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Generalized Symmetric (f, g) – Biderivations on Lattices

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Abstract

In this paper, we introduce the notion of generalized symmetric (f, g)-biderivations on lattices, also some properties of generalized symmetric (f, g)- biderivations we studies.

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1. Introduction

Lattices play an important role in many different domains such as information theory, information retrieval, information access controls and cryptanalysis [see1,2,5,11].

In ring theory, the properties of derivations are very consequential topic to studying, many researchers study the derivation theory on different algebraic structures, lately the notion of derivations displayed in ring and near rings has been studies by various researchers in contextually of lattices [see 1,7,9,10,13,15].

The notion of symmetric bi-derivations, generalized symmetric bi – derivations, symmetric f-biderivations and symmetric (f, g) – biderivations in latticesare studies and proved some results [3,6,7], Y. ceven introduced the notion of symmetric bi-derivations and generalized symmetric bi derivations of lattices [see 14,15] .

In this paper , we introduce the notion of generalized symmetric (f, g)- biderivations ,which more commonalty than the notion of generalized symmetric bi- derivations and symmetric (f, g)- biderivations in lattices which is introduced in [see 3,6] , also we give some interesting results about generalized symmetric (f, g)-biderivations of lattices , we apply the notions to lattices and looking for some related properties which are discussed in [see 6 , 8] .

2. Preliminaries

Definition 2.1[6]: Let L be anon-empty set endowed with operations ∧ and∨ , then $(L, \wedge, \vee)$ is called a lattice if it satisfying the following conditions for all $x, y, z \in L$

$(i) x \wedge x = x , x \vee x = x$

$(ii) x \wedge y = y \wedge x , x \vee y = y \vee x$

$(iii) (x \wedge y) \wedge z = x \wedge (y \wedge z) , (x \vee y) \vee z = x \vee (y \vee z)$

$(iv) (x \wedge y) \vee x = x , (x \vee y) \wedge x = x$

Definition 2.2[4]: A lattice $(L, \wedge, \vee)$ is called distributive lattice if one of the following identities hold for all $(L, \wedge, \vee)$

$(v)\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$(vi)\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Remark 2.3 [8]: In any lattice , the properties (v) and (vi) are equivalent .

Definition 2.4[4]: let $(L, \wedge, \vee)$ be a lattice , a binary relation $\leq$ on L is defined by

$x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$

Definition 2.5 [6]: A lattice $(L, \wedge, \vee)$ is called modular if for $x, y, z \in L$ satisfies the following condition :

(vii) if $x \leq y$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$

Lemma 2.6 [8]: let $(L, \wedge, \vee)$ be a lattice , let the binary relation $\leq$ be as in definition 2.4 , then $(L, \leq)$ is partially ordered set (poset) and for any $x, y \in L$, $x \wedge y$ is the g.l.b of $\{x, y\}$ and $x \vee y$ is the l.u b. of $\{x, y\}$.

Definition 2.7[15] : let $(L, \wedge, \vee)$ be a lattice , a mapping $D(.,.): L \times L \longrightarrow L$ is called symmetric if $D(x, y) = D(y, x)$ for all $x, y \in L$.

Definition 2.8[15]: let $(L, \wedge, \vee)$ be a lattice , a mapping $d: L \longrightarrow L$ defined by $d(x) = D(x, x)$ is called the trace of $D(.,.)$ .where $D(.,.): L \times L \longrightarrow L$ is symmetric mapping .

Definition 2.9[15] : let $(L, \wedge, \vee)$ be a lattice , a mapping $D(.,.): L \times L \longrightarrow L$ is called symmetric bi derivation on L if

$$D(x \wedge y, z) = (D(x, z) \wedge y) \vee (x \wedge D(y, z))$$

for all $x, y, z \in L$.

obviously , a symmetric biderivation on L satisfies the relation

$$D(x, y \wedge z) = (D(x, y) \wedge z) \vee (y \wedge D(x, z))$$

for all $x, y, z \in L$.

Definition 2.10[14]: let $(L, \wedge, \vee)$ be a lattice , be symmetric biderivation $D(.,.): L \times L \longrightarrow L$ and $\nabla(.,.) L \times L \longrightarrow L$ be a symmetric mapping , we call ∇ a generalized symmetric biderivation related to D, if it satisfies the following condition

$$\nabla(x \wedge y, z) = (\nabla(x, z) \wedge y) \vee (x \wedge D(y, z))$$

for all $x, y \in L$.

Definition 2.11[6] : let $(L, \wedge, \vee)$ be a lattice and $D(.,.): L \times L \longrightarrow L$ be a symmetric mapping , D is called symmetric (f, g) – biderivation on L if there exist two functions $f, g: L \longrightarrow L$ such that

$$D(x \wedge y, z) = (D(x, z) \wedge f(y)) \vee (g(x) \wedge D(y, z))$$

for all $x, y, z \in L$.

proposition 2.12[7]: let $(L, \wedge, \vee)$ be a lattice and $f: L \longrightarrow L$ be a mapping . Let d be the trace of symmetric f-biderivation D, then the following hold for all $x, y \in L$:

(i) $D(x, y) \leq f(x)$ and $D(x, y) \leq f(y)$
(ii) $D(x, y) \leq f(x) \wedge f(y)$
(iii) $d(x) \leq f(x)$

Theorem2.13 [8] : let $(L, \wedge, \vee)$ be a lattice and $f: L \longrightarrow L$ be a mapping satisfying $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in L$. Let $\triangle$ be a generalized symmetric f-biderivation related to a symmetric f-biderivation D, δ be the trace of D. Then

(i) $D(x, y) \leq \triangle(x, y)$ $for\ all\ x, y \in L$

(ii) $\triangle(x,y) \leq f(x)$ *and* $\triangle(x,y) \leq f(y)$

(iii) $\triangle(x,y) \leq f(x) \wedge f(y)$

(iv) $d(x) \leq \delta(x) \leq f(x)$

3. Generalized symmetric $(\mathbf{f},\mathbf{g})$-biderivations

Definition 3.1: let $(L,\wedge,\vee)$ be a lattice , be symmetric (f,g)- biderivation $D(.,.): L \times L \longrightarrow L$ and $\nabla(\,.\,,.): L \times L \longrightarrow L$ be a symmetric mapping , we call ∇ a generalized symmetric (f,g)- biderivation related to D, if it satisfies the following condition

$$\nabla(x \wedge y, z) = (\nabla(x,z) \wedge f(y)) \vee (g(x) \wedge D(y,z))$$

for all $x,y,z \in L$.

Definition 3.2 : let $(L,\wedge,\vee)$ be a lattice and $\nabla(\,.\,,.): L \times L \longrightarrow L$ be a generalized symmetric (f,g)- biderivation on L.The mapping $\delta: L \longrightarrow L$ defined by $\delta(x) = \nabla(x,x)$ is called the trace of generalized symmetric (f,g)-biderivation ∇ .

It is clear that a generalized symmetric (f,g)-biderivation ∇ satisfies the following relations

$$\nabla(x, y \wedge z) = (\nabla(x,y) \wedge f(z)) \vee (g(y) \wedge D(x,z))$$

for all $x,y,z \in L$.

Example 3.3: let $(L,\wedge,\vee)$ be a lattice with least element 0 and $f: L \longrightarrow L$ and $g: L \longrightarrow L$ are mappings satisfying $f(x \wedge y) = f(x) \wedge f(y)$ and $g(x \wedge y) = g(x) = g(y)$ for all $x,y \in L$(respectively)

The mapping $D(\,.\,,.): L \times L \longrightarrow L$ defined by $D(x,y) = 0$ for all $x,y \in L$ is symmetric(f,g)- biderivation on L, then the mapping $\nabla(x,y): L \times L \longrightarrow L$ defined by $\nabla(x,y) = f(x) \wedge f(y)$ for all $x,y \in L$ is generalized symmetric(f,g)- biderivation on L

Remarks 3.3:

1) If $f = 1$ and $g = 1$ the identity on L , then generalized symmetric (1,1)-biderivation is generalized symmetric bi derivation on L.

2) If $\nabla = D$ then ∇ is symmetric (f,g) – biderivation .

Proposition 3.4 : Let ∇ be a generalized symmetric (f,g)-biderivation on a lattice$(L,\wedge,\vee)$ related to a symmetric (f,g)-biderivation D, then the mappings $\delta_1: L \longrightarrow L$ and $\delta_2: L \longrightarrow L$ defined by $\delta_1(x) = \nabla(x,z)$ and $\delta_2(y) = \nabla(x,y)$ are generalized (f,g)- derivation on L .

Proof :

$$\begin{aligned}\delta_1(x \wedge y) &= \nabla(x \wedge y, z)\\ &= (\nabla(x,z) \wedge f(y)) \vee (g(x) \wedge D_1(y,z))\\ &= (\delta_1(x) \wedge f(y)) \vee (g(x) \wedge d_1(y))\end{aligned}$$

Where $d_1: L \longrightarrow L$ defined by $d_1(y) = D(y,z)$ is (f,g)- derivation on L and D is symmetric (f,g)- biderivation on L .

Hence δ_1 is generalized (f,g) – derivation .

Proposition 3.5 : let $(L,\wedge,\vee)$ be a lattice and $\nabla(\,.\,,.): L \times L \longrightarrow L$ be a generalized symmetric (f,g)-biderivation on L related with symmetric (f,g)- biderivation D , then

$$\delta(x) \leq f(x) \vee g(x) \text{ for all } x \in L$$

where δ is the trace of ∇.

Proof :since $\nabla(x,x) \wedge f(x) \leq f(x)$ and $D(x,x) \wedge g(x) \leq g(x)$

Then since $(\nabla(x,x) \wedge f(x)) \vee (D(x,x) \wedge g(x)) \leq f(x) \vee g(x)$

Hence $\nabla(x \wedge x, x)) \leq f(x) \vee g(x)$

since $x \wedge x = x$ then

$$\delta(x) = \nabla(x,x) = \nabla(x \wedge x, x)$$

Proposition 3.6 : let $(L, \wedge, \vee)$ be a lattice and $\nabla(.\,,.): L \times L \longrightarrow L$ be a generalized symmetric (f,g)-biderivation on L related with symmetric (f,g)- biderivation D, then

$\nabla(x,y) \leq f(x) \vee g(x)$ and $\nabla(x,y) \leq f(y) \vee g(y)$ for all $x, y \in L$.

Proof: since $x \wedge x = x$ for all $x \in L$

Then for all $x \in L$

$$\nabla(x,y) = \nabla(x \wedge x, y)$$
$$= (\nabla(x,y) \wedge f(x)) \vee (g(x) \wedge D(x,y))$$

And since $\nabla(x,y) \wedge f(x) \leq f(x)$ and $D(x,y) \wedge g(x) \leq g(x)$

We can conclude that $\nabla(x,y) \leq f(x) \vee g(x)$

Similarly $\nabla(x,y) \leq f(y) \vee g(y)$ for all $x, y \in L$

Corollary 3.7 : let $(L, \wedge, \vee)$ be a lattice and $\nabla(.\,,.): L \times L \longrightarrow L$ be a generalized symmetric (f,g)- biderivation on L related with symmetric (f,g)- biderivation D, then when $g(x) \leq f(x)$ for all $x \in L$, we have .

$\nabla(x,y) \leq f(x)$ and $\nabla(x,y) \leq f(y)$ for all $x, y \in L$.

Proposition 3.8: let $(L, \wedge, \vee)$ be a lattice and $\nabla(.\,,.): L \times L \longrightarrow L$ be a generalized symmetric (f,g)- biderivation on L related with symmetric (f,g)- biderivation D, if L has a least element 0 such that $f(0) = 0$ and $g(0) = 0$ then $\nabla(0,y) = 0$.

Proof : by proposition 3.6 we have

$$\nabla(x,y) \leq f(x) \vee g(x)$$

And sine 0 is the least element of , then

$$0 \leq \nabla(0,y) \leq f(0) \vee g(0) = 0$$

Hence $\nabla(0,y) = 0$.

Theorem 3.9: let $(L, \wedge, \vee)$ be a lattice and $\nabla(.\,,.): L \times L \longrightarrow L$ be a generalized symmetric (f,g)- biderivation on L related with symmetric (f,g)- biderivation D, then when $g(x) \leq f(x)$ for all $x \in L$, then the following identities are holds $\forall\, x, y, w \in L$.

i) $\nabla(x,y) \wedge \nabla(w,y) \leq \nabla(x \wedge w, y) \leq \nabla(x,y) \vee \nabla(w,y)$.

ii) $\nabla(x \wedge w, y) \leq f(x) \vee f(w)$.

Proof:

i) For all $x, y, w \in L$, we have

$$\nabla(x \wedge w, y) = (\nabla(x, y) \wedge f(w)) \vee (g(x) \wedge D(w, y))$$

Which implies that

$$\nabla(x, y) \wedge f(w) \leq \nabla(x \wedge w, y)$$

And since $\nabla(w, y) \leq f(w)$ for all $\in L$, we have

$$\nabla(x, y) \wedge \nabla(w, y) \leq \nabla(x, y) \wedge f(w)$$

So ,we get

$$\nabla(x, y) \wedge \nabla(w, y) \leq \nabla(x \wedge w, y) \qquad ....(1)$$

Now

$$\nabla(x, y) \wedge f(w) \leq \nabla(x, y)$$

and since $g(x) \wedge D(w, y) \leq D(w, y)$, $D(w, y) \leq \nabla(w, y)$

we get $g(x) \wedge D(w, y) \leq \nabla(w, y)$

hence $(\nabla(x, y) \wedge f(w)) \vee (g(x) \wedge D(w, y)) \leq \nabla(w, y) \vee \nabla(w, y)$

$$\nabla(x \wedge w, y) \leq \nabla(x, y) \vee \nabla(w, y) \qquad ...(2)$$

From (1) and (2)

$$\nabla(x, y) \wedge \nabla(w, y) \leq \nabla(x \wedge w, y) \leq \nabla(x, y) \vee \nabla(w, y)$$

(ii) since $\nabla(x, y) \wedge f(w) \leq f(w)$

and since $g(x) \wedge D(w, y) \leq f(x) \wedge D(w, y) \leq f(x)$

hence $(\nabla(x, y) \wedge f(w)) \vee (g(x) \wedge D(w, y)) \leq f(x) \vee f(w)$

so that $(\nabla(x \wedge w, y) \leq f(x) \vee f(w)$

Proposition 3.10: let $(L, \wedge, \vee)$ be a lattice and $\nabla(.,.): L \times L \longrightarrow L$ be a generalized symmetric (f, g)-biderivation on L related with symmetric (f, g)- biderivation D , if L has a greatest element 1 such that $f(1) = 1$ and $g(x) \leq D(1, y)$ then $\nabla(x, y) \geq D(1, y)$.

Proof :

For all $x, y \in L$

$$\nabla(x, y) = \nabla(x \wedge 1, y)$$

$$= (\nabla(x, y) \wedge f(1)) \vee (g(x) \wedge D(1, y))$$

$$= \nabla(x, y) \vee D(1, y)$$

Hence $\nabla(x, y) \geq D(1, y)$.

Theorem 3.11: let $(L, \wedge, \vee)$ be a modular lattice and $f, g: L \longrightarrow L$ are mappings . Let $\nabla(.,.): L \times L \longrightarrow L$ be a generalized symmetric (f, g)- biderivation on L related with symmetric (f, g)- biderivation D , δ be the trace of ∇ and d be the trace of D . Then

$$\delta(x \wedge y) = \big(\delta(x) \wedge f(y)\big) \vee (g(x) \wedge D(X, y) \wedge f(y)) \vee (g(x) \wedge d(y))$$

For all $x, y \in L$.

Proof :

$$\delta(x \wedge y) = \nabla(x \wedge y, x \wedge y)$$
$$= \big(\nabla(x, x \wedge y) \wedge f(y)\big) \vee \big(g(x) \wedge D(y, x \wedge y)\big)$$
$$= \{[(\nabla(x,x) \wedge f(y)) \vee (g(x) \wedge D(x,y))] \wedge f(y)\}$$
$$\vee \{ g(x) \wedge [(D(x,y) \wedge f(y)) \vee (g(x) \wedge D(y,y))]\}$$

Since $(L, \wedge, \vee)$ is a modular lattice , then

$$\delta(x \wedge y) = [(\delta(x) \wedge f(y)) \vee (g(x) \wedge D(x,y) \wedge f(y))]$$
$$\vee [(g(x) \wedge D(x,y) \wedge f(y)) \vee (g(x) \wedge d(y))]$$

Hence

$$\delta(x \wedge y) = (\delta(x) \wedge f(y)) \vee (g(x) \wedge D(x,y) \wedge f(y)) \vee (g(x) \wedge d(y))$$

For all $x, y \in L$

Corollary 3.12:: let $(L, \wedge, \vee)$ be a modular lattice and $f, g: L \longrightarrow L$ are mappings . Let $\nabla(\,.\,,.): L \times L \longrightarrow L$ be a generalized symmetric (f, g)- biderivation on L related with symmetric (f, g)- biderivation $D$with traces δ and d respectively ,then for all $x, y \in L$

1) $g(x) \wedge D(x,y) \wedge f(y) \leq \delta(x \wedge y)$
2) $g(x) \wedge d(y) \leq \delta(x \wedge y)$
3) $\delta(x) \wedge f(y) \leq \delta(x \wedge y)$.

Proof : (1) , (2) and (3) are easly proved from theorem 3.11 .

Theorem 3.13: let $(L, \wedge, \vee)$ be a distributive lattice, ∇_1 and ∇_2 are generalized symmetric (f, g)- biderivations related to same symmetric (f, g)- biderivation D. Then $\nabla_1 \wedge \nabla_2$ defined by $(\nabla_1 \wedge \nabla_2)(x, y) = \nabla_1(x, y) \wedge \nabla_2(x, y)$ generalized symmetric (f, g)- biderivation related to symmetric (f, g)- biderivation Don L.

Proof :

$$(\nabla_1 \wedge \nabla_2)(x \wedge y, z) = \nabla_1(x \wedge y, z) \wedge \nabla_2(x \wedge y, z)$$
$$= \{(\nabla_1(x,z) \wedge f(y)) \vee (g(x) \wedge D(y,z))\}$$
$$\wedge \{(\nabla_2(x,z) \wedge f(y)) \vee (g(x) \wedge D(y,z))\}$$
$$= \{(\nabla_1(x,z) \wedge f(y)) \wedge (\nabla_2(x,z) \wedge f(y))\} \vee (g(x) \wedge D(y,z))$$
$$= \{(\nabla_1(x,z) \wedge (\nabla_2(x,z)) \wedge f(y)\} \vee (g(x) \wedge D(y,z))$$
$$= \{(\nabla_1 \wedge \nabla_2)(x,z) \wedge f(y))\} \vee (g(x) \wedge D(y,z))$$

Hence $\nabla_1 \wedge \nabla_2$ is a generalized symmetric (f, g)- biderivation related to symmetric (f, g)- biderivation D on a lattice L .

Theorem 3.14: let $(L, \wedge, \vee)$ be a distributive lattice, ∇_1 and ∇_2 are generalized symmetric (f, g)- biderivations related to same symmetric (f, g)- biderivation D. Then $\nabla_1 \vee \nabla_2$ defined by $(\nabla_1 \vee \nabla_2)(x, y) = \nabla_1(x, y) \vee \nabla_2(x, y)$ generalized symmetric (f, g)- biderivation related to symmetric (f, g)- biderivation D on L.

Proof :

$$(\nabla_1 \vee \nabla_2)(x \wedge y, z) = \nabla_1(x \wedge y, z) \vee \nabla_2(x \wedge y, z)$$
$$= \{(\nabla_1(x,z) \wedge f(y)) \vee (g(x) \wedge D(y,z))\}$$
$$\vee \{(\nabla_2(x,z) \wedge f(y)) \vee (g(x) \wedge D(y,z))\}$$
$$= \{(\nabla_1(x,z) \wedge f(y)) \vee (\nabla_2(x,z) \wedge f(y))\} \vee (g(x) \wedge D(y,z))$$
$$= \{(\nabla_1(x,z) \vee (\nabla_2(x,z)) \wedge f(y)\} \vee (g(x) \wedge D(y,z))$$
$$= \{(\nabla_1 \vee \nabla_2)(x,z) \wedge f(y))\} \vee (g(x) \wedge D(y,z))$$

Hence $\nabla_1 \vee \nabla_2$ is a generalized symmetric (f,g)- biderivation related to symmetric (f,g)- biderivation D on a lattice L .

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