Strongly Coretractable Modules and Some Related Concepts

[1] Inaam Mohammed Ali Hadi and [2] Shukur Neamah Al-aeashi

[1] Department of Mathematics, College of Education for Pure Sciences (Ibn-Al-Haitham), University of Baghdad, Iraq

[2] Department of Urban Planning, College of Physical Planning, University of Kufa, Iraq

shukur.mobred@uokufa.edu.iq, Innam1976@yahoo.com

Abstract

Let R be a ring with identity and M be an R-module with unite. The module M is called strongly coretractable module if for each proper submodule N of M, there exists a nonzero R-homomorphism $f:M/N\to M$ such that $\mathrm{Im}f+N=M$. In this paper we investigate some relationships between strongly coretractable module and other related concepts.

Keywords: Coretractable Module, Strongly Coretractable Module, Mono-Coretractable Module, Epi-Coretractable Module, Rickart Module and Nonsingular Module.

1. Introduction

Throughout this paper all modules are unital right R-modules, where R is a ring with identity. An R-module M is called coretractable if for each a proper submodule N of M, there exists a nonzero R-homomorphism $f:M/N\to M$. Equivalently M is called coretractable if for each proper submodule N of M , there exists a nonzero mapping $f\in \mathrm{End}_R(M)$ such that $f(N)=0$; that is $N\subseteq \ker f$ [1]. The class of strongly coretractable is introduced and studied in [12], where an R-module M is called strongly coretractable if for each proper submodule N of M, there exists a nonzero R-homomorphism $f:M/N\to M$ such that $\mathrm{Im}f+N=M$. This paper consists of three sections. In section two, many characterizations of strongly coretractable under certain classes of module as finitely generated Artinian, finitely generated projective, finitely multiplication and Rickart are studied. In section three, we present many characterizations of strongly coretractable under other classes of module such as: finitely generated faithful multiplication and finitely generated faithful module over Noetherian ring also under nonsingular modules.

2. Strongly Coretractable Modules with Rickart Module

In this section , we introduce some results concerned with strongly coretractable modules and Rickart modules where an R-module M is called a Rickart module if for all $f\in S=\mathrm{End}(M)$, $\ker f <^{\oplus} M$ and M is called a dual-Rickart (or a d-Rickart) if for all $f\in S$, $\mathrm{Im}f <^{\oplus} M$ [15] . First, we need to recall that an R-module M is called endoregular if S is von Neumann regular ring [9]. But by [9, Theorem (1.3.27)], S is von Neumann regular ring equivalent to kerf and Imf are direct summand of M, for each $f\in S$. Hence by definitions of Richart and dual Rickart module, M is endoregular coincides with M is Richart and dual Rickart.

Proposition (2.1): Let M be a Noetherian R-module. Then the following statements are equivalent:

(1) M is a strongly coretractable module;

(2) M is a semisimple module;

(3) M is a Rickart and dual Rickart module;

(4) $\mathrm{End}_R(M)$ is a von Neumann regular ring (M is endoregular module).

Proof: **(1⟺2)** Since M is Noetherian module. So M is finitely generated, and hence the result follows by [12, Corollary (2.16)].

(2⟹3), **(3⟺4)** are clear.

(3⟹2) Since M is a dual Rickart and Noetherian, thus M is a semisimple by [9, Proposition (3.3.13), P.105]

Remark (2.2): A Noetherian and Rickart R-module may be not coretractable module. For example, consider Z as Z-module is Noetherian and Rickart module, but it is not coretractable module.

Proposition (2.3): Let R be a commutative Noetherian ring (or R is an Artinian). Then the following statements are equivalent:

(1) R is a strongly coretractable ring;

(2) R is a semisimple ring;

(3) R is a Rickart and dual Rickart ring;

(4) R is a von Neumann regular ring.

Proof: It follows directly by Proposition (2.1).

Corollary (2.4): Let M be a finitely generated R-module over Noetherian ring R. Then the following statements are equivalent:

(1) M is a Rickart and dual Rickart module;

(2) M is semisimple module;

(3) M is strongly coretractable module.

(4) $End_R(M)$ is a von Neumann regular ring.

Proof: Since M is a finitely generated module over Noetherian ring. So M is Noetherian module and hence by Proposition (2.1), the result is obtained

Proposition (2.5): Let M be a finitely generated Artinian R-module over a commutative ring. Then the following statements are equivalent:

(1) M is an endoregular module;

(2) M is a semisimple module;

(3) M is a strongly coretractable module.

Proof: **(1⇔2)** It follows by [9, Proposition (4.2.12), P.125].

(2⇔3) It follows by [12, Corollary (2.16)].

Remark (2.6): The condition M is a finitely generated module cannot be dropped in Proposition (2.5), for example the Z-module $Z_{p\infty}$ is not finitely generated and Artinian strongly cortractable, but it is not Rickart and so it is not endoregular.

Lemma (2.7): Let M be a finitely generated multiplication R-module over commutative von Neumann regular ring R . Then $S=End_R(M)$ is a von Neumann regular ring.

Proof: Since M is a finitely generated multiplication module, hence M is a scalar R-module by [11, Corollary (1.1.11)]. Hence M is a scalar faithful $\bar{R}$-module where$\bar{R}$=R/annM, and by [13, Lemma (6.1)], $End_R(M)\cong$ R/annM which is a von Neumann regular ring.

Corollary (2.8): Let M be a finitely generated multiplication R-module. Then the following statements are equivalent:

(1) M is a coretractable module;

(2) M is a semisimple module;

(3) M is a strongly coretractable module;

Proof: It follows by Lemma (2.7) and [12, Proposition (2.1)].

Proposition (2.9): Let M be a finitely generated projective R-module over von Neumann regular ring. Then the following statements are equivalent:

(1) M is a coretractable module;

(2) M is a semisimple module;

(3) M is a strongly coretractable module.

Proof: By [9, Corollary (2.2.22), P.41], M is a Rickart module satisfying C_2 condition, hence by [9, Corollary (2.2.20), P.41], S=End(M) is a von Neumann regular. Thus, the result follows by [12, Proposition (2.1)].

Recall that " An R-module M is called co-epi-retractable if it contains a copy of any of its factor modules " [6]. However, for more convenient, we call it mono-coretractable module.

Proposition (2.10): Let M be a Rickart R-module. Then M is mono-coretractable module if and only if M is semisimple module.

Proof: **(⟹)** Let K be a proper submodule of M. Since M is mono-coretractable module, then there exists f:M⟶M, f ≠0, f(K)=0 and K=kerf. But M is Rickart module, then $kerf <^{\oplus} M$. Therefore $K <^{\oplus} M$ which implies that M is a semisimple module.

(⟸) It is clear.

Corollary (2.11): Let M be a Rickart R-module with C_2 condition. Then the following statements are equivalent:

(1) M is a mono-coretractable module;

(2) M is a semisimple module;

(3) M is a coretractable module;

(4) M is a strongly coretractable module.

Proof: **(1 ⟺ 2)** By Proposition (2.10).

(2⟺3⟺4) Since M is a Rickart with C_2 condition by [9, Theorem (2.2.20), P.40], $End_R(M)$ is a Von Neumann regular. Hence the statements are equivalent by [12, Proposition (2.1)].

The condition M is Rickart is necessary in Corollary (2.11), for example Z-module Z_4 is not Rickart, Z_4 is coretractable module but it is not strongly coretractable module also it is mono-coretractable module, but it is not semisimple.

Corollary (2.12): Let R be a von Neumann regular ring. Then the following statements are equivalent:

(1) R is a coretractable ring;

(2) R is a semisimple ring;

(3) R is a strongly coretractable ring;

(4) R is a mono-coretractable ring.

Proof: Since R is von Neumann regular ring. Then R is Rickart ring with C_2 condition by [9, Corollary (2.2.12)], and hence we get the result by Corollary (2.11).

Corollary (2.13): Let R be a Rickart ring. Then the following statements are equivalent:

(1) R is a mono-coretractable ring;

(2) R is a semisimple ring;

(3) R is a coretractable ring;

(4) R is a strongly coretractable ring.

Proof: **(1⟺2)** It follows by Proposition (2.10).

(2⟺3) Since R is Rickart ring, hence R is nonsingular by [9, P.24], and so coretractable module coincides with semisimple by [1, Propsition2.3].

(2⟺4) It is clear by [12, Corollary (2.16)], since R is finitely generated.

The following Lemma will be needed in the next Proposition;

Lemma (2.14): Let M be an R-module such that for all proper submodule N of M such that $M/N \cong L \leq^{\oplus} M$, then M is mono-coretractable module.

***Proof*:** Let N and L be submodules of an R-module M. Since $M/N \cong L$, so there exists $\alpha: M/N \to L$, α is an isomorphism. Let $\pi: M \to M/N$ be the natural projection and $j: L \to M$ be the inclusion mapping. Put $\beta = j\alpha\pi$, $\beta \in End_R(M)$, $\beta \neq 0$ and $\ker\beta = N$. Thus, M is a mono-coretractable module.

Recall that an R-module M is called C-Rickart if kerf is closed submodule of M for all $f \in End_R(M)$ [15].

Proposition (2.15): Let M be a C-Rickart module such that for all N proper submodule of M and $M/N \cong L \leq^{\oplus} M$. Then M is a semisimple and hence M is strongly coretractable module.

***Proof*:** Let N be a proper submodule N of M such that $M/N \cong L \leq^{\oplus} M$, then M is mono-coretractable module by Lemma (2.14) and hence there exists $\beta \in End_R(M)$, $\beta \neq 0$ such that $\ker\beta = N$. But M is C-Rickart module, So N is closed submodule, thus every submodule of M is closed which implies M is semisimple by [2, Exercise(6(c)), P.139]. Thus, M is a strongly coretractable module.

Recall that an R-module M is called a strongly Rickart if and only if kerf is fully invariant direct summand for all $f \in End_R(M)$ [10]. Recall that a submodule N of a module M is called stable if for each $f \in Hom(N, M)$, $f(N) \subseteq N$ and M is called fully stable if every submodule of M is stable [41].

Proposition (2.16): Let M be a strongly Rickart R-module. Then M is mono-coretractable module if and only if M is fully stable semisimple module and hence strongly coretractable module.

***Proof*:** **(⟹)** Since M is strongly Rickart module, $\ker\varphi$ is stable direct summand submodule for each $\varphi \in End_R(M)$ by [10, Corollary (1.19)]. But M is mono-coretractable module, so for each proper submodule N of M, there exists $\varphi \in End_R(M)$ such that $N = \ker\varphi$, thus N is a stable direct summand. Therefore, M is a fully stable and semisimple.

(⟸) If M is semisimple, then it is clear that M is mono-coretractable module.

3. More About Strongly Coretractable Modules and Related Concepts

In this section, more properties about strongly coretractable module and related concepts are introduced such as finitely generated faithful multiplication, nonsingular, projective, dual-Baer and κ-nonsingular modules.

Proposition (3.1): Let M be a finitely generated faithful multiplication R-module. Then the following statements are equivalent:

(1) R is a strongly coretractable ring;
(2) M is a strongly coretractable module;
(3) M is a semisimple module;
(4) R is a semisimple ring.

***Proof*:** **(1⟹2)** Since R is strongly coretractable ring. R is a semisimple. Thus, M is semisimple R-module by [2, Corollary (8.2.2), P.196], and hence M is a strongly coretractable module.

(2⟹1) Let M be a strongly coretractable module. Since M is finitely generated module, then M is semisimple module by [12, Corollary (2.16)]. Now, Let I be an ideal of R, then MI is a submodule of M and so $MI \oplus W = M$ for some $W \leq M$. But W=MJ for some ideal J in R, since M is a multiplication module. Thus $MI \oplus MJ = M(I \oplus J) = MR$ and as M is finitely generated faithful multiplication module. $I \oplus J = R$; that is, I is a direct summand of R and hence R is semisimple ring. Therefore, R is strongly coretractable ring.

(2⟺3) and **(1⟺4)** It follows by [12, Corollary (2.16)].

Recall that an R-module M is said to be regular (sometimes called F-regular) if R/ann(x) is regular ring for all nonzero $x \in M$ [8, P.29]. Equivalently, an R-module M is said to be regular (F-regular) if every submodule of M is a pure submodule [8, Theorem (1.7), P.35], where a submodule N of M is pure if $MI \cap N = NI$ for each ideal I of R " [2].

Corollary (3.4): Let M be a finitely generated faithful multiplication R-module. If M is strongly coretractable module, then M is F-regular module and R is von Neumann regular ring.

Proof: Since M is a finitely generated and strongly coretractable R-module. Then M is semisimple R-module and so R is semisimple ring by Proposition (3.1), then M is F-regular module and R is von Neumann regular by [8, Proposition2.8, p.41].

Remark (3.5): The condition that M is finitely generated is necessary in Corollary (3.4). For example, $M=Z_{p\infty}$ as Z-module is a strongly coretractable, but it is not finitely generated and not multiplication. But R=Z is not regular ring , also $M=Z_{p\infty}$ is not F-regular module , since if I=PZ and N=<1/P + Z> , $IZ_{p\infty} \cap N = PZ_{p\infty} \cap < 1/P + Z > = Z_{p\infty} \cap < 1/P + Z > = <1/P + Z>$. But $IN = PZ< 1/P + Z > = 0_M$.

Theorem (3.6): Let M be a finitely generated faithful R-module over Noetherian ring R. Then the following statements are equivalent:

(1) M is a strongly coretractable module;
(2) M is a semisimple module;
(3) R is a regular ring;
(4) M is an F-regular module;
(5) R is a semisimple ring;
(6) R is a strongly coretractable ring.

Proof: **(1⟺2**) and **(5⟺6)** follow by [12, Corollary (2.16)].

(2⟹4) It follows by [8, Proposition (2.8), P.41].

(4⟺3) Since M is a finitely generated module, so by [8, Theorem1.10, P37], M is F-regular if and only if R/annM is regular and hence M is F-regular if and only if R is regular since annM=0.

(5⟹3) It is clear.

(3⟹5) Since R is Noetherian, and R is a regular ring, then R is a semisimple ring. **(5⟹1)** Since R is semisimple ring. Then M is semisimple R-module by [2, Corollary (8.2.2), P.196], and so M is strongly coretractable module

Theorem (3.7): Let M be a finitely generated faithful R-module over local ring R. Then the following statements are equivalent:

(1) M is a strongly coretractable module;
(2) M is a semisimple module;
(3) M is an F-regular module;
(4) R is a regular ring.

Proof: **(1⟺2)** Clear.

(2⟹3) and **(3 ⟹2)** By Proposition (2.8) and Proposition (2.9) in [8, P.41].

(3⟺4) Since M is a finitely generated module and $ann_R(M)=0$, then R=R/annM is regular if and only if M is F-regular module by [8, Theorem (1.10), P.37].

Proposition (3.8): Let R be a local ring. Then the following conditions are equivalent:

(1) Every R-module is strongly coretractable;
(2) Every finitely generated R-module is strongly coretractable;
(3) Every finitely generated R-module is semisimple module;
(4) R is semisimple ring;
(5) Every R-module is regular module.

Proof: (**1⟹2⟹3⟹4)** It is clear.

(4⟹5) Since R is semisimple, so every R-module is semisimple by [2, Corollary (8.2.2), P.196], and hence every R-module is F-regular by [8, Proposition (2.8), P.41].

(5⟹1) Since every F-regular R-module over a local ring is semisimple by [8, Proposition (2.8), P.41], and so every semisimple R-module is strongly coretractable.

Theorem (3.9): Let M be an R-module such that M has a non-torsion element (That is there exists x∈M such that ann(x)=0) and R is Noetherian ring. Then the following statements are equivalent:

(1) R is a Von Neumann regular ring;
(2) M is an F-regular module;
(3) R is a semisimple ring;
(4) R is a strongly coretractable ring.

Proof: **(1⟹2)** Since R is Von Neumann regular ring. Then M is F-regular module by [8, Remark (1.2(2)), P.29].

(2⟹1) Since M is F-regular module. So, R/ann(x) is Von Neumann regular ring for all x∈M. But M has a non-torsion element, then ann(x)=0 for some x ∈ M. Therefore, R is a Von Neumann regular ring.

(1⟹3) Since R is Von Neumann regular ring and Noetherian ring, then R is semisimple ring.

(3⟹2) It is clear, since every module over semisimple ring is semisimple module by [2, Corollary (8.2.2), P.196], and hence M is a regular module.

(3⟺4) Clear.

Corollary (3.10): Let M be a free module over a Noetherian ring R. Then the following statements are equivalent:

(1) R is a Von Neumann regular ring;
(2) R is a semisimple ring;
(3) R is a strongly coretractable ring;
(4) M is a F-regular module.

Proof: It is clear by Theorem (3.9).

Recall that a module M is called continuous if every submodule of M is essential in a direct summand of M and every submodule L of M is isomorphic to a direct summand of M, then L is a direct summand of M [9].

Recall that an R-module M is called epi-retractable if every submodule of M is a homomorphic image of M [4].

Proposition (3.11): Let M be a nonsingular R-module. Then the following statements are equivalent:

(1) M is a coretractable module;
(2) M is a semisimple module;
(3) M is a strongly coretractable module;
(4) M is a continuous and epi-retractable module;
(5) M is a mono-coretractable module.

Proof: **(1⟺2)** By [1, Corollary (2.4)].

(2⟹3) and (**3⟹1)** are clear.

(2⟺4) It follows by [4, Proposition 3.1].

(2⟺5) Since M is semisimple module. Then it is clear that M is mono-coretractable module

Proposition (3.12): Let R be a nonsingular ring and M be a projective R-module. Then the following statements are equivalent:

(1) M is a coretractable module;
(2) M is a semisimple module;
(3) M is a strongly coretractable module;
(4) M is a mono-coretractable module.

Proof: (**1⟺2)** By [1, Corollary 2.4].

(2⟹3) and **(3⟹1)** Clear.

(3⟹4) Since R is nonsingular ring and M is projective R-module, then M is nonsingular module. But M is strongly coretractable module, so M is semisimple module by Proposition (3.11). Therefore, M is mono-coretractable module.

(4⟹3) Since M is mono-coretractable module implies M is coretractable. But M is nonsingular module. Thus, M is semisimple by [1, Corollary (2.4)]. Therefore, M is strongly coretractable.

Proposition (3.13): Let M be a quasi-injective R-module and $J(End_R(M)) = 0$. Then the following statements are equivalent:

(1) M is a coretractable module;
(2) M is a semisimple module;
(3) M is a strongly coretractable module.

Proof: **(1⟹2)** Since M is a coretractable module, every nonzero proper submodule N of M is not quasi-invertable submodule, then by [7, Theorem (3.8), P.17], N is not essential submodule of M. Thus, M has no proper essential submodule. Therefore, M is a semisimple module.

(2⟹3) and (3⟹1) are clear

Recall that a ring R is a completely coretractable ring (denoted by CC-ring) if every R-module is coretractable [1].

Proposition (3.14): Let R be a ring. Then the following statements are equivalent:

(1) $\oplus_{i\in I} R_i$ is a strongly coretractable ring for every index set I, where $R_i = R$ for all $i\in I$.
(2) Every projective R-module is a strongly coretractable.
(3) R is a strongly coretractable ring;
(4) R is a semisimple ring;
(5) Each R-module M is a strongly coretractable;
(6) Each proper ideal I in R, annI=eR for some nonzero idempotent element e in R.
(7) All R-modules are nonsingular modules and all R-modules are coretractable; that is R is a CC-ring;
(8) R is a nonsingular and coretractable ring.

Proof: **(1⟹2)** Let M be a projective R-module. Then there exists a free R-module F such that f: F→M is an epimorphism. Since F is a free module, then $F \cong \oplus_{i\in I} R_i$, $R_i = R$. But by hypothesis $\oplus_{i\in I} R_i$ is strongly coretractable module. Thus, F is strongly coretractable module, and so M is strongly coretractable module.

(2⟹1), (5⟹3), (4⟹6), (7⟹8) and **(3⟺4)** are clear.

(4⟹5) Since R is semisimple ring. Then every R-module is semisimple by [2, Corollary (8.2.2), P.196] implies every R-module is strongly coretractable. **(6⟹4)** Let I be a proper ideal of R. Since 1=e+(1-e), then $eR\oplus(1-e)R = R$. Then it easy to check that I=(1-e) R and so $I <^{\oplus} R$. Therefore, R is semisimple.

(4⟹7) Since R is a semisimple ring. Then every R-module is a nonsingular, also R is semisimple implies that every R-module is a coretractable.

(8⟹1) It follows by proposition (3.11).

Recall that an R-module M is called dual-Baer if for a submodule N of M, there exists an idempotent e in $S=End_R(M)$ such that D(N) = eS where D(N)= {f∈S: Imf⊆N} [9].

Proposition (3.15): Let M be a dual-Baer and nonsingular R-module. Then the following statements are equivalent:

(1) M is a retractable module;
(2) M is a coretractable module;
(3) M is a semisimple module;
(4) M is a strongly coretractable.

Proof: **(1⟹3)** By [5, Corollary (2.19)].

(3⟹1) It is clear for all proper submodule N of M, $N <^{\oplus} M$. Then N⊕W = M for some submodule W of M, so there exists f: N→M, f(n)=(n,0). Then M is retractable module.

(2⟺3⟺4) Since M is nonsingular module, so the result follows by Proposition (3.11).

Recall that an R-module M is called purely Baer if for each left ideal I of $S=End_R(M)$, $r_M(I)$ is a pure submodule of M [14].

Proposition (3.16): Let M be a finitely generated over principal ideal domain. Then the following statements are equivalent:

(1) M is a purely Baer module;
(2) M is a semisimple;
(3) M is a strongly coretractable.

Proof: **(1⟺2)** It follows by [14, Theorem (3.13)] and **(3⟺2)** It follows by [12, Corollary (2.16)].

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