent all Metallic Means.

Hence, the aim of this paper is to introduce the classical geometric properties of 1:2:$\sqrt{5}$ triangle, which prove this triangle to be the very origin of the Golden Ratio in geometry. Also, the aim of this paper is to provide generalised geometric substantiation of all Metallic Ratios. Further objective of this work is to highlight the peculiar relationship between two very special right angled triangles, viz. 1:2:$\sqrt{5}$ triangle and 3-4-5 Pythagorean triple, which in turn reveals the august link between nature's two important constants, viz. Pi (π) and the Golden Ratio (φ). The paper presents the author's investigations and findings.

1:2:$\sqrt{5}$ Triangle: The Geometric Expression of Golden Ratio

The right triangle, with its catheti in ratio 1:2, is observed to possess several peculiar geometric features. Noticeably, the three sides of this triangle provide the fractional expression for Golden Ratio: $\frac{1+\sqrt{5}}{2}$; which is the solution to the quadratic equation $x^2 - x - 1 = 0$. Moreover, all three sides of this triangle are the precise expression of Golden Proportion. Touchpoints of the Incircle, which are also the vertices of Gergonne Triangle, divide the sides of 1:2:$\sqrt{5}$ triangle in terms of Golden ratio φ, as shown below.

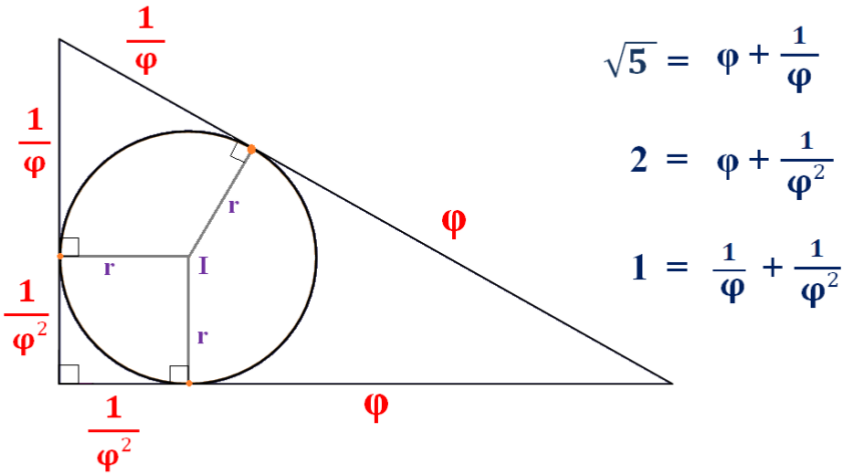


Figure 1: The side lengths of 1:2:$\sqrt{5}$ Triangle in terms of Golden Ratio φ

Similarly, the Extouch points i.e. points of tangency of the triangle's sides with its 3 Excircles, which are also the vertices of Nagel Triangle (PQR in following Figure), similarly divide the triangle sides in terms of Golden ratio, but in exactly inverse manner**.** This can be easily calculated as these extouch-points are the perimeter-splitters of reference triangle.

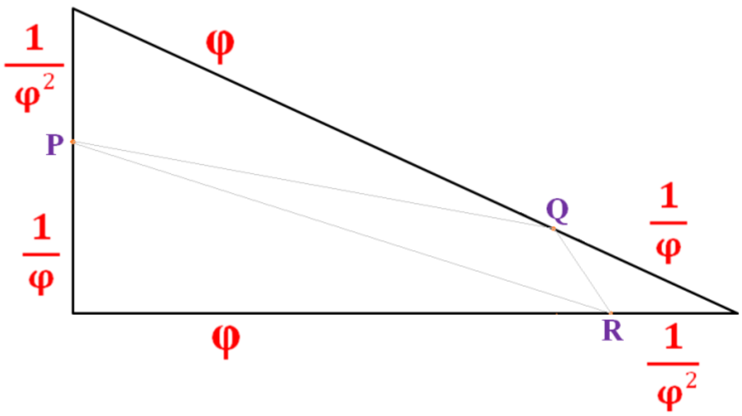
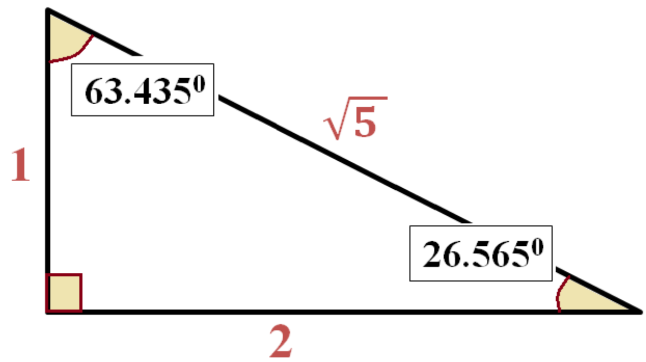


Figure 2: Inverse Division of side lengths by Touchpoints of 3 Excircles

Moreover, the angles corresponding to 1, 2 and $\sqrt{5}$ sides also reflect the precise value of Golden Ratio. The angles opposite to the hypotenuse and the shorter cathetus, viz. 90^0 and "26.565.....degrees" add up to "116.565....degrees". And, the angle opposite to the longer cathetus measures "63.4349.....degrees". Remarkably, these angles **116.565⁰** and **63.4349⁰** are precisely **the angles of the Golden Rhombus**, whose diagonals are in the Golden Ratio, 1: φ.

Furthermore, 63.435^0 is the Supplimantary angle of $2\tan^{-1}\varphi$

And, 26.565^0 is the Complementary angle of $2\tan^{-1}\frac{1}{\varphi}$



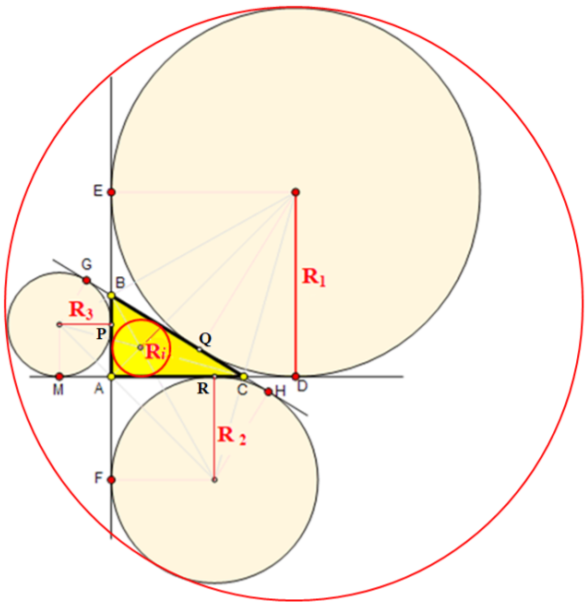
$$63.435^0 = 2\tan^{-1}\frac{1}{\varphi}$$

$$26.565^0 = 2\tan^{-1}\frac{1}{\varphi^3}$$

$$90^0 + 26.565^0 = 116.565^0 = 2\tan^{-1}\varphi$$

Figure 3: The Angles of 1:2:√5 Triangle in terms of Golden Ratio φ

The Golden Ratio is ubiquitous in this 1:2:√5 Triangle ABC, as shown below in **Figure 4**.



Semiperimeter (S) of 1:2:√5 Triangle ABC = φ^2

Inradius (Ri) $= \frac{1}{\varphi^2}$

Exradii R_1, R_2, R_3 are φ^2, φ & $\frac{1}{\varphi}$: (2.618….; 1.618….; 0.618….)

Distances of Vertices from Tangency Points:

$AD = AE = BF = BH = CM = CG = \varphi^2$

$AF = AR = BE = BQ = \varphi$

$AM = AP = CQ = CD = \frac{1}{\varphi}$

$BG = BP = CR = CH = \frac{1}{\varphi^2}$

Radius of Apollonius Circle, which is tangent to all 3 Excircles (Shown in Red here); $R_A = \frac{7}{4}\varphi^2$

Figure 4: Golden Ratio embedded in all aspects of 1:2:√5 Triangle

Also, the angles between three angle bisectors of 1:2:√5 Triangle, the angles between its angle bisectors and the opposite sides, angles between 3 Inradii as well as the angles between Inradii and angle bisectors, all are well expressed in terms of the Golden Ratio φ. The Golden Ratio also manifests in the division of the area of this 1:2:√5 triangle by its 3 inradii perpendicular to triangle sides, and also in the division of triangle area by 3 angle bisectors. Moreover, applying the angle bisector theorem, three angle bisectors are also found to divide the triangle sides in a peculiar manner, as shown below in **Figure 5**.

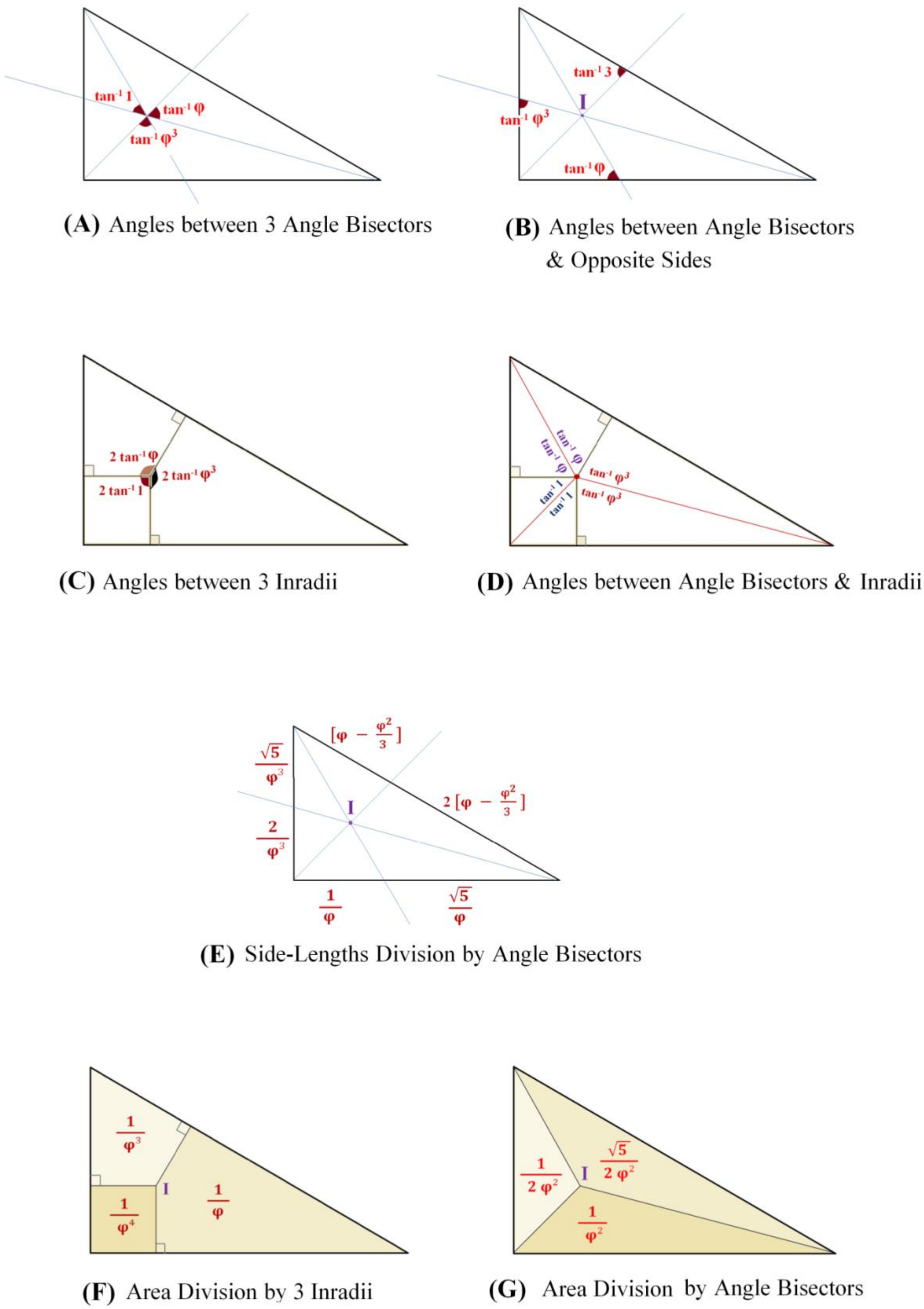
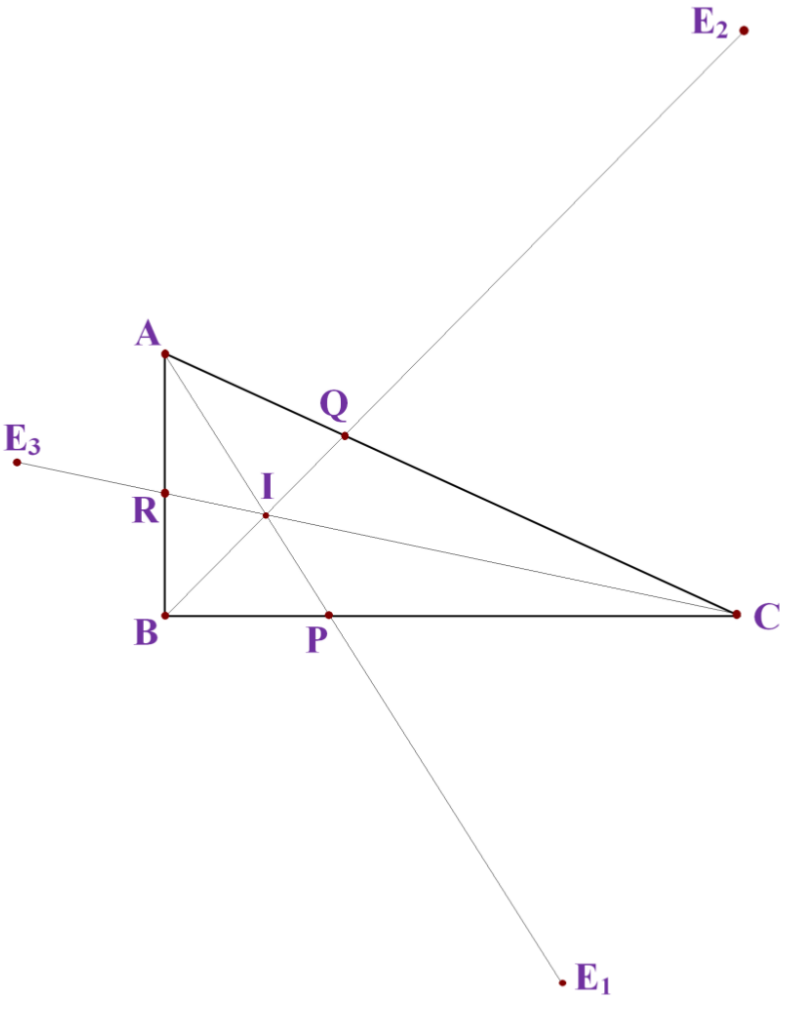


Figure 5: (A) Angles between 3 Angle Bisectors at the Incenter I, **(B)** Angles between angle bisector & opposite side, **(C)** Angles between 3 Inradii perpendicular to triangle sides, **(D)** Angles between Inradii and Angle Bisectors, **(E)** Division of side-lengths of triangle by Angle Bisectors, **(F)** Division of Triangle Area by Three Inradii, **(G)** Division of Triangle Area by Three 'Vertices to Incenter' segments.

Moreover, consider the 1:2:$\sqrt{5}$ Triangle ABC in **Figure 6**, point I being its Incenter X(1) and E_1, E_2 & E_3 being the Excenters. This **Incenter-Excenters Orthocentric System of 1:2:$\sqrt{5}$ Triangle** is simply the geometric utterance of Golden Ratio. Before the illustration in following Figure, it must be mentioned here that the angles **18^0, 36^0, 54^0, 72^0** and **108^0** are the angles associated with the regular pentagon, pentagram, and the so called golden ratio triangle and golden gnomon, the geometric shapes which are full of Golden Ratio. And also, the Golden Ratio can be expressed in terms of the trigonometric ratios of these angles, such as;

$$\varphi = 2\cos 36 \text{ or } 2\sin 54 = \frac{1}{2\cos 72} \text{ or } \frac{1}{2\sin 18} = \frac{\cos 18}{\sin 36} \text{ or } \frac{\sin 72}{\cos 54}$$

Now, consider the 1:2:$\sqrt{5}$ Triangle ABC in following Figure, its Incenter and Excenters, its angle bisectors, and the Golden Geometry embedded therein;



$$AE_1 = \tan 72$$

$$AI = \tan 36$$

$$IE_1 = 4 \sin 36$$

$$BE_2 = \sqrt{2}\,\varphi^2$$

$$BI = \frac{\sqrt{2}}{\varphi^2}$$

$$IE_2 = \sqrt{2}\,\left(\varphi + \frac{1}{\varphi}\right)$$

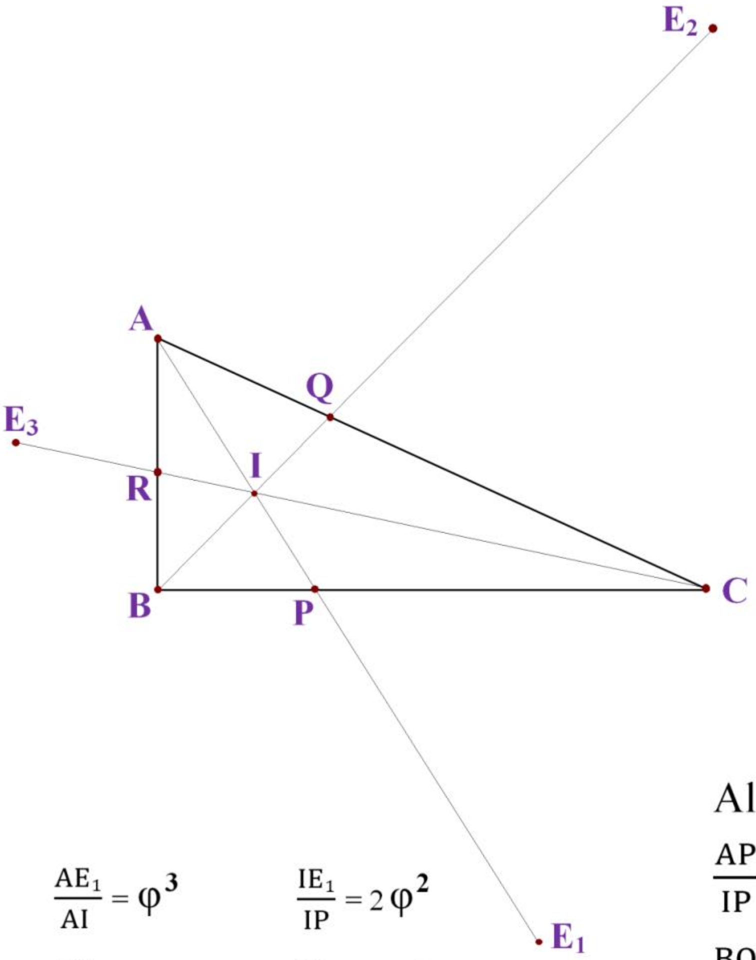
$$CE_3 = 2\sqrt{2}\,\cos 18$$

$$CI = 2\sqrt{2}\,\sin 36$$

$$IE_3 = \sqrt{2}\,\tan 36$$

Figure 6: The Golden Ratio in all segments of Angle Bisectors

Also, the Incenter and the Excenters being harmonic conjugates, the internal as well as the external divisions are in form of $\frac{a+b}{c}$; as shown below.



$$\frac{AI}{IP} = \frac{AE_1}{PE_1} = \varphi$$

$$\frac{BI}{IQ} = \frac{BE_2}{QE_2} = \varphi - \frac{1}{\sqrt{5}\,\varphi}$$

$$\frac{CI}{IR} = \frac{CE_3}{RE_3} = \varphi^3$$

Moreover,

$$\frac{AP}{AI} = \varphi$$

$$\frac{BQ}{BI} = \frac{2}{3}\varphi^2$$

$$\frac{CR}{CI} = \frac{2}{\varphi}$$

$$\frac{AE_1}{AI} = \varphi^3 \qquad \frac{IE_1}{IP} = 2\varphi^2$$

$$\frac{BE_2}{BI} = \varphi^4 \qquad \frac{IE_2}{IQ} = 3\varphi^2$$

$$\frac{CE_3}{CI} = \varphi \qquad \frac{IE_3}{IR} = \varphi^2$$

Also,

$$\frac{AP}{IP} = \varphi^2$$

$$\frac{BQ}{IQ} = \frac{2}{\sqrt{5}}\varphi^2$$

$$\frac{CR}{IR} = 2\varphi^2$$

Also,

$$\frac{AE_1}{AP} = \varphi^2$$

$$\frac{BE_2}{BQ} = 1.5\,\varphi^2$$

$$\frac{CE_3}{CR} = 0.5\,\varphi^2$$

$$\frac{PE_1}{IP} = \varphi^3 \qquad \frac{AE_1}{IE_1} = \frac{\varphi^2}{2} \qquad \frac{AP}{PE_1} = \frac{1}{\varphi} \qquad \frac{AE_1}{IP} = \varphi^4$$

$$\frac{QE_2}{IQ} = \varphi^4 \qquad \frac{BE_2}{IE_2} = \frac{\varphi^2}{\sqrt{5}} \qquad \frac{BQ}{QE_2} = \frac{2}{\sqrt{5}\,\varphi^2} \qquad \frac{BE_2}{IQ} = \frac{3}{\sqrt{5}}\varphi^4$$

$$\frac{RE_3}{IR} = \varphi \qquad \frac{CE_3}{IE_3} = \varphi^2 \qquad \frac{CR}{RE_3} = 2\varphi \qquad \frac{CE_3}{IR} = \varphi^4$$

Figure 7: The Golden Geometry of Incenter and Excenters of the 1:2:$\sqrt{5}$ triangle.

Now, consider the triangle formed by three Excenters $\Delta E_1E_2E_3$, for which the reference triangle ΔABC is Pedal Triangle (or Orthic Triangle). The side lengths of this Excenters-Triangle are divided by the vertices A, B and C of reference triangle in a characteristic manner.

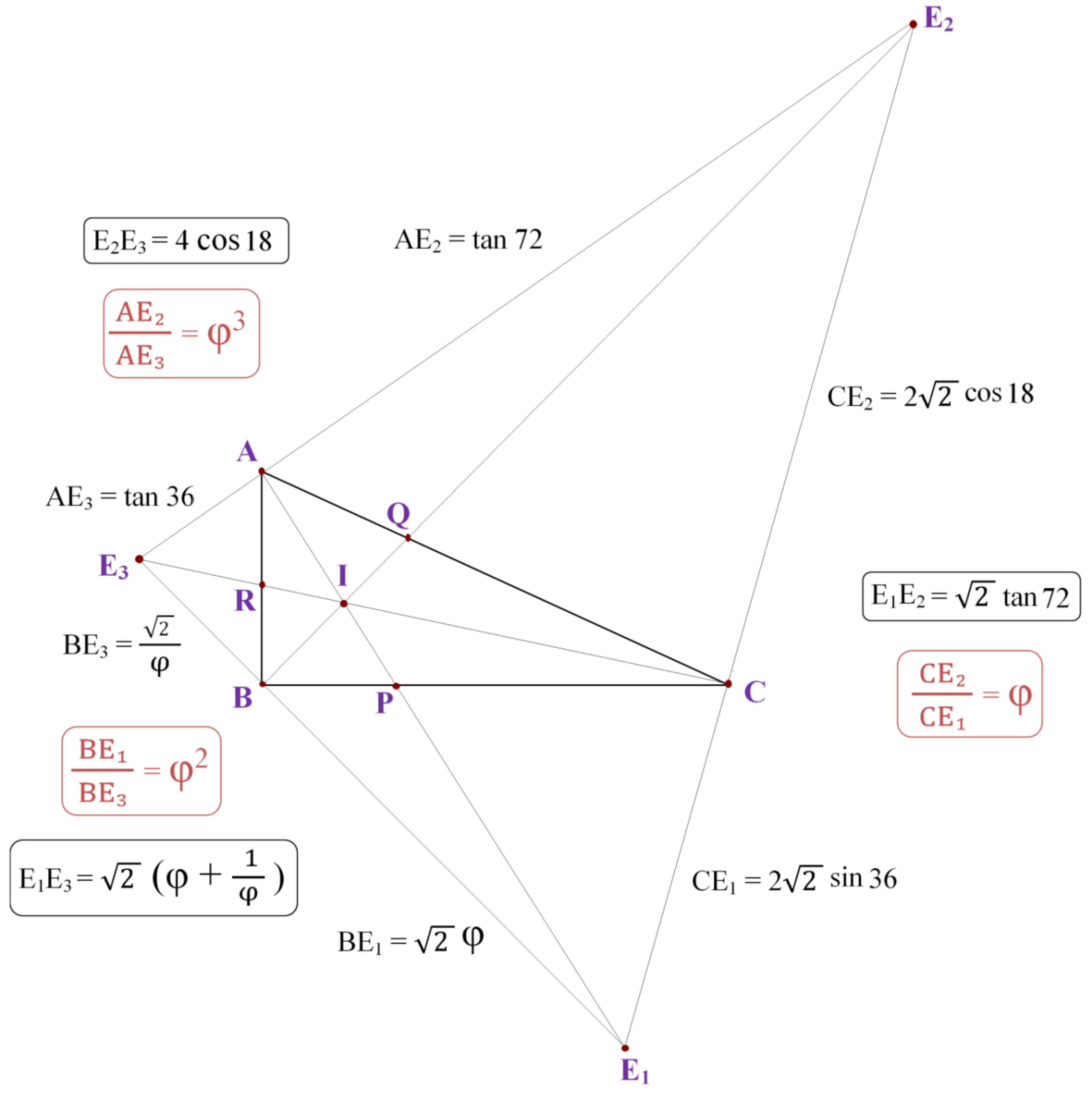


Figure 8: Excenters Triangle & the Golden Ratio

Also, all angles in such geometric formation are the precise expressions of Golden Ratio.

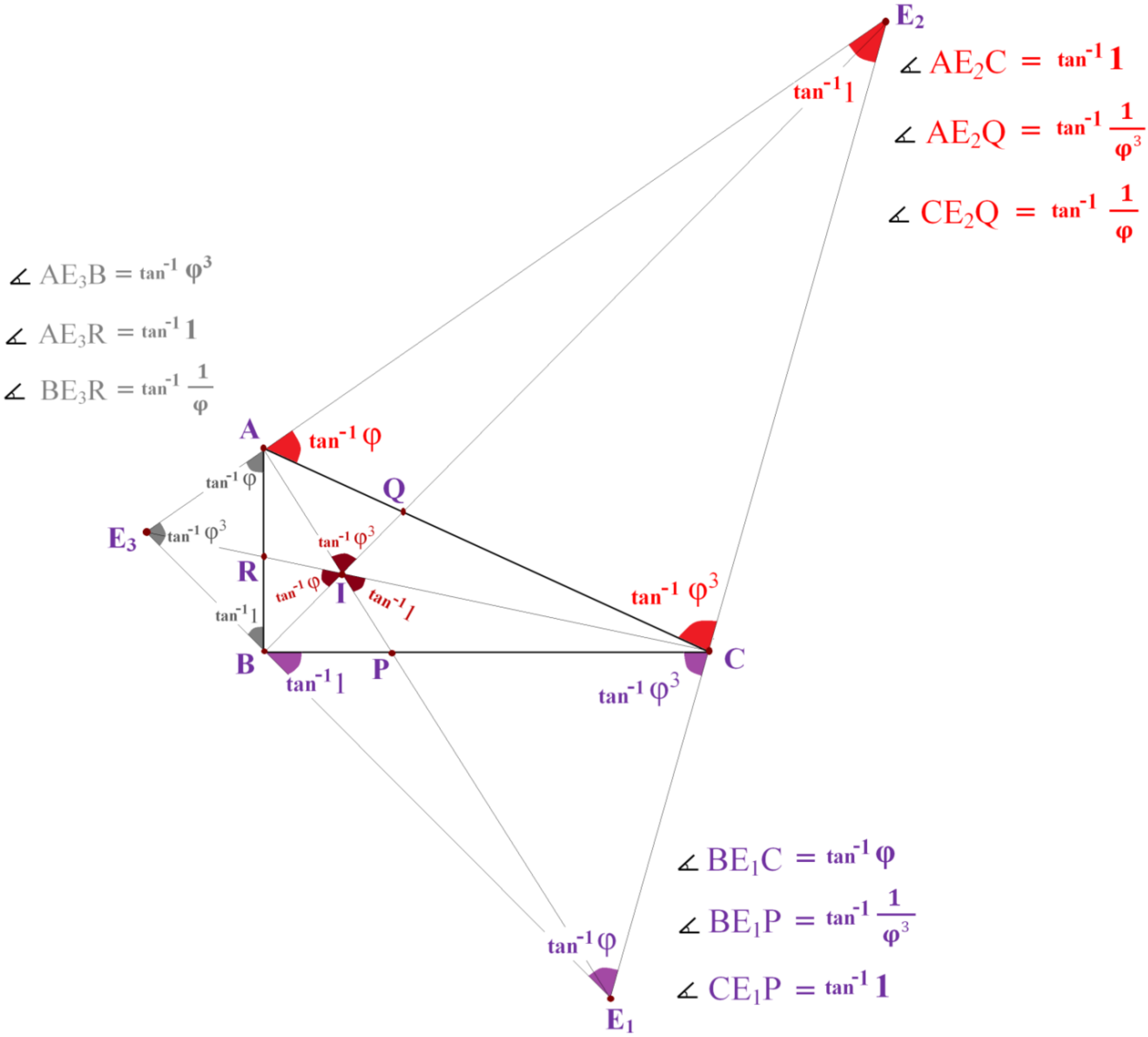


Figure 9: All angles in terms of Golden Ratio.

Moreover, the distances between the vertices of 1:2:$\sqrt{5}$ reference triangle and its Excenters **E_1**, **E_2** and **E_3** exhibit specific proportions, in terms of φ

$AE_1 : AE_2 : AE_3 = \varphi^3 : \varphi^3 : 1$

$BE_1 : BE_2 : BE_3 = \varphi^2 : \varphi^3 : 1$

$CE_1 : CE_2 : CE_3 = 1 : \varphi : \varphi$

Also, the Golden Ratio is reflected in the areas of various triangles formed in **Figure 8**.

$\Delta BE_1C = \varphi$

$\Delta AE_2C = \frac{\sqrt{5}}{2}\varphi^2$

$\Delta AE_3B = \frac{1}{2\varphi}$

And, the area of the Excenters Triangle as a whole: **$\Delta E_1E_2E_3$** = $\sqrt{5}\ \boldsymbol{\varphi}^2$

Noticeably, in 1:2:$\sqrt{5}$ reference triangle, the area of both, its Gergonne Triangle (Intouch Triangle) as well as its Nagel Triangle (Extouch Triangle) is $\dfrac{1}{\sqrt{5}\ \varphi^2}$

Moreover, in 1:2:$\sqrt{5}$ triangle, a specific proportion is observed between the side lengths of the its Gergonne Triangle and the side lengths of the triangle formed by its 3 Excenters;

$$\frac{\textbf{Side Length of } E_1E_2E_3 \textbf{ Triangle}}{\textbf{Corresponding Side Length of Gergonne Triangle}} = \sqrt{5}\ \varphi^2$$

Golden Ratio is also reflected in the Gergonne and Nagel Triangles of the 1:2:$\sqrt{5}$ triangle, as demonstrated in **Figure 10** below. ΔABC is the 1:2:$\sqrt{5}$ reference triangle, and, ΔPQR and ΔLMN are its Gergonne triangle (Intouch triangle) and Nagel triangle (Extouch triangle), respectively.

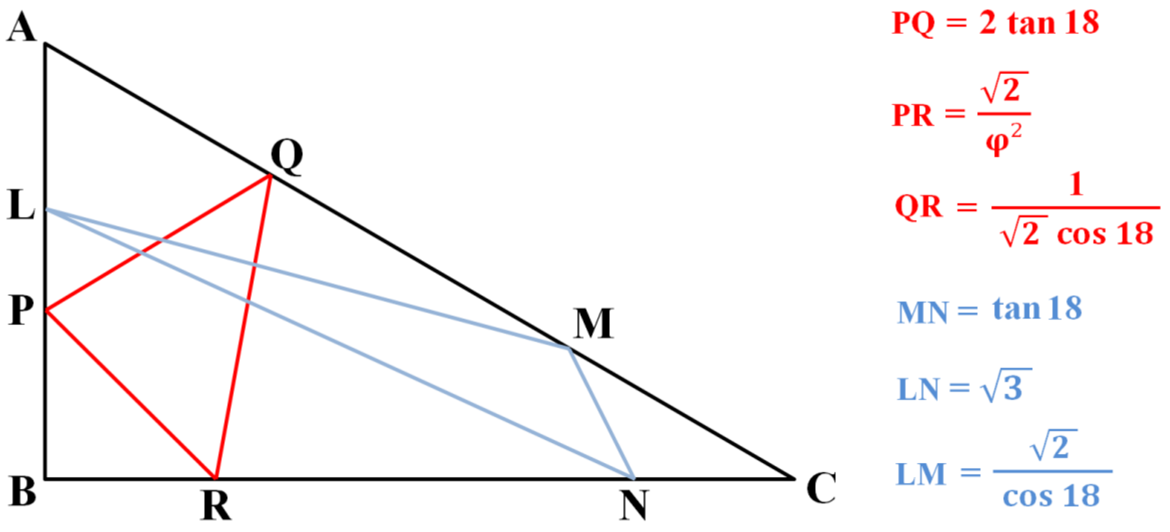


Figure 10: Gergonne triangle and Nagel triangle of the 1:2:$\sqrt{5}$ triangle.

Moreover, the Golden Proportions between Gergonne and Nagel triangle is also observed, such as;

$$\frac{(QR)^2}{(PQ)(MN)} = \boldsymbol{\varphi}^2\ ; \quad \frac{(QR)(LM)}{(PQ)^2} = \boldsymbol{\varphi}^2 \quad \text{and} \quad \frac{QR}{MN} = \frac{LM}{PQ} = \sqrt{2}\ \boldsymbol{\varphi}, \text{ etc.}$$

The **Gergonne triangle** of 1:2:$\sqrt{5}$ triangle exhibits Golden Ratio embedded in its angles and sides. In **Figure 11** below, **ΔPQR** is the **Gergonne triangle** of ΔABC, and **Ge** is the **Gergonne point**: Kimberling Center **X(7)**.

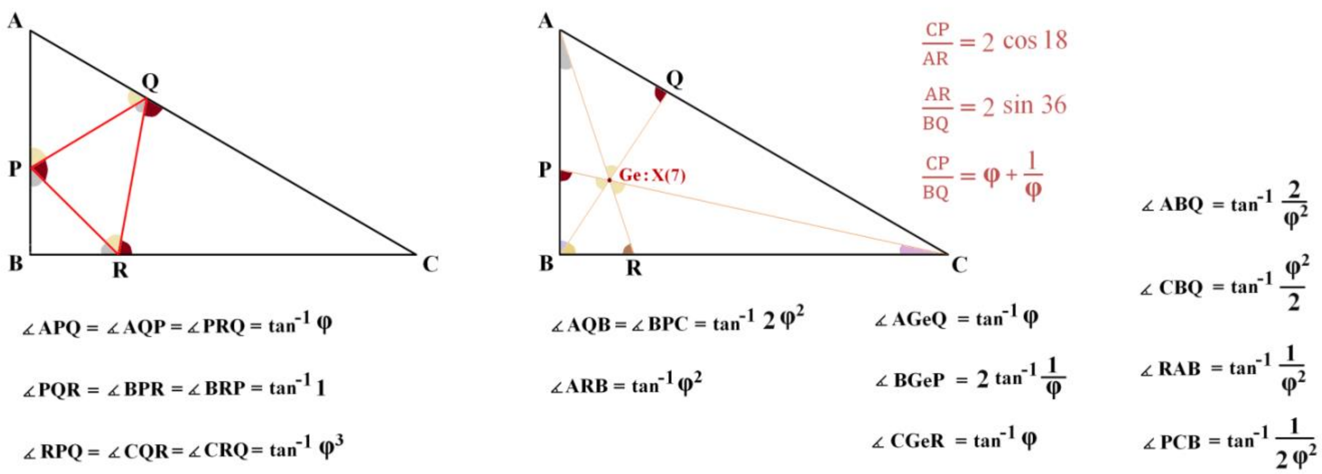


Figure11: Golden Ratio in Gergonne triangle of the 1:2:$\sqrt{5}$ triangle.

Similarly, the **Nagel triangle** of 1:2:$\sqrt{5}$ triangle is also full of Golden Proportions, as shown below in **Figure 12**, the **ΔLMN** here is **Nagel triangle** of ΔABC, and **Ng** is the **Nagel point**: Kimberling Center **X(8)**.

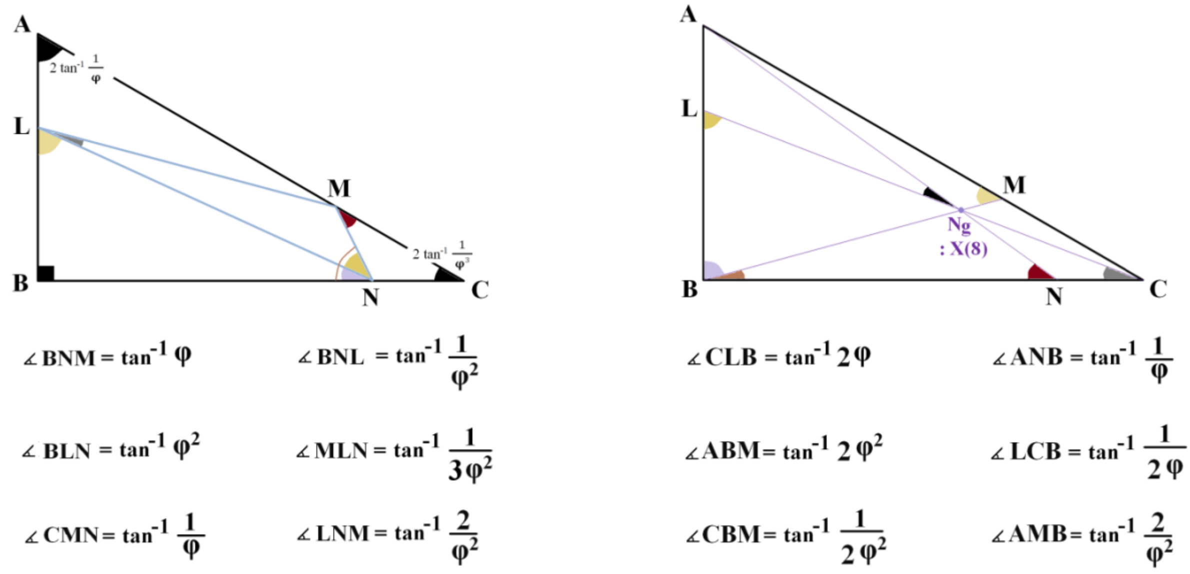


Figure12: Golden Ratio in Nagel triangle of the 1:2:$\sqrt{5}$ triangle.

It is noteworthy here, in above Figure 12, the angle BNgN = 137.47^0, which is remarkably close to the Golden Angle 137.5^0, however its mention in the Figure has been deliberately avoided, since only precise values and accurate numerical results are being illustrated in this work.

Furthermore, Golden Ratio is also expressed as the trilinear coordinates of various triangle centers of 1:2:$\sqrt{5}$ triangle; for example;

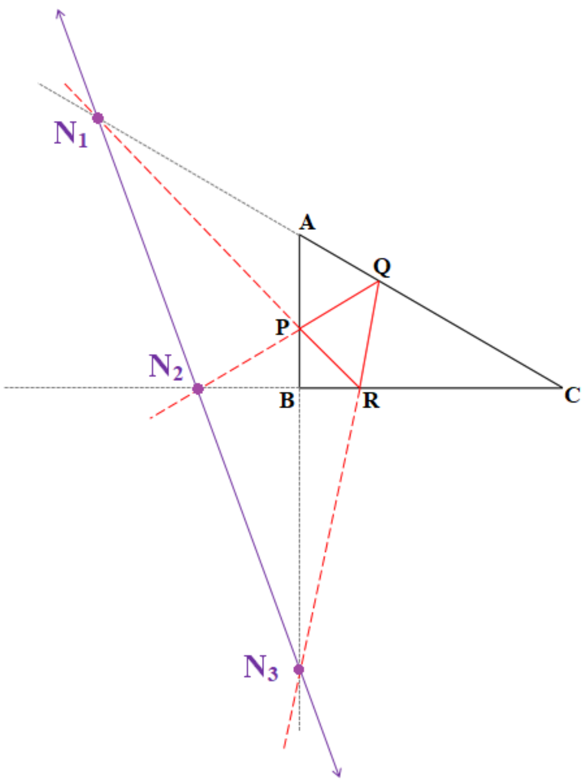
Trilinear coordinates for the Gergonne point X(7) are: $\varphi^2 : \frac{\sqrt{5}\,\varphi}{2} : \frac{\sqrt{5}}{\varphi}$

Trilinear coordinates for the Nagel point X(8) are: $2\,\varphi : \frac{1}{\varphi} : \frac{2}{\sqrt{5}\,\varphi^2}$

Barycentric coordinates for the Nagel point: $\varphi : \frac{1}{\varphi} : \frac{1}{\varphi^2}$

Similarly, the mittenpunkt X(9) has trilinear coordinates: $2\varphi : \frac{2}{\varphi} : \frac{2}{\varphi^2}$

The Golden Ratio is also conspicuous in the **Gergonne Line** and the **Nobbs' Points** of 1:2:$\sqrt{5}$ triangle. In **Figure 13** below, ΔPQR is the Gergonne triangle and Line N_1N_3 is the Gergonne line for 1:2:$\sqrt{5}$ Triangle ABC, and N_1, N_2 and N_3 are the Nobbs' points. Various angles formed therein reflect the precise value of Golden Ratio, as shown here.



$$\measuredangle BPN_2 = \tan^{-1}\varphi \qquad \measuredangle N_1PN_2 = \tan^{-1}\varphi^3$$

$$\measuredangle PN_2B = \tan^{-1}\frac{1}{\varphi} \qquad \measuredangle PN_1N_2 = \tan^{-1}\frac{1}{\sqrt{5}}$$

$$\measuredangle BN_2N_3 = \tan^{-1}\varphi^2 \qquad \measuredangle N_1N_2P = \tan^{-1}2\varphi^2$$

$$\measuredangle BN_3N_2 = \tan^{-1}\frac{1}{\varphi^2} \qquad \measuredangle PAN_1 = 2\tan^{-1}\varphi$$

$$\measuredangle BN_3R = \tan^{-1}\frac{1}{\varphi^3} \qquad \measuredangle PN_1A = \tan^{-1}\frac{1}{3}$$

$$\measuredangle BRN_3 = \tan^{-1}\varphi^3 \qquad \measuredangle APN_1 = \tan^{-1}1$$

Figure 13 : Gergonne Line and Nobbs' Points of the 1:2:$\sqrt{5}$ Triangle ABC

In above Figure, Distance between the Nobbs' points:

$N_1N_2 = N_2N_3 = \sqrt{3}$

Distance between the vertices of reference triangle ABC and the Nobbs' points:

$N_3A = \varphi^2$ $\qquad$ $N_2C = \varphi^2$

$N_3B = \varphi$ $\qquad$ $N_2B = \frac{1}{\varphi}$

$N_1A = \frac{\sqrt{5}}{\varphi}$ $\qquad$ $N_2A = 2\sin 36$

$N_1C = \sqrt{5}\,\varphi$

Distance between Nobbs' points and the vertices of Gergonne triangle PQR:

$N_1P = \frac{\sqrt{2}}{\cos 36}$ $\qquad$ $N_2P = \tan 36$ $\qquad$ $N_3P = \varphi + \frac{1}{\varphi^2}$

$N_1Q = \varphi + \frac{1}{\varphi^2}$ $\qquad$ $N_2Q = \tan 54$ $\qquad$ $N_3Q = \frac{2}{\sqrt{2}\sin 36}$

$N_1R = \sqrt{2}\,\varphi$ $\qquad$ $N_2R = \frac{1}{\varphi} + \frac{1}{\varphi^2}$ $\qquad$ $N_3R = 2\sqrt{2}\sin 36$

The ratio between the sides of Gergonne triangle and their extension up to Nobbs' Points:

$\frac{N_1P}{PR} = 2\varphi$; $\qquad$ $\frac{N_2P}{PQ} = \frac{\sqrt{5}\cos 36}{\varphi}$; $\qquad$ $\frac{N_3R}{QR} = \varphi + \frac{1}{\varphi}$

Now, most importantly, this **1:2:$\sqrt{5}$ "Golden Triangle"** is also found to be closely associated with the **Regular Pentagon**, which is full of **Golden Ratio** embedded in its various lengths proportions. The 1:2:$\sqrt{5}$ triangle not only provides a novel method for construction of regular pentagon, but the **1, 2** & $\sqrt{5}$ length proportions are invariably observed in the geometry of **Regular Pentagon**. Consider the following **Figure 14,**

triangle ABC being the 1:2:$\sqrt{5}$ triangle, having its Incenter X(1) at point I, and ID being the Inradius perpendicular to Longer Cathetus BC i.e. D is the touchpoint of triangle's Incircle on side BC. Now, if a circle is drawn with centre C and radius R = BC, and the Inradius ID of the triangle is extended to intercept that circle at points P and Q, the length of chord PQ provides the Side of the Regular Pentagon PQRST inscribed in this circle.

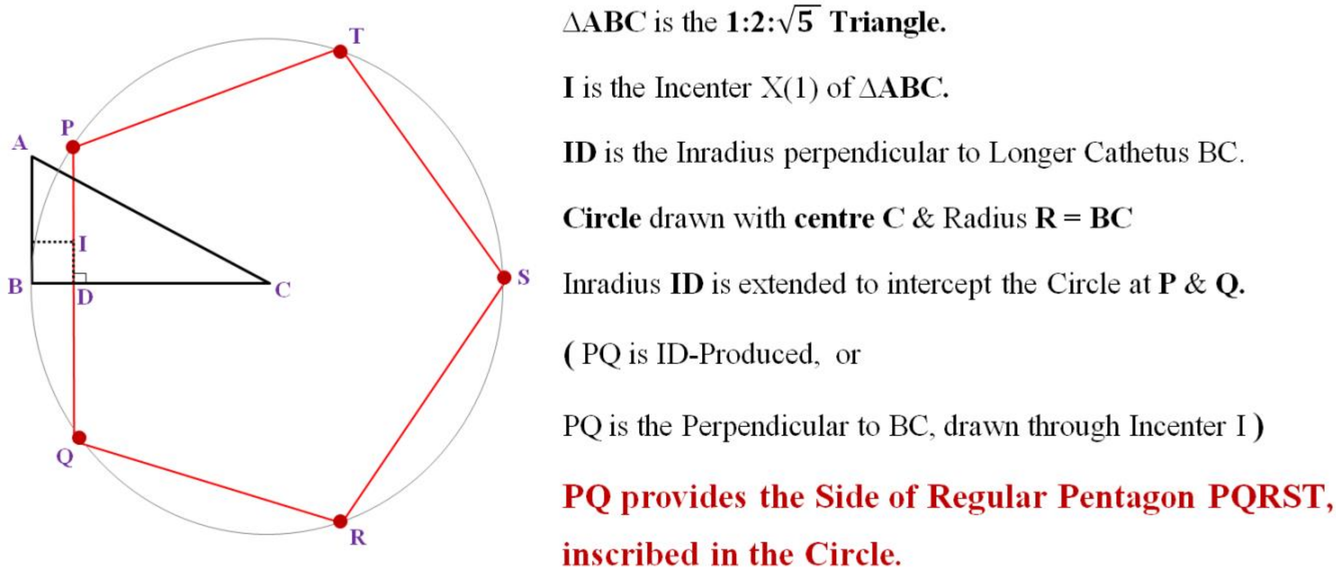


Figure 14: Construction of Regular Pentagon with 1:2:$\sqrt{5}$ Triangle

In above Pentagon PQRST,

If Circumradius of Pentagon $\mathbf{R = BC = 2}$, then,

Inradius or Apothem of pentagon $(\mathbf{a_0}) = \boldsymbol{\varphi}$

Height of Pentagon $(\mathbf{R + a_0}) = \sqrt{5}\,\boldsymbol{\varphi}$

Side of Pentagon $(\mathbf{S}) = \mathbf{4\sin 36} = \mathbf{2\varphi\tan 36} = 2\times\frac{\cos 18}{\cos 36} = \frac{\sqrt{5}}{\cos 18} = 2\sqrt{\frac{\sqrt{5}}{\varphi}} = 2\sqrt{1+\frac{1}{\varphi^2}}$

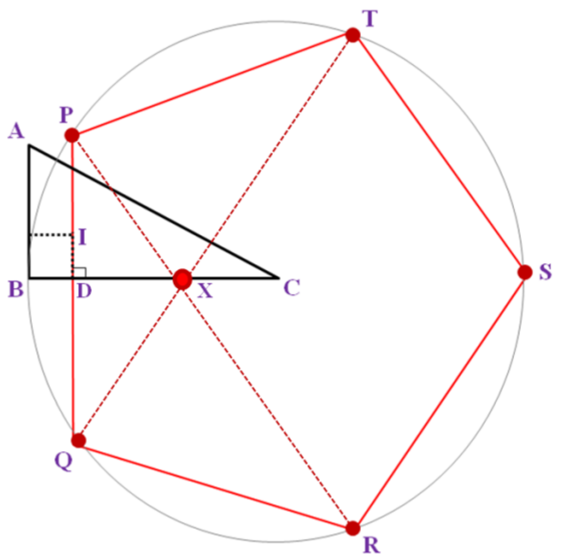
And Diagonal of Pentagon $(\mathbf{D}) = \mathbf{Side \times \varphi} = \mathbf{4\cos 18} = 2\times\frac{\sin 36}{\sin 18}$

Remarkably, the side of this Pentagon **(S)** is exactly equal to the diastance between Incenter **I** and Excenter $\mathbf{E_1}$, in **Figure 8**. And the Diagonal of this Pentagon **(D)** is exactly equal to the diastance between two Excenters, $\mathbf{E_2}$ and $\mathbf{E_3}$, in **Figure 8**.

And, not only the 1:2:$\sqrt{5}$ Triangle provides this distinctive method of constructing Regular Pentagon, but also inversely, the 1, 2 & $\sqrt{5}$ lengths proportions are found to be embedded in the geometry of Regular Pentagon. **The Diagonals and the Side of a Regular Pentagon are observed to divide its Circumadius into 1, 2 & $\sqrt{5}$ proportion.** As illustrated in **Figure 15** below, the regular pentagon PQRST has BC as its circumradius, PR and TQ as its diagonals those intercept BC at point X, and Pentagon's side PQ intercepts BC at point D, it is observed that: **BD : DX : CX = 1 : $\sqrt{5}$: 2**

Moreover, the point X, at which the two diagonals of pentagon intercept the circumradius BC, divides BC in Golden Proportion; $\frac{BC}{BX} = \frac{BX}{CX} = \varphi$

And remarkably, the length **BX** = $\frac{2}{\varphi}$: provides the Side Length of the **Regular Decagon** that can be inscribed in the same circle. It is noteworthy here that the ratio between the Circumradius of a Regular Decagon and its Side is the Golden Ratio.



$$\mathbf{BD} = \frac{1}{\varphi^2}\ ; \qquad \mathbf{CD} = \text{Apothem} = \varphi$$

$$\mathbf{DX} = \frac{\sqrt{5}}{\varphi^2}\ ; \qquad \mathbf{BT} = \mathbf{BR} = 2\varphi$$

$$\mathbf{CX} = \frac{2}{\varphi^2}\ ; \qquad \frac{\mathrm{BC}}{\mathrm{BX}} = \frac{\mathrm{BX}}{\mathrm{CX}} = \varphi$$

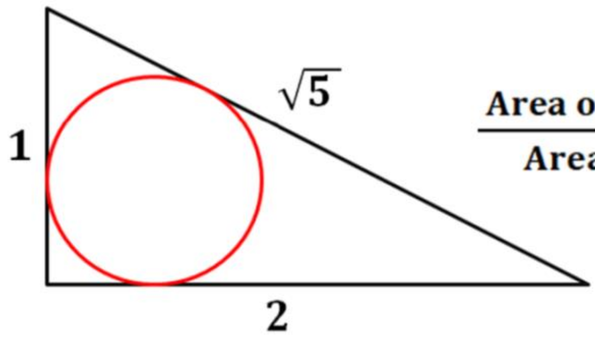
$$\mathbf{BX} = \frac{2}{\varphi} = \text{Side of Regular Decagon Inscribed in the same Circle}$$

Figure 15: The 1:2:√5 Triangle, the Pentagon, the Decagon and the Golden Ratio.

And noticeably, the area enclosed between the Incircle and the Circumcircle of this Pentagon precisely equals $\sqrt{5} \times \frac{\pi}{\varphi}$ **:** the geometric expression of the precise correlation between **Pi (π)** and the **Golden Ratio (φ) !**

Hence, although **Pi(π)** is a **transcendental number** and **Phi(φ)** is an **algebraic number**, i.e. the solution to the polynomial equation x^2-x-1= 0, and even though the Golden Ratio **φ** is the "most irrational" of all irrational numbers, the 1:2:√5 triangle can be exploited to derive precise **π:φ Correlation** by various methods, for instance by incorporation of **Incircle**, that naturally causes advent of **Pi (π)** into the picture, in addition to the **Phi(φ)** that is ubiquitous in this 1:2:√5 triangle.

And thence, the precise correlation between **Pi (π)** and **Golden Ratio (φ)** can be expressed geometrically, as the proportion between the 1:2:√5 Triangle and its Incircle, as shown below.



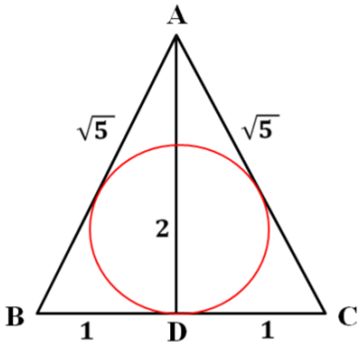
$$\frac{\textbf{Area of 1:2:}\sqrt{5}\textbf{ Triangle}}{\textbf{Area of Its Incircle}} = \frac{\varphi^4}{\pi}$$

Figure 16: The 1:2:√5 triangle, its Incircle, and Pi (π) : Phi (φ) correlation.

As the Inradius of 1:2:√5 triangle is $\frac{1}{\varphi^2}$, extremely precise geometric proportions are observed between the 1:2:√5 Triangle and its Incircle;

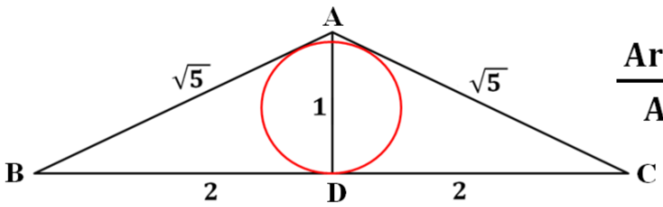
$$\frac{\textbf{Area of 1:2:}\sqrt{5}\textbf{ Triangle}}{\textbf{Area of Its Incircle}} = \frac{\textbf{Perimeter of 1:2:}\sqrt{5}\textbf{ Triangle}}{\textbf{Circumference of Its Incircle}} = \frac{\varphi^4}{\pi}$$

Moreover, if a couple of 1:2:$\sqrt{5}$ Triangles are connected along one of their catheti, the similar precise correlations are observed in the geometric ratios therein. As shown in **Figure 17** below, if two equal sized 1:2:$\sqrt{5}$ triangles, viz. Δ**ABD** and Δ**ACD**, are connected along their: **(A) Common Longer Cathetus**, and **(B) Common Shorter Cathetus,** the exact ratios between areas of the combined triangle Δ**ABC** and that of its Incircle are the very peculiar and extremely precise **π: φ** correlations; and naturally, same are the ratios between Perimeter of the triangle Δ**ABC** and circumference of its Incircle, in each case.



(A) Common Longer Cathetus

$$\frac{\text{Area of Triangle ABC}}{\text{Area of Its Incircle}} = \frac{2\,\varphi^2}{\pi}$$



(B) Common Shorter Cathetus

$$\frac{\text{Area of Triangle ABC}}{\text{Area of Its Incircle}} = \frac{\varphi^6}{2\pi}$$

Figure 17: Two 1:2:$\sqrt{5}$ triangles, with (A) Common Longer Leg, and (B) Common Shorter Leg

Also, in case of two different sized 1:2:$\sqrt{5}$ triangles, having one common cathetus, such that the longer cathetus of one triangle is the shorter cathetus of other triangle;

in such case; $\frac{\text{Area of Combined Triangle}}{\text{Area of Its Incircle}} = \frac{\varphi^4}{\pi}$

Similarly, the ratios between area of 1:2:$\sqrt{5}$ triangle and the areas of its 3 Excircles can also be expressed in terms of precise **π:φ** correlation, since all 3 Exradii are nothing but the different powers of Phi (φ).

Thus, every geometric aspect of 1:2:$\sqrt{5}$ right triangle is the assertion of Golden Ratio. Such golden geometry of the 1:2:$\sqrt{5}$ triangle explicates why the Pythagorean triples derived from Fibonacci series, approach the 1:2:$\sqrt{5}$ triangle's proportions, as the series advances. Pythagorean triples can be formed with the alternate Fibonacci numbers as the hypotenuses, like **5**-4-3, **13**-12-5, **34**-30-16, **89**-80-39, **233**-208-105, **610**-546-272 and so on. And, as such series of **Fibonacci-Pythagorean Triples** advances, the triples so formed, invariably approach 1:2:$\sqrt{5}$ triangle proportions, exactly in the same manner as the ratio between consecutive Fibonacci numbers approaches the Golden Ratio; $\lim_{n\to\infty} \frac{\text{Fn}}{\text{Fn}-1} \cong \varphi$. And hence, it clearly endorses: while **φ** is the Golden Ratio in nature, the 1:2:$\sqrt{5}$ triangle is truly the Golden Trigon in geometry, in every sense of the term.

A Right Angled Triangle for each Metallic Mean

Just as, this 1:2:$\sqrt{5}$ Triangle is the obvious manifestation of Golden Ratio, similar Right Triangles can provide for the geometric substantiation of all Metallic Means. The n^{th} Metallic Mean can be represented by the Right Triangle having catheti 1 and $\frac{2}{n}$. Hence, the right triangle with one of its catheti = 1 may substantiate any

Metallic Mean, having its second cathetus = $\frac{\mathbf{2}}{\mathbf{n}}$, where n = 1 for Golden Ratio, n = 2 for Silver Ratio, n = 3 for Bronze Ratio, and so on.

Such Right Triangle gives the precise value of n^{th} Metallic Mean by the generalised formula:

The n^{th} Metallic Mean $\delta_n = \frac{\text{Cathetus 1 + Hypotenuse}}{\text{Second Cathetus}} = \frac{\text{1 + Hypotenuse}}{2/n}$

For example, the right triangle 1:1: $\sqrt{2}$ represents the Silver Ratio $\delta_2 = \frac{1+\sqrt{2}}{1} = 2.41421356....$

And remarkably, just as $\frac{1}{\varphi+1}$ is the Inradius of the Golden Ratio triangle, $\frac{1}{\delta_2+1}$ is the Inradius of Silver ratio triangle.

Further, $\frac{\text{Area of the Silver Ratio Triangle}}{\text{Area of Its Incircle}} = \frac{(\text{Silvr Ratio})^2}{\pi}$

Moreover, just like the Fibonacci-Pythagorean Triples, the Pythagorean triples derived from **Pell Numbers series**, approach the 1:1:$\sqrt{2}$ triangle's proportions, as the series advances. The Pythagorean triples can be formed with the alternate Pell numbers as the hypotenuses, like <u>5</u>-4-3, <u>29</u>-21-20, <u>169</u>-120-119, <u>985</u>-697-696, <u>5741</u>-4060-4059 and so on. And, as such series of **Pell-Pythagorean Triples** advances, the triples so formed invariably approach **1:1:$\sqrt{2}$** limiting triangle proportions, exactly in the same manner as the ratio between consecutive Pell numbers approaches the Silver Ratio, $\lim_{n\to\infty} \frac{Pn}{Pn-1} \cong \delta_2$ (=2.41421356……). And hence, while δ_2 is the Silver Ratio, the **1:1: $\sqrt{2}$ triangle** is **the Silver Ratio Triangle**.

Similarly, the right triangle with its catheti 1 and $\frac{\mathbf{2}}{\mathbf{3}}$ is the Bronze Ratio Triangle, and so on.

Certain generalised features, like specific side lengths and peculiar measures of the angles, are evident in such generalised 'Metallic Ratio Triangle' representing the n^{th} Metallic Ratio $\boldsymbol{\delta_n}$, as illustrated here in **Figure 18.**

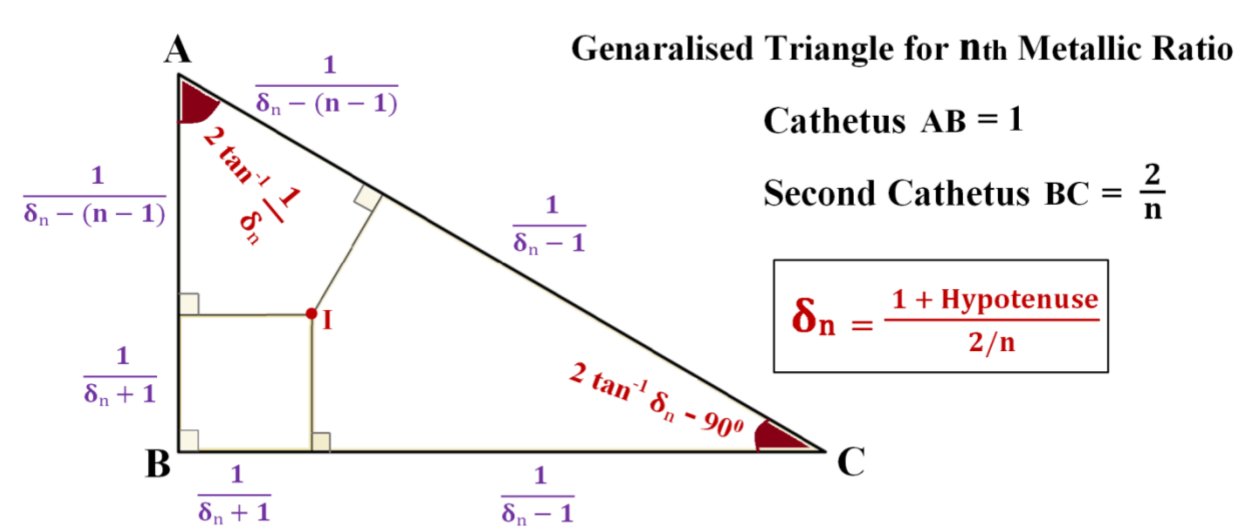


Figure 18: The 'Metallic Ratio Triangle' for n^{th} Metallic Mean.

Noticeably, the Area of such Metallic Ratio Triangle for n^{th} Metallic Mean = $\frac{1}{n}$

And its Semiperimeter (s) = $\frac{\delta_n}{\delta_n - 1}$; and the Inradius (r) = $\frac{1}{\delta_n + 1}$

$\frac{\text{Area of this Metallic Ratio Triangle}}{\text{Area of Its Incircle}} = \frac{(\boldsymbol{\delta}\mathbf{n}+\mathbf{1})^2}{\mathbf{n}\boldsymbol{\pi}}$; where $\boldsymbol{\delta_n}$ is the n^{th} Metallic Ratio.

The 1:2:$\sqrt{5}$ Triangle and 3-4-5 Pythagorean Triple

The **1:2:$\sqrt{5}$** triangle is also found to exhibit a classical geometric relationship with the **3-4-5** right triangle, which is the first of primitive Pythagorean triples. The **1:2:$\sqrt{5}$ triangle** and **3-4-5 triple** are invariably formed together, as described below. But more importantly, these two right triangles exhibit an outstanding synergy between all their corresponding sides and angles, that provides for the phenomenal expression of Golden Ratio.

Before illustrating such Golden Synergy between these two right triangles, let us consider how this couple of right triangles emerge together in various geometric constructions.

1) Median and Altitude on the Hypotenuse of 1:2:$\sqrt{5}$ triangle form together a 3-4-5 triangle. Consider following **Figure 19**, the reference triangle ABC is a 1:2:$\sqrt{5}$ triangle, having BP as the Bisector of its right angle, BO as the Median on hypotenuse, and BQ as the Symmedian which is also the Altitude on Hypotenuse. Remarkably, the triangle BOQ formed here is the 3-4-5 Pythagorean triple. This is obvious because the median BO coincides with the Euler Line which makes 53.13^0 angle with the hypotenuse, and it is precisely the angle of the 3-4-5 triangle.

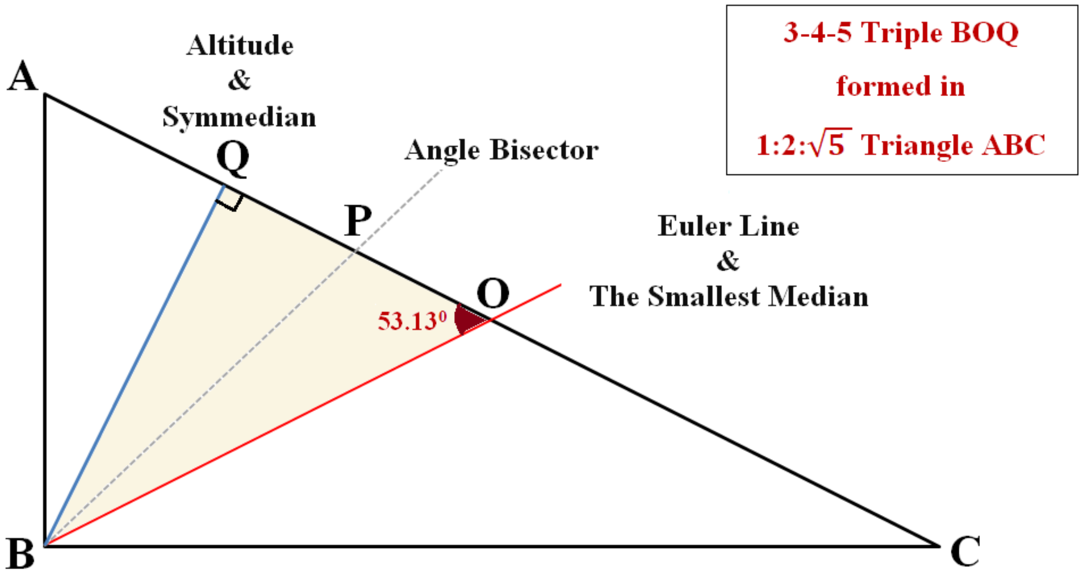


Figure 19: Emergence of **3-4-5** Triple in the **1:2:$\sqrt{5}$** Triangle.

It is worth mentioning here that the Nine Point Circle of the reference triangle ABC in above Figure, is also the Circumcircle of this 3-4-5 triangle BOQ. Also, noticeably, the formation of this 3-4-5 triangle BOQ is accompanied by the formation of another 1:2:$\sqrt{5}$ triangle ABQ.

2) Another geometric method for concomitant formation of 1:2:$\sqrt{5}$ triangle and 3-4-5 triple involves the Euler Line of 1:2:$\sqrt{5}$ triangle. Consider following **Figure 20(A)**. The reference triangle ABC has point B as its Orthocenter X(4), and point O as its Circumcenter X(3); and hence Line BO as the Euler line. If a perpendicular AP is drawn from vertex A on Euler line, it produces 3-4-5 triangle APO, alongside the 1:2:$\sqrt{5}$ triangle APB.

   Moreover, as illustrated in **Figure 20(B),** the perpendicular CQ on Euler line from Vertex C, and the segment connecting vertex C to de Longchamps Point D [i.e. Kimberling Center X(20) which is the reflection of orthocenter B about the circumcenter O] give rise to 3-4-5 triangle COQ, alongside the 1:2:$\sqrt{5}$ triangle CDQ**.**

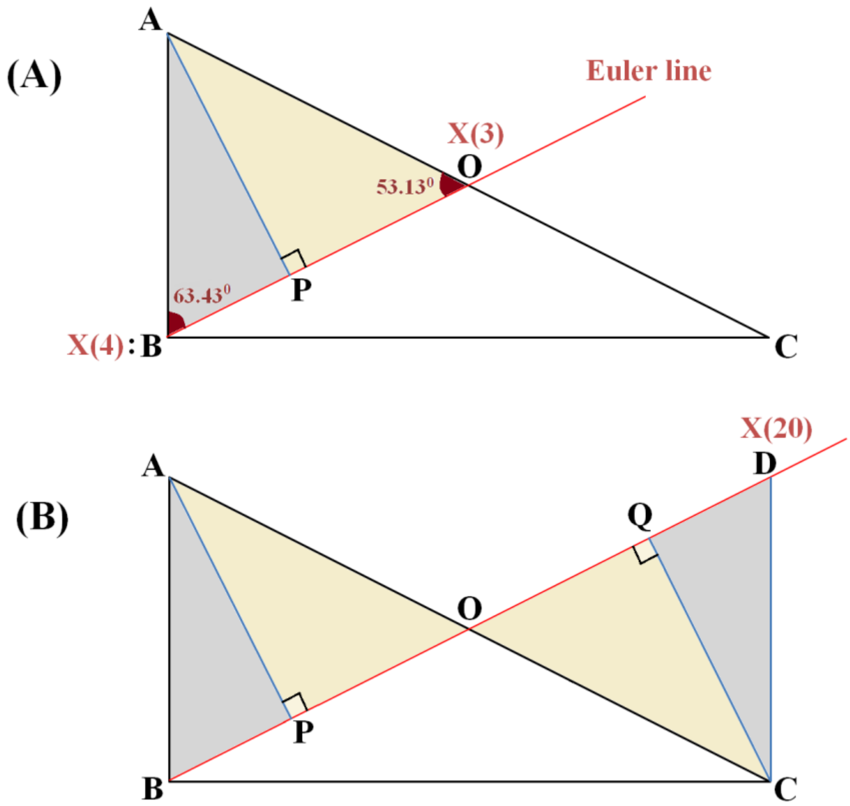


Figure 20: Perpendiculars on the Euler Line of 1:2:√5 triangle

Noticeably, here the quadrilateral ABCD is a 1: 2 Rectangle, which is also called as a Domino; each diagonal half of it is a 1:2:√5 triangle.

3) Also, the dissection of each of the two right triangles produces a couple of 1:2:√5 triangles along with one 3-4-5 triple, as shown below.

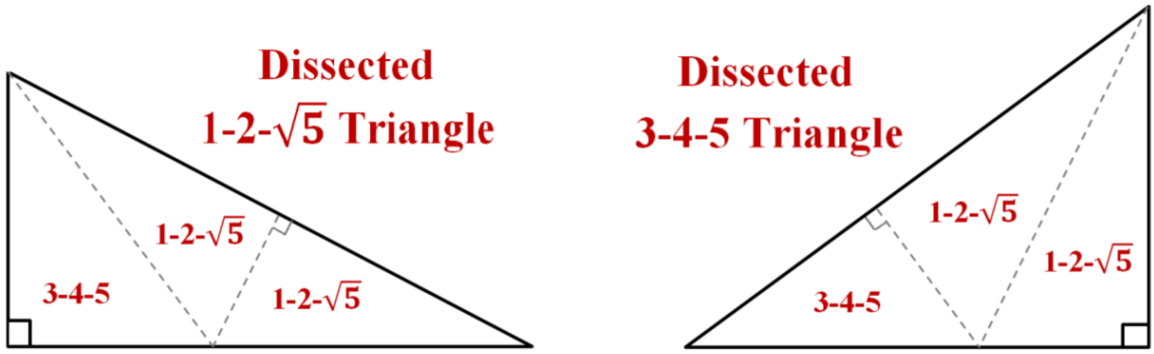


Figure 21: Dissection of two triangles: into Complementary and Self-Similar Triangles.

4) Beside several such geometric methods those impart such concurrent formation of 1:2:√5 triangle and 3-4-5 triangle, the simplest and the most important method is the dissection of a square. The 3-4-5 right triangle can be geometrically produced by dissection of a square by three line segments, as shown below in **Figure 22**. Remarkably, such dissection of a square invariably produces multiple 1:2:√5 triangles alongside a central Pythagorean triple.

In square ABCD, points P and R are the midpoints of side AD and DC, respectively. Connecting point P to the vertices B and C, and point R to the vertex B, produces the triangle PBQ in center; which is a 3-4-5 Pythagorean triangle. And remarkably, all other triangles formed in the figure, namely, ΔCQR, ΔCQB, ΔAPB, ΔCDP and ΔCBR; all are 1:2:√5 triangles of various sizes.

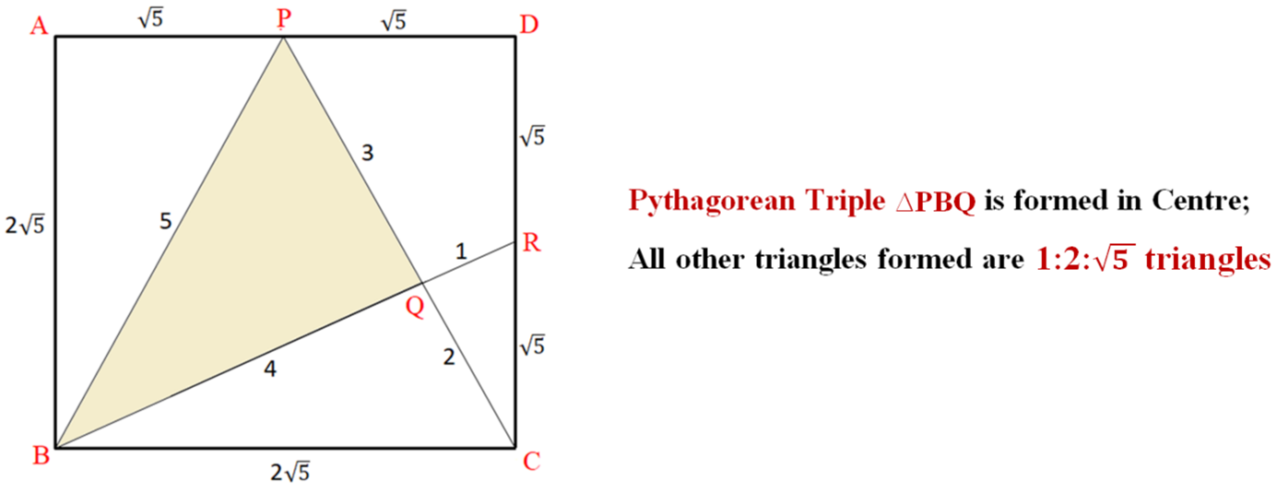


Figure 22: Dissection of the Square imparts 3-4-5 Pythagorean triple & multiple 1:2:$\sqrt{5}$ triangles.

The abovementioned method of the dissection of a square can divide any square into a 3-4-5 Pythagorean triple and multiple 1:2:$\sqrt{5}$ triangles. And the square, so dissected, possess several unique geometric features, as mentioned below.

The square, so dissected, consists of following three types of geometric figures;

1. The central 3-4-5 Pythagorean triangle ΔPQB
2. The 1:2:$\sqrt{5}$ triangles, namely, ΔCQR, ΔCQB and ΔAPB.

   Remarkably, these three 1:2:$\sqrt{5}$ triangles are in a definite proportion with each other. They have their corresponding sides, and hence also their semiperimeters, their inradii and circumradii; all in the **1:2:$\sqrt{5}$** proportion that is explicit between these three triangles.
3. And, the irregular quadrilaterals □ABQP and □PQRD: The quadrilateral □ABQP consists of the 3-4-5 Pythagorean triangle PQB and an equivalent sized 1:2:$\sqrt{5}$ triangle PAB, having their common hypotenuse PB. The internal angles of these quadrilaterals □ABQP and □PQRD, beside two opposite right angles, are

   $\angle$ ABQ = $\angle$ DPQ = $\mathbf{63.435^0}$

   and, $\angle$ APQ = $\angle$ DRQ = $\mathbf{116.565^0}$

   Remarkably, these $\mathbf{63.435^0}$ and $\mathbf{116.565^0}$ are exactly **"the angles of the Golden Rhombus"**.

Also, the perimeters of □PQRD and □ABQP are invariably in the proportion **1: φ**, where **φ** is the Golden Ratio.

Moreover, in above dissected square, beside all 1:2:$\sqrt{5}$ triangles, in which the Golden Ratio is ubiquitous, the irregular quadrilaterals □PQRD and □ABQP, and the square □ABCD as a whole, are also full of the Golden Ratios, embedded in their very structure, for example, the two Quadrilaterals □ABQP and □PQRD possess Golden Ratio in their side lengths as:

$$\frac{\mathbf{BA+BQ}}{\mathbf{PA+PQ}} = \frac{\mathbf{PQ+PD}}{\mathbf{QR+DR}} = \boldsymbol{\varphi}$$

Now, most importantly, the 1:2:$\sqrt{5}$ triangle is not just a geometric structure formed alongside the 3-4-5 triangle in the abovementioned dissected square, but it is also observed that there exists a precise complementary relationship between these two right triangles. And the classical correspondence between these two triangles culminates in some unconventional geometric outcomes.

All corresponding angles and sides of these two triangles, synergize with each other to reflect the Golden Ratio, as follows. The angles of 1:2:$\sqrt{5}$ triangle and 3-4-5 triangle are found to be entangled with each other in a remarkable way, and hence they add up to very peculiar values, as illustrated in **Figure 23**.

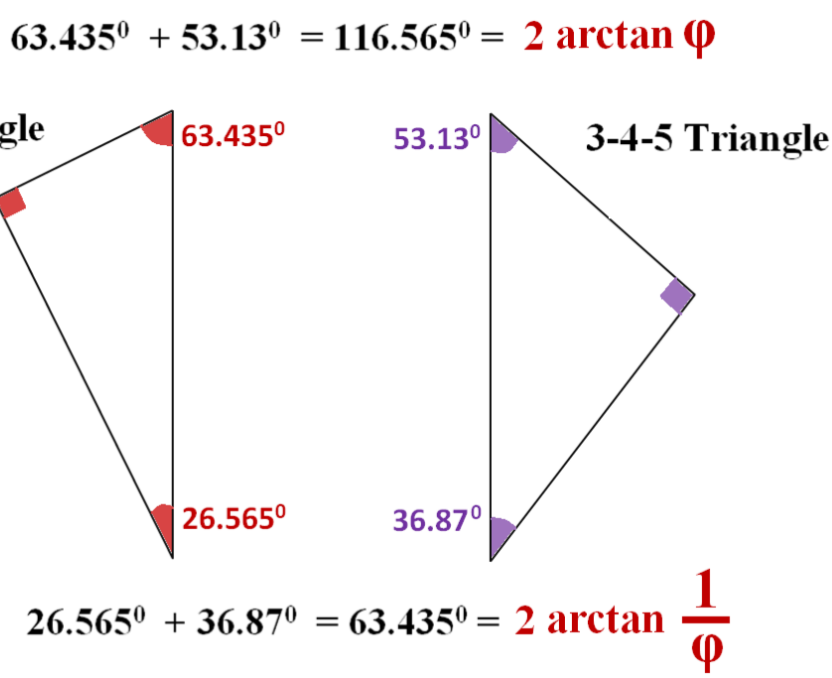


Figure 23: The corresponding angles of the two triangles add up, to reflect **the Golden Ratio.**

As shown in above diagram, the corresponding angles of the two triangles add up to reflect the precise value of **Golden Ratio**, noticeably they add up to the **angles of Golden Rhombus**, whose diagonals are perfectly in Golden Proportion.

$$53.13^0 + 63.435^0 = 116.565^0 = 2 \arctan \varphi$$

and also, $$36.87^0 + 26.565^0 = 63.435^0 = 2 \arctan \frac{1}{\varphi}$$

In other words, the average of the corresponding angles of these two right triangles equals the arctangent of Golden Ratio.

Remarkably, beside Golden Rhombus, the these angles, formed by combining the corresponding angles of these two triangles, are also associated with all those polyhedrons whose geometry is full of Golden Ratio, such as, $53.13^0 + 63.435^0 = \mathbf{116.565^0}$ is the dihedral angle of Regular Dodecahedron, Dodecadodecahedron, Small Stellated Dodecahedron as well as Small Icosihemidodecahedron. Similarly, $36.87^0 + 26.565^0 = \mathbf{63.435^0}$ is also the dihedral angle of Great dodecahedron and Great Stellated Dodecahedron.

Noteworthy here, the angle $\mathbf{36.87^0}$ of 3-4-5 triangle is the **complementary angle** for **twice the** $\mathbf{26.565^0}$ angle of $1:2:\sqrt{5}$ triangle.

And, the angle $\mathbf{53.13^0}$ of 3-4-5 triangle is the **supplementary angle** for **twice the** $\mathbf{63.435^0}$ angle of $1:2:\sqrt{5}$ triangle. The 53.13^0 - 63.435^0 - 63.435^0 triangle is an unique isosceles triangle that has its unequal side equal to the height or altitude on that side (such as triangle PBC in Figure 22). And noticeably, the semiperimeter of such isosceles triangle is **Unequal Side × φ**.

Further, two smaller angles of $1:2:\sqrt{5}$ and 3-4-5 triangles, $\mathbf{26.565^0}$ & $\mathbf{36.87^0}$, add up to angle $\mathbf{63.435^0}$ of $1:2:\sqrt{5}$ triangle, while the angle $\mathbf{53.13^0}$ of Pythagorean triple is twice the $\mathbf{26.565^0}$ angle of $1:2:\sqrt{5}$ triangle. Hence, dissection of these two triangles, as illustrated in **Figure 21** above, invariably causes the emergence of a couple of $1:2:\sqrt{5}$ triangles along with one Pythagorean triple.

Beside angles, the classical geometric synergy between $1:2:\sqrt{5}$ triangle and 3-4-5 triangle is naturally manifested in their side lengths also. As shown below in **Figure 24,** the $1:2:\sqrt{5}$ triangle ABC merged with an equivalent sized the 3-4-5 triangle ADC, with their common hypotenuse AC, reveals the Golden Ratio, precisely embedded in such blend of the two triangles.

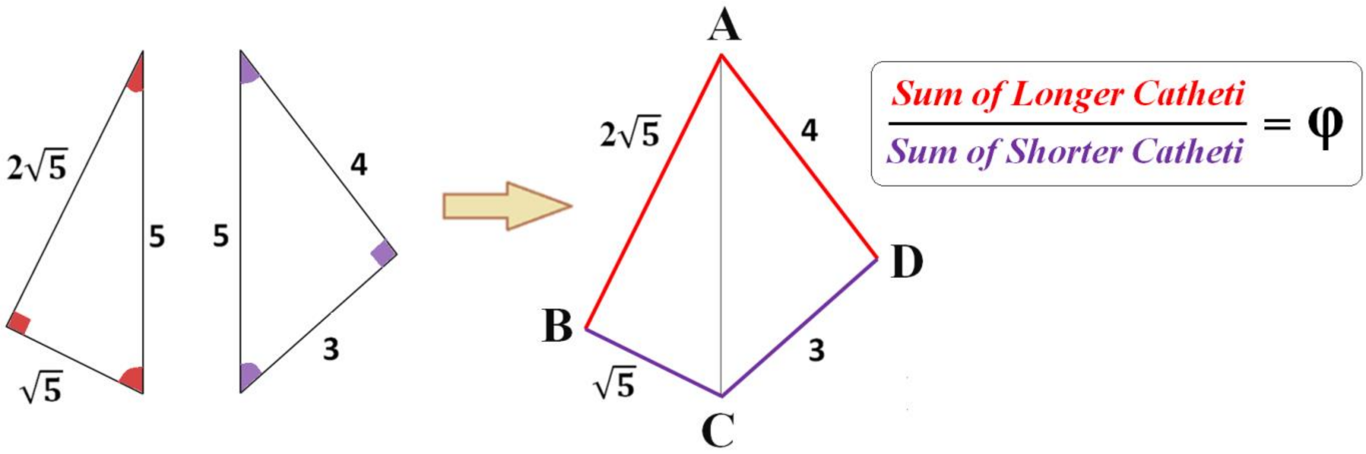


Figure 24: The Golden Ratio in amalgam of 1:2:$\sqrt{5}$ triangle and 3-4-5 triangle.

In these **equivalent sized** 1:2:$\sqrt{5}$ and 3-4-5 triangles, with **equal hypotenuse AC = 5**, the Longer Catheti of two triangles (AB + AD) add up to $2\varphi^3$, and the Shorter Catheti (CB + CD) add up to $2\varphi^2$

Hence, in the quadrilateral □ABCD formed by merger of the two triangles,

$$\frac{\textbf{The Sum of the Longer Catheti of the Two Triangles}}{\textbf{Sum of their Shorter Catheti}} = \frac{\textbf{AB+AD}}{\textbf{CB+CD}} = \varphi$$

The above quadrilateral ABCD, formed by the merger of 1:2:$\sqrt{5}$ triangle and Pythagorean triple, is exactly same as □ABQP in **Figures 22.**

Hence, to summarise the Golden Synergy in **Figure 24**,

in the 1:2:$\sqrt{5}$ and 3-4-5 triangles having common hypotenuse:

: The Average of the corresponding longer catheti; $2\sqrt{5}$ & 4 is $\boldsymbol{\varphi^3}$

: The Average of the corresponding shorter catheti; $\sqrt{5}$ & 3 is $\boldsymbol{\varphi^2}$

: The Average of the corresponding larger angles; 53.13^0 & 63.435^0 is **arctan** $\boldsymbol{\varphi}$

: The Average of the corresponding smaller angles; 36.87^0 & 26.565^0 is **arctan** $\boldsymbol{\frac{1}{\varphi}}$

In other words, the right triangle with its catheti in Golden Ratio **1:** $\boldsymbol{\varphi}$, occupies the **exact mean position** between the 1:2:$\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple.

Several geometric features signify the classical correspondence and the interconnectedness between the 1:2:$\sqrt{5}$ triangle and the Pythagorean triple. Consider □ABQP in following **Figure 25**, which is formed by the merger of 1:2:$\sqrt{5}$ triangle ΔPAB and equivalent sized Pythagorean triangle PQB, with their common hypotenuse PB which is also the diagonal of □ABQP. Interestingly, if the second diagonal AQ is incorporated in □ABQP, this diagonal AQ divides the right angle in each triangle (angle PAB and angle PQB) into two smaller angles of the complementary triangle.

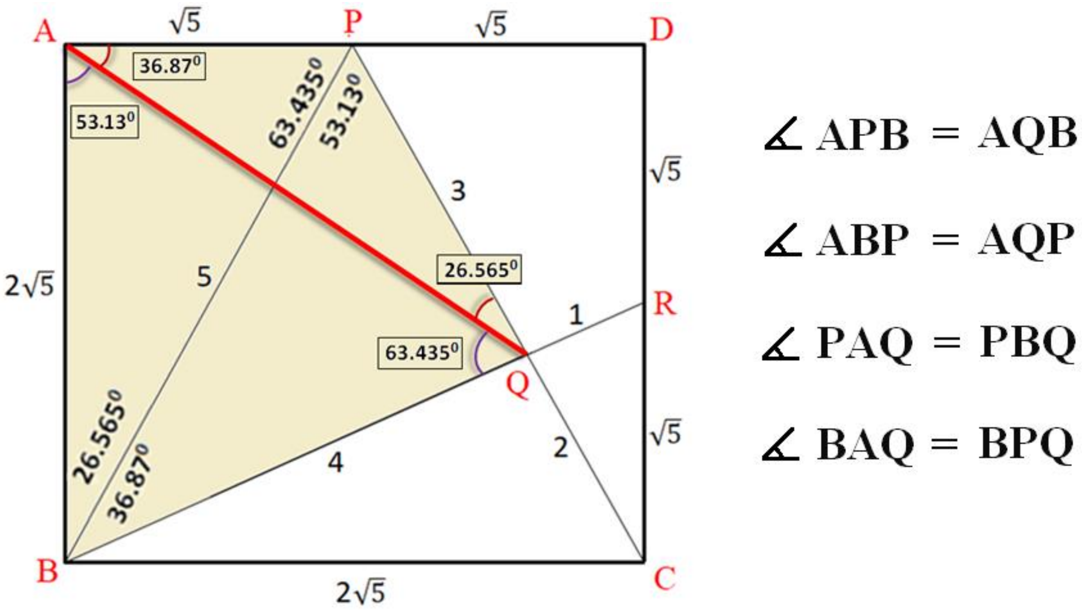


Figure 25: The diagonal AQ divides the right angle of each triangle into reciprocal smaller angles.

Diagonal AQ divides the right angle PAB of 1:2:$\sqrt{5}$ triangle, into reciprocal smaller angles, angle PAQ and angle BAQ those measure 36.87^0 and 53.13^0, which are precisely the angles of 3-4-5 Pythagorean triangle; and vice versa.

Furthermore, all angles and side lengths of 1:2:$\sqrt{5}$ triangle as well as 3-4-5 triangle, are the exact expressions of Golden Ratio, as illustrated below. Noticeably, all concerned Angles are the precise arctangent functions of the Odd Powers of Golden Ratio: φ^n and $\frac{1}{\varphi^n}$; where n = 1, 3, 5.

Angles in Degrees	Angles in terms of φ
63.435^0	$= \; 2\arctan\frac{1}{\varphi} \; = \; \tan^{-1}\varphi^3 - \tan^{-1}\frac{1}{\varphi^3}$
26.565^0	$= \; 2\arctan\frac{1}{\varphi^3} \; = \; \tan^{-1}\varphi - \tan^{-1}\frac{1}{\varphi}$
36.87^0	$= \tan^{-1}\frac{1}{\varphi^5} + \tan^{-1}\frac{1}{\varphi} \; = \; 2[\tan^{-1}\frac{1}{\varphi^5} + \tan^{-1}\frac{1}{\varphi^3}]$ $= 2[\tan^{-1}\frac{1}{\varphi} - \tan^{-1}\frac{1}{\varphi^3}] \; = \; 2[\tan^{-1}\varphi^3 - \tan^{-1}\varphi]$
53.13^0	$= \; \tan^{-1}\varphi - \tan^{-1}\frac{1}{\varphi^5} \; = \; \tan^{-1}\varphi^5 - \tan^{-1}\frac{1}{\varphi} \; = \; 4\arctan\frac{1}{\varphi^3}$

116.565⁰	$= 2\tan^{-1}\varphi = \tan^{-1}\frac{1}{\varphi^3} + [180^0 - \tan^{-1}\varphi^3]$

Also, the side lengths of both triangles are the assertion of Golden Ratio, as shown below.

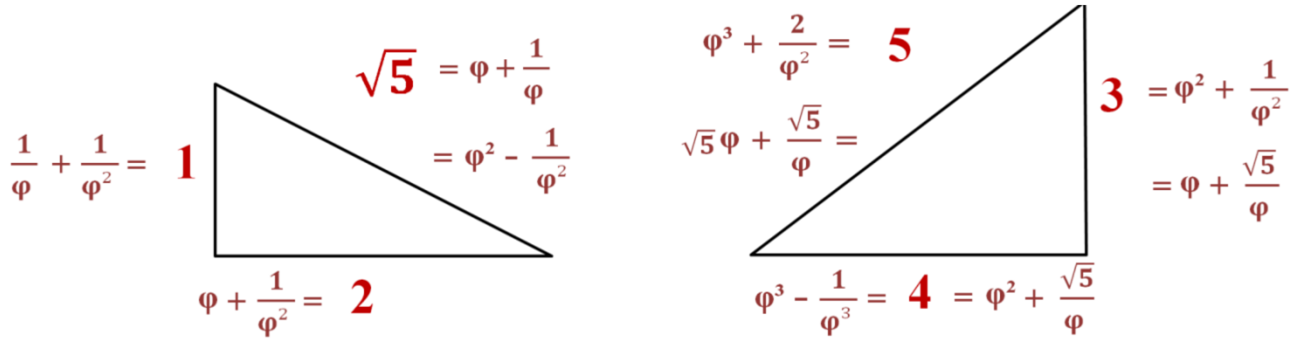


Figure 26: Precise expression of Sides and Angles of both triangles in terms of the Golden Ratio

More importantly, just like the 1:2:$\sqrt{5}$ triangle and its Incircle, the merger of 1:2:$\sqrt{5}$ triangle - Pythagorean triple and the Incircle of such combined triangle also exhibit the precise geometric ratios, in terms of $\boldsymbol{\pi:\varphi}$ correlation, which is the paramount validation of the geometric intimacy between these two right triangles.

As shown below in **Figure 27**, the 1:2:$\sqrt{5}$ triangle ABD and the 3-4-5 Pythagorean triangle ACD, of equivalent sizes, share one of their catheti: AD, and the precise ratios observed therein are as follows;

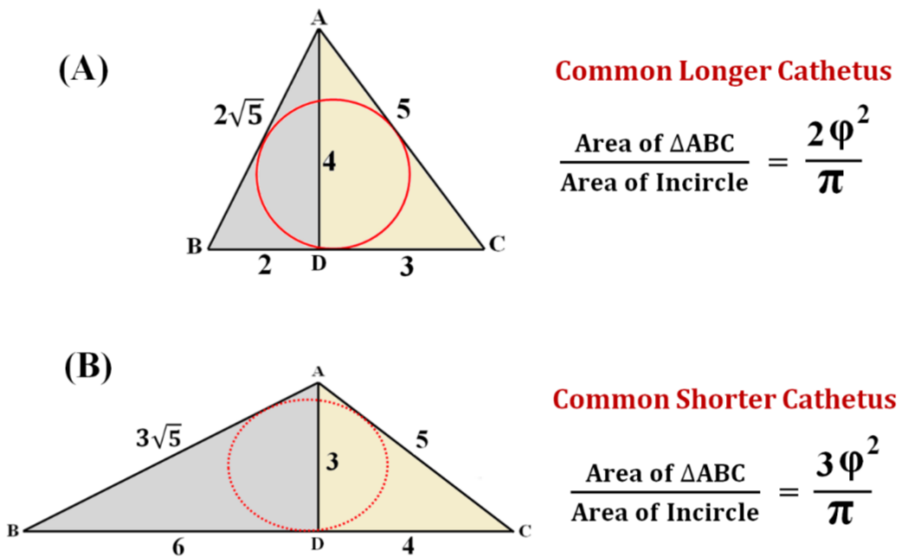


Figure 27: The two right triangles, with **(A)** Common Longer Leg, and **(B)** Common Shorter Leg

Such extremely precise results finally corroborate the classical synergy between these two triangles.

A closely resembling, but inaccurate correlation is worth mentioning here. The oft-cited, coincidental relationship: $\frac{4}{\pi} \approx \sqrt{\varphi}$ is the ratio between the area of a Square and its Incircle. However, this fluky correlation is far from precise, due to the so called Squaring the Circle, or Quadrature of Circle problem. Hence, the accurate $\boldsymbol{\pi:\varphi}$ correlations, which the ratios between a Square and its Incircle fail to impart, can be provided by the golden geometry of 1:2:$\sqrt{5}$ triangle. The incorporation of Incircle into the 1:2:$\sqrt{5}$ triangle alone, or into the unique combination of 1:2:$\sqrt{5}$ and 3-4-5 triangles, and thence by bringing the Pi($\boldsymbol{\pi}$) into picture, the precise correlation between a transcendental number Pi($\boldsymbol{\pi}$), and an algebraic number Phi($\boldsymbol{\varphi}$) can be expressed in terms of the accurate geometric ratios between the 1:2:$\sqrt{5}$ triangle, or its amalgam with 3-4-5 triangle, and the Incircle therein.

Such distinguished results further authenticate the classical geometric relationship between these two triangles. Noticeably, the catheti of these two triangles **(2, 1)** and **(3, 4)** are the first four Lucas Numbers. Note the Lucas Numbers are more closely associated with the Golden Ratio than the Fibonacci Numbers; like $\lim_{n\to\infty} L_n \cong \varphi^n$, or more precisely $L_n = \varphi^n + (1-\varphi)^n$ And, the hypotenuse $\sqrt{5}$ is nothing but the coarse form of Golden Ratio; like $\lim_{n\to\infty} \frac{L_n}{F_n} \cong \sqrt{5}$ or in the Binet's formula $F_n = \frac{\varphi-(-\varphi)^{-n}}{\sqrt{5}}$. Remarkably, the classical relationship mentioned so far is exclusively observed between the 1:2:$\sqrt{5}$ triangle and the 3-4-5 triangle. It is NOT observed with such other right tringle-couples, having any other four consecutive Lucas or Fibonacci numbers as their catheti, although the average of corresponding angles in such triangle-couples are very close to arctan φ & arctan $\frac{1}{\varphi}$, but NOT precisely, unlike in case of the 1:2:$\sqrt{5}$ and 3-4-5 unique couple. This is simply because $\frac{F_n}{F_{n-1}}$ or $\frac{L_n}{L_{n-1}}$ never precisely equals the φ. And this highlights the uniqueness and significance of the classical couple of two right triangles, viz. the 1:2:$\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple.

Further, although several other 'right triangle-couples' are formed on same geometric principles as this couple of triangles, such a precise complementary relationship is the distinctive and characteristic feature of 1:2:$\sqrt{5}$ and 3-4-5 triangles. It is well known that, on doubling an acute angle of a right triangle, its hypotenuse gets squared. Hence, while the angle 26.565^0 of 1:2:$\sqrt{5}$ triangle is doubled to 53.13^0, the hypotenuse $\sqrt{5}$ gets squared, giving rise to 3-4-5 Pythagorean triple. Beside 1:2:$\sqrt{5}$ and 3-4-5 triangles, several such pairs of right triangles are formed on this same geometric principle; like 2-3-$\sqrt{13}$ and 5-12-13 triangles, 1-3-$\sqrt{10}$ and 8-6-10 triangles, 1-4-$\sqrt{17}$ and 8-15-17 triangles, 2-5-$\sqrt{29}$ and 20-21-29 triangles, and so on. However, the classical geometric relationship described in this paper, is the unique and exclusive feature of the 1:2:$\sqrt{5}$ and 3-4-5 triangle-couple and its multiples, like 2-4-$\sqrt{20}$ and 12-16-20 triangles, 3-6-$\sqrt{45}$ and 27-36-45 triangles, and so on.

On a last note, author wants to mention a couple of thought provoking concepts, regarding these 1:2:$\sqrt{5}$ and 3-4-5 triangles, in context of the so called $\boldsymbol{\pi:\varphi}$ correlations.

Noticeably, the assessment of a much celebrated but imprecise equation $\frac{\varphi^2}{\pi} \approx \frac{5}{6}$, can also be done on basis of the geometry of 1:2:$\sqrt{5}$ triangle. Just as the 1:2:$\sqrt{5}$ triangle provides the fractional expression for the Golden Ratio: $\frac{1+\sqrt{5}}{2}$, the idiosyncratic geometry of this triangle can also approximate the value of Pi as;

$(\sin 63.435^0 + \sin 26.565^0) + (\sin 63.435^0 + \sin 26.565^0)^2 \approx \pi$

or, in terms of catheti and hypotenuse: $\left(\frac{1+2}{\sqrt{5}}\right) + \left(\frac{1+2}{\sqrt{5}}\right)^2 \approx \pi$

Of course, these equations do not provide the accurate value of Pi, however, this value, approximating the Pi($\boldsymbol{\pi}$), delivers the precise value of Golden Ratio($\boldsymbol{\varphi}$) through the fluky equation $5\pi \approx 6\varphi^2$.

In other words, $\frac{5}{6}[(\sin 63.435^0 + \sin 26.565^0) + (\sin 63.435^0 + \sin 26.565^0)^2] = \varphi^2$ precisely. Remarkably, this **5:6** is the ratio between the areas of equivalent sized 1:2:$\sqrt{5}$ and 3-4-5 triangles, having their equal hypotenuse, as in Figure 24.

Lastly, a well-known and interesting calculator trick is worth mentioning here. This famed trick exploits the fact that **Sine** or **Tangent Function** of $\frac{1}{55555\ldots\ldots}$ gives **Pi-like values** on calculator screen, in degree mode. This is due to degrees-radians conversion and the "small angle approximation", $\frac{1}{55555\ldots\ldots n \text{ times}} \approx 1.8 \times 10^{-n}$, and this figure multiplied by $\frac{\pi}{180}$ imparts 'π - like values' in form of $3.141592\ldots \times 10^{-(n+2)}$

But more importantly, it is noteworthy here that, replacing **5**s by **6**s in above trick imparts 'φ^2 -like values';

sin $\frac{1}{666666......\text{n times}} \approx \varphi^2 \times 10^{-(n+2)}$

Also, such result with digit **6** can be generalised for any digit **k**, as follows

sin or **tan** $\frac{1}{\text{kkkkkk......n times}} \approx \frac{6}{k}\varphi^2 \times 10^{-(n+2)}$

And again, these equations of course does not impart the accurate value of the Square of Golden Ratio, however, this value approximating the φ^2**,** delivers the value of Pi(π) **precise up to (n-1) decimal places,** through the arbitrary equation $5\pi \approx 6\varphi^2$.

In other words, $\frac{6}{5}$ **× sin** $\frac{1}{66666......\text{n times}} \approx \pi \times 10^{-(n+2)}$

: the value of π obtained here is precise up to (n-1) decimal places, and not to mention, this **5:6** is the ratio between the areas of equivalent sized 1:2:√5 and 3-4-5 triangles

Conclusion:

This paper illustrated the unique geometry of the 1:2:√5 triangle, which makes it the real "Golden Ratio Triangle" in every sense of the term. The Golden Ratio is observed to be embedded in every geometric aspect of this right triangle. Such special right triangle, impregnated with Golden Proportion in its very geometry, is naturally found to be closely associated with regular pentagon. It not only provided the unique method of constructing regular pentagon, but the regular pentagon is also observed to possess the 1:2:√5 proportions in its own geometry.

Moreover, just like the 1:2:√5 triangle is for Golden Ratio, this paper introduced the concept of a special right triangle that accurately represents each Metallic Mean. All metallic Ratios can be perfectly illustrated by the generalised right triangle, described in this paper.

Further, this paper introduced the hidden link in geometry, namely, the precise complementary relationship between the 1:2:√5 triangle and the 3-4-5 Pythagorean triple. Remarkably, these two right triangles are not only formed together by various geometric methods, but they also manifest as the perfect complementary geometric entities for each other. These two triangles, by communion with each other, engender the Golden Ratio in a distinctive manner. When equivalent sized 1:2:√5 triangle and 3-4-5 triple are combined together along their common hypotenuse, their corresponding catheti as well as their corresponding angles add up to reveal the exact value of Golden Ratio. And, when connected along their shorter or longer catheti, these two triangles impart the accurate Pi:Phi Correlation, as the precise ratios between the combined triangle and its incircle therein. This is the paramount validation of the classical geometric intimacy between these two right triangles.

Hence, the 1:2:√5 Golden Triangle, alone as well as in amalgamation with 3-4-5 Pythagorean triple, not only provided for the ultimate geometric substantiation of Golden Ratio, but it also revealed the Golden Link in Geometry; that is the accurate Pi:Phi correlation, with an unprecedented level of precision, and which is firmly premised upon the classical geometric principles.

Finally, author leaves it to readers' wisdom and fine sense of judgement: for which triangle the appellation "Golden Ratio Triangle" is more suitable, the generally accepted 36^0-72^0-72^0 triangle OR the *sui generis* 1:2:√5 Triangle**?**

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