

DOI: <https://doi.org/10.24297/jam.v17i0.8498>**A Classical Geometric Relationship That Reveals The Golden Link in Nature**

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Abstract

This paper introduces the perfect complementary relationship between the $1:2:\sqrt{5}$ right triangle and the 3-4-5 Pythagorean triangle. The classical geometric intimacy between these two right triangles not only provides for the ultimate geometric substantiation of Golden Ratio, but it also reveals the fundamental Pi:Phi correlation ($\pi:\varphi$), with an extreme level of precision, and which is firmly based upon the classical geometric principles.

Keywords: Golden Ratio, Fibonacci sequence, Pi, Phi, Pythagorean triangle, Golden triangle, Divine Proportion

Introduction

One of the greatest geometers of all times; Johannes Kepler had mentioned "The Two Great Treasures of Geometry", namely, "The Theorem of Pythagoras" and "The Golden Ratio", the former he compared to 'a measure of gold', and the later he named 'a precious jewel'! And, both of them were brought together in Kepler's triangle.

However, Kepler's golden postulate is now observed to be truly corroborated by the classical geometric relationship between 3-4-5 Pythagorean triple and $1:2:\sqrt{5}$ triangle, which is found to be the very origin of the Golden Ratio in geometry.

The right angled triangle $1:2:\sqrt{5}$, with its catheti in ratio 1:2, is described in this paper, to possess several unique geometric features, which make it a perfect complementary triangle for the 3-4-5 Pythagorean triple.

The astonishing properties of this $1:2:\sqrt{5}$ triangle also prove it to be the very origin of the Golden Ratio (φ) in geometry. Moreover, this triangle provides a very important link between π and φ , for example; as an "extremely precise" and stunning relationship, the ratio of the area of $1:2:\sqrt{5}$ triangle to the area of its Incircle is exactly $\frac{\varphi^4}{\pi}$; where φ is the Golden Ratio, and this is **the ultimate geometric substantiation of φ** , and this is **the extremely precise Pi:Phi correlation**, premised upon classical geometric ratios, revealed by this unique $1:2:\sqrt{5}$ triangle.

This paper introduces the intimate geometric relationship between the $1:2:\sqrt{5}$ triangle and the 3-4-5 triangle, which is the first of the primitive Pythagorean triples. These two right angled triangles, with all their corresponding sides and angles collaborating with each other, conjointly provide for the ultimate geometric substantiation of Golden Ratio, in a phenomenal way described in this paper. Also, the confluence of these two triangles disclose the precise geometric correlation between Pi(π) and the Golden Ratio (Phi; φ). The classical correspondence between $1:2:\sqrt{5}$ triangle and the first primitive Pythagorean triple provides the most precise method for the mathematical determination of the Pi:Phi correlation, firmly premised upon classical geometric principles.

Hence, the aim of this paper is to introduce the peculiar relationship between two very special right angled triangles, which in turn reveals the august link between the two most important constants in the geometry of nature; the π and the φ .

The paper presents the author's investigations and findings.

The 1:2: $\sqrt{5}$ triangle:

The right angled triangle, with its catheti in ratio 1:2, is observed to possess several peculiar geometric properties. Noticeably, three sides of this 1:2: $\sqrt{5}$ triangle provide the fractional expression for Golden Ratio: $\frac{1+\sqrt{5}}{2}$; which is the solution to the quadratic equation $x^2 - x - 1 = 0$.

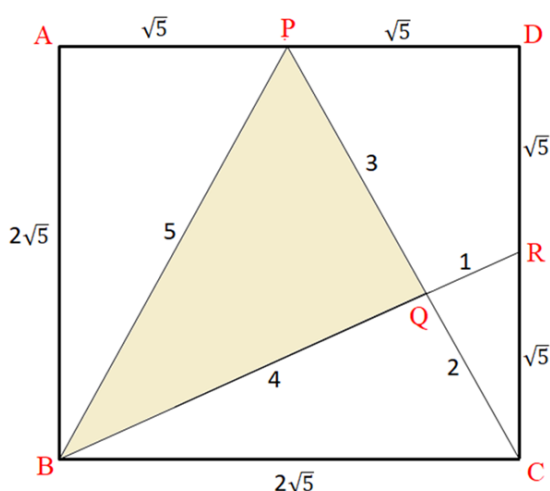
A perfect classical geometric relationship is observed between this 1:2: $\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple, and it is elaborated in this paper, on basis of following precise points;

- 1) Concomitant Formation of the two triangles
- 2) Classical intimacy between the angles of two triangles
- 3) Side Lengths Synergism; and the Golden Ratio embedded in
- 4) The Fibonacci-Pythagorean Triples
- 5) π and φ : The General Correlation
- 6) π and φ : The Special Correlation

Formation of the Two Triangles

As a well-known geometric method of producing 3-4-5 triangle, a square can be dissected by three line segments, as shown here, in figure 1. Remarkably, such dissection of a square invariably produces multiple 1:2: $\sqrt{5}$ triangles alongside the central Pythagorean triple.

In square ABCD, points P and R are the midpoints of side AD and DC, respectively. Connecting point P to the vertices B and C, and point R to the vertex B, produces the triangle PBQ in centre; which is a 3-4-5 Pythagorean triangle. And remarkably, all other triangles formed in the figure, namely, ΔCQR , ΔCQB , ΔAPB , ΔCDP and ΔCBR ; all are 1:2: $\sqrt{5}$ triangles of various sizes.



Pythagorean Triple ΔPBQ is formed in Centre;
All other triangles formed are 1:2: $\sqrt{5}$ triangles

Figure 1: Dissecting the Square; the triangle PBQ produced in centre is the 3-4-5 Pythagorean triple, and all other triangles produced are 1:2: $\sqrt{5}$ triangles of various sizes.

The abovementioned method of the dissection of a square can divide any square into a 3-4-5 Pythagorean triple and multiple $1:2:\sqrt{5}$ triangles. And the square, so dissected, possess several unique geometric properties, as mentioned below.

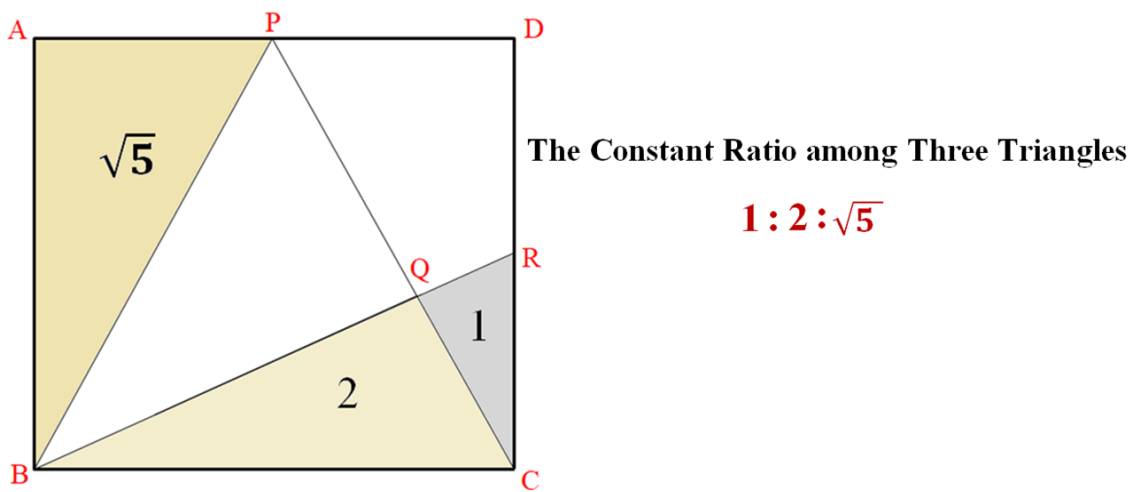


Figure 2: The constant ratio $1:2:\sqrt{5}$ between the Sides, Perimeters, Inradii and Circumradii of three triangles

The square, so dissected, consists of following three types of geometric figures;

1. The 3-4-5 Pythagorean triangle ΔPQB
2. The $1:2:\sqrt{5}$ triangles, namely, ΔCQR , ΔCQB and ΔAPB .

Remarkably, these three $1:2:\sqrt{5}$ triangles are in a definite proportion with each other. They have their corresponding sides, and hence also their perimeters, as well as the their inradii and circumradii; all in the **ratio $1:2:\sqrt{5}$** , as shown above in Figure 2.

3. And, the irregular quadrilaterals $\square ABQP$ and $\square PQRD$.

The internal angles of these quadrilaterals, beside two opposite right angles, as shown below in Figure 3, are

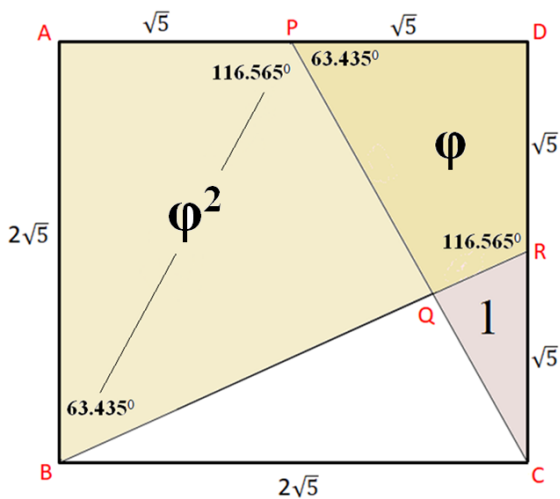
angle $ABQ =$ angle $DPQ = \underline{63.435^\circ}$,

and angle $APQ =$ angle $DRQ = \underline{116.565^\circ}$,

which are remarkably same as **“the angles of the Golden Rhombus”**.

Also, it is conspicuous in Figure 3, that the quadrilateral $ABQP$ has the same internal angles and same shape as $\square PQRD$. This quadrilateral $ABQP$ consists of the 3-4-5 Pythagorean triangle PQB and an equivalent sized $1:2:\sqrt{5}$ triangle PAB , having their common hypotenuse PB .

Also noticeably, the perimeters of ΔCQR , $\square PQRD$ and $\square ABQP$ are invariably in the proportion $1: \varphi : \varphi^2$, where φ is the Golden Ratio.



**Perimeters of
1:2:√5 Triangle & the Quadrilaterals
Invariably 1 : φ : φ²**

Figure 3: The 1:2:√5 triangle & the two Quadrilaterals: the Constant Perimeters’ Ratio

In such dissected square, the bigger quadrilaterals, namely, ABQP, ABRD and CBPD also have same internal angles as the Golden Rhombus 63.435° & 116.565°.

And, while the perimeters of ΔCQR, □PQRD and □ABQP are invariably in the proportion 1 : φ : φ², each of the bigger quadrilaterals has the comparative perimeter of 10φ .

Also, the bigger quadrilaterals, □ABQP, □CBPD and □ABRD, have Golden Ratio embedded in their side lengths, such as for □ABRD; $\frac{AD+DR+BR}{AB} = \phi^2$

All these 1:2:√5 triangles, the irregular quadrilaterals □PQRD and □ABQP, and the square □ABCD as a whole, are full of the Golden Ratios, embedded in their very structure; which will be elaborated subsequently.

The 1:2:√5 Triangle and the First Pythagorean Triple: “A Classical Geometric Relationship” :

The 1:2:√5 triangle is not just a geometric structure formed alongside the 3-4-5 triangle in the abovementioned dissected square, but it is observed that there exists a precise complementary relationship between these two right angled triangles. The word ‘complementary’ here, should not be confused with the ‘complementary angles’ which add up to 90 degrees. Here, the ‘complementary relationship’ between 1:2:√5 triangle and Pythagorean triple signifies the classical synergism between the two triangles that culminates in unconventional geometric outcomes. All corresponding angles and sides of these two triangles, synergize with each other to reflect the Golden Ratio, as follows.

The angles of the 1:2:√5 triangle and the 3-4-5 triangle are closely associated with each other, and they add up to very peculiar values.

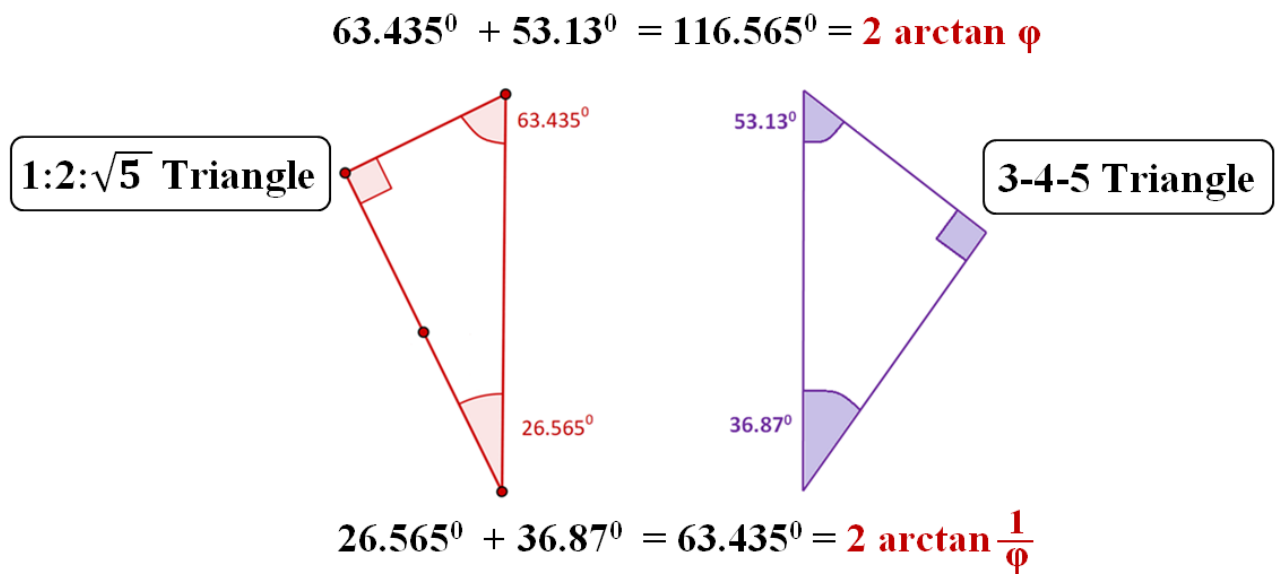


Figure 4: The corresponding angles of 1:2:√5 triangle and Pythagorean triangle add up to reflect the precise Golden Ratio

As shown in the figure 4, the corresponding angles of the two triangles add up, to reflect **the Golden Ratio**;

$$\frac{63.435^{\circ} + 53.13^{\circ}}{2} = \arctan \phi$$

and also,
$$\frac{26.565^{\circ} + 36.87^{\circ}}{2} = \arctan \frac{1}{\phi}$$

Noticeably, the corresponding angles of the two triangles add up to the **angles of Golden Rhombus**, whose diagonals are perfectly in Golden Ratio;

$$26.565^{\circ} + 36.87^{\circ} = 63.435^{\circ}$$

and,
$$53.13^{\circ} + 63.435^{\circ} = 116.565^{\circ}.$$

The close correspondence between the angles of 1:2:√5 triangle and Pythagorean triangle is also reflected in the following interesting relationships. Two smaller acute angles of these two right triangles (26.565° & 36.87°) add up to angle 63.435° of 1:2:√5 triangle, while the angle 53.13° of Pythagorean triangle is twice the angle 26.565° of 1:2:√5 triangle. Moreover, the angle 53.13° is the Supplementary angle for twice of the angle 63.435° of 1:2:√5 triangle, and the angle 36.87° of 3-4-5 Pythagorean triangle is the Complimentary angle for twice of the angle 26.565° of 1:2:√5 triangle.

Also, remarkably, $53.13^{\circ} + 63.435^{\circ} = 116.565^{\circ}$ is also the dihedral angle of Regular Dodecahedron.

Such peculiar values of the angles of 1:2:√5 triangle and the 3-4-5 triangle, and their interconnectedness are bound to produce the outstanding results, as follows. The following Figure 5 shows the 3-4-5 triangle ADC and equivalent sized 1:2:√5 triangle ABC, with their common hypotenuse AC, and surprisingly the figure reveals the Golden Ratio, precisely embedded in this blend of the two triangles.

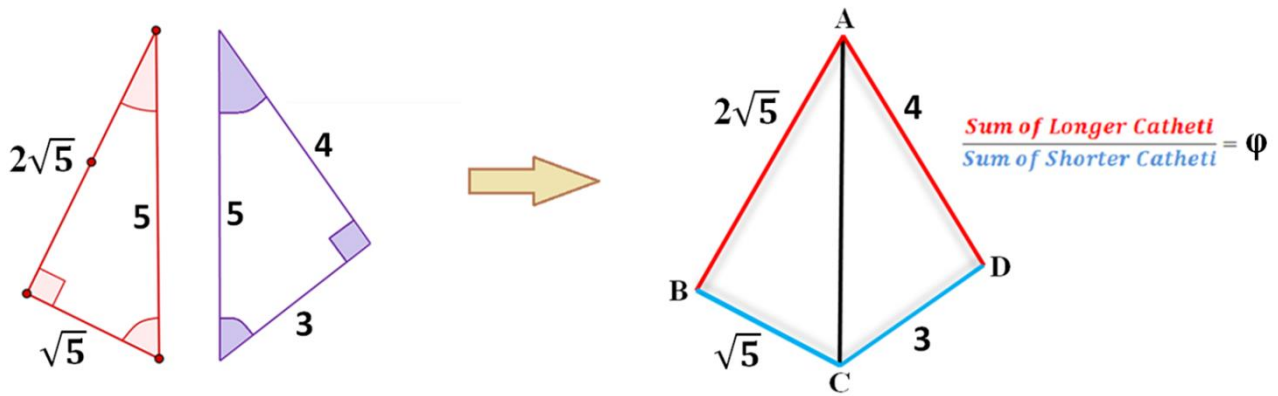


Figure 5: The Golden Ratio in Confluence of $1:2:\sqrt{5}$ triangle and 3-4-5 triangle.

In quadrilateral ABCD,

$$\frac{\text{The Sum Of the Longer Catheti of the Two Triangles}}{\text{Sum of their Shorter Catheti}} = \frac{AB+AD}{BC+CD} = \phi$$

Noticeably, the above quadrilateral ABCD formed by the merger of $1:2:\sqrt{5}$ triangle and Pythagorean triple, is exactly same as $\square ABQP$ in Figures 1, 2 and 3.

The close association between the corresponding angles and sides of the $1:2:\sqrt{5}$ triangle and the Pythagorean triple is also reflected in the dissection of the two triangles. Each of the two triangles can be dissected into its complementary triangle and self-similar smaller triangle.

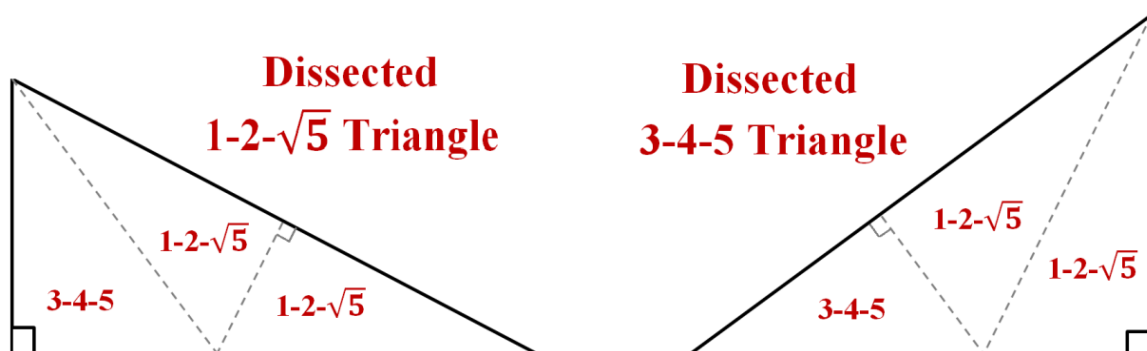


Figure 6: Dissection of two triangles: into Complementary and Self-Similar Triangles.

Moreover, the Golden Ratio is not just restricted to above blend of the two triangles, but it is all embedded in all geometric formations in the dissected square in figures 1,2 and 3: in all the $1:2:\sqrt{5}$ triangles, and in the combinations of triangles, and also in the irregular quadrilaterals.

As shown in Figure 7 below, the two Quadrilaterals $\square ABQP$ and $\square DRQP$ possess Golden Ratio, as:

$$\frac{BA+BQ}{PA+PQ} = \frac{PQ+PD}{QR+DR} = \phi$$

Also, the five $1:2:\sqrt{5}$ triangles, namely, ΔCQR , ΔCQB , ΔAPB , ΔCDP and ΔCBR retain Golden Ratio in their side lengths, as: $\frac{\text{Hypotenuse} + \text{Shorter Cathetus}}{\text{Longer Cathetus}} = \phi$

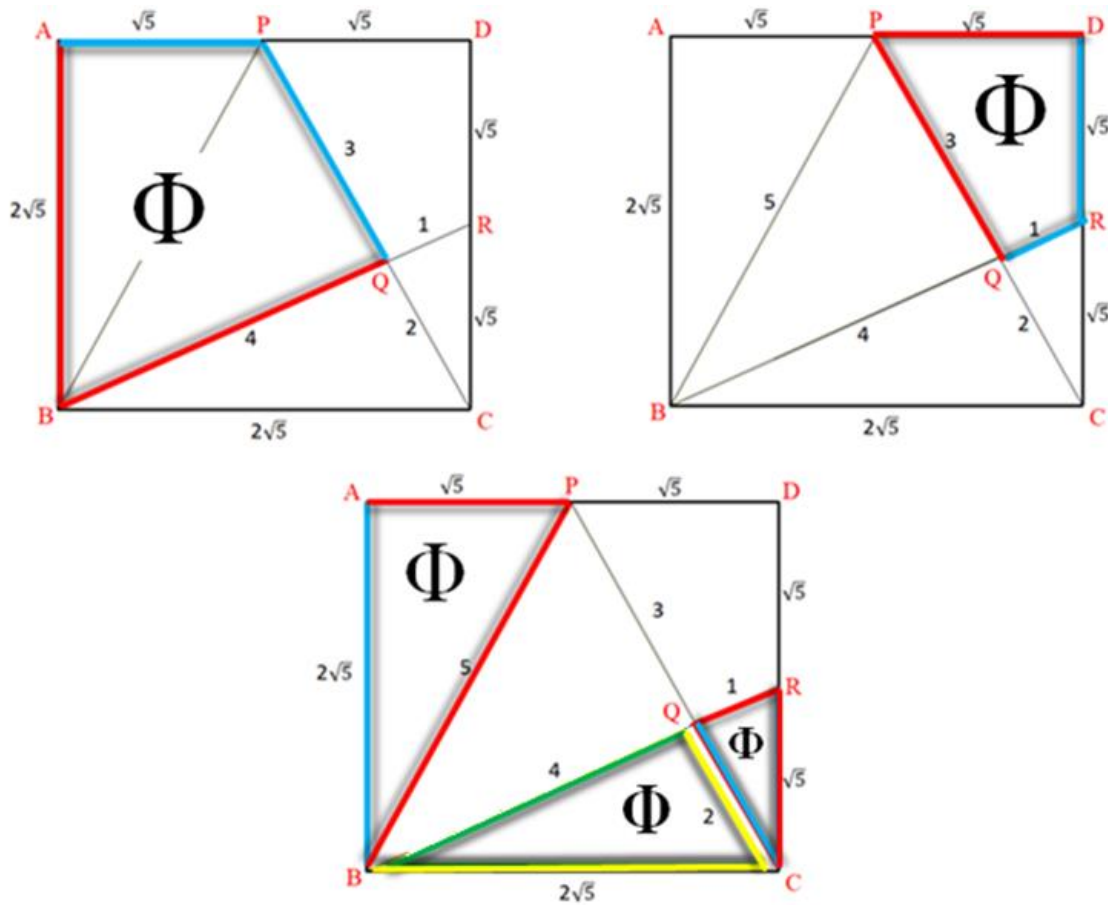


Figure7: Golden Ratio embedded in the formations.

Furthermore, certain geometric features emphasize the close correspondence between the angles of $1:2:\sqrt{5}$ triangle and the Pythagorean triple. Consider $\square ABQP$ in following Figure 8, the quadrilateral $ABQP$ is formed by the merger of Pythagorean triangle PQB and equivalent sized $1:2:\sqrt{5}$ triangle ΔPAB , with their common hypotenuse PB which is also the first diagonal of $\square ABQP$. Interestingly, if the second diagonal AQ is incorporated in $\square ABQP$, the AQ divides the right angle in each triangle into two smaller angles of its complementary triangle.

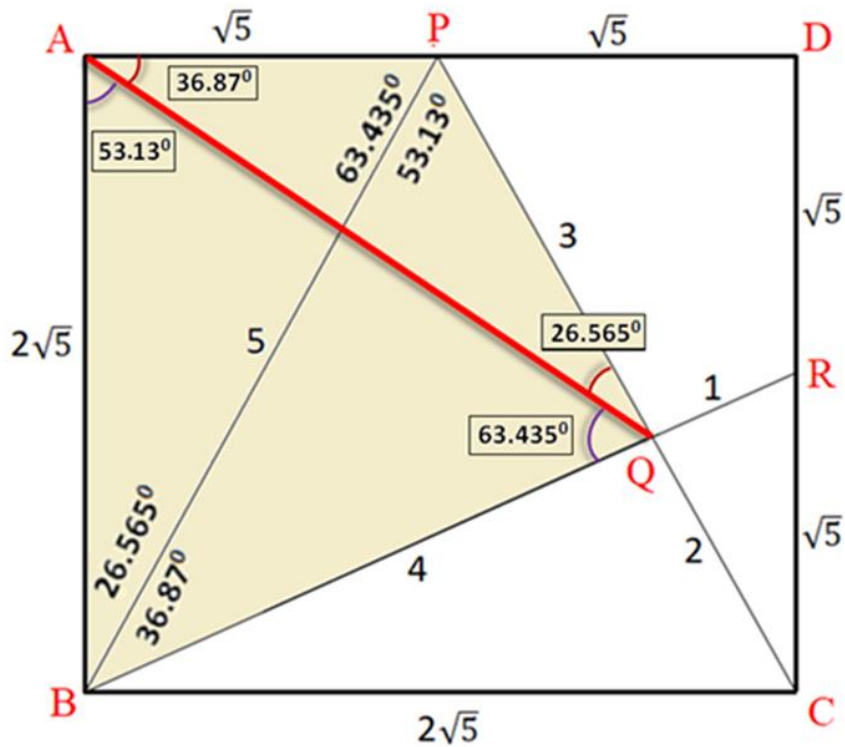


Figure 8 : The second diagonal AQ dividing the right angle of each triangle into reciprocal smaller angles.

Diagonal AQ divides right angle PAB of $1:2:\sqrt{5}$ triangle, into reciprocal smaller angles, 53.13° and 36.87° , which are the angles of Pythagorean triangle; and vice versa.

The Fibonacci Pythagorean Triples:

The idiosyncratic geometry of the $1:2:\sqrt{5}$ triangles is also high lightened by the well-known methods of generating Pythagorean triples from Fibonacci and Fibonacci-like sequences. And noticeably, the Pythagorean triples formed using the Fibonacci or Fibonacci-like sequence, tend to approach the $1:2:\sqrt{5}$ triangles. The Pythagorean triples so derived from Fibonacci series, with the alternate Fibonacci numbers, starting from 5, as the hypotenuse, approach the $1:2:\sqrt{5}$ triangles proportions, as the series advances.

Following method can generate infinite number of the Pythagorean triples from Fibonacci sequence.

The Alternate Fibonacci Number as Hypotenuse, for example, say **13**

Sum of all three sides of previous triple gives Longer Cathetus: $3+4+5 = \mathbf{12}$

The missed Fibonacci number before 13 minus the Shorter Leg of previous triple gives the Shorter Cathetus: $8-3 = \mathbf{5}$; and hence, the triple formed is **5-12-13**.

The successive Fibonacci-Pythagorean Triples, with $3-4-\mathbf{5}$ as the first of them, and every second Fibonacci number as hypotenuse, form the series of Pythagorean triples.

starting with 3-4-5 as the first triple, invariably approaches the $1:2:\sqrt{5}$ triangles proportions, and hence endorse the classical geometric relationship between $1:2:\sqrt{5}$ triangles and 3-4-5 triangle revealed in this paper.

Such Fibonacci-Pythagorean triples inevitably approaches $1:2:\sqrt{5}$ triangle ratios. As such series advances, like **5**-4-3, **13**-12-5, **34**-30-16, **89**-80-39, **233**-208-105, **610**-546-272 and so on, the triples so formed approach $1:2:\sqrt{5}$ triangle proportions, and attain the side length ratio **$1:2:\sqrt{5}$** , exactly in the same manner as the ratio between consecutive Fibonacci numbers, $\frac{F_n}{F_{n-1}}$ approaches the Golden Ratio, $\lim_{n \rightarrow \infty} \frac{F_n}{F_{n-1}} \cong \phi$. And hence, it clearly endorses: while ϕ is the Golden Ratio in nature, the **$1:2:\sqrt{5}$ triangle** is truly the Golden Trigon in geometry, in every sense of the term.

The $1:2:\sqrt{5}$ Triangle and Pi(π): Phi(ϕ) Correlation

The $1:2:\sqrt{5}$ triangle is a very special triangle with the Golden Ratio embedded in all its geometry, and it is also the perfect complementary triangle for the 3-4-5 Pythagorean triple. Moreover, this unique triangle also provides the geometric link between nature's two most important constants, namely, the Pi(π) and the Golden Ratio Phi(ϕ), in following couple of manners.

Noticeably, a coincidental but very peculiar relationship between π and ϕ is revealed by **the ratios between the areas of $1:2:\sqrt{5}$ triangle and equivalent sized Pythagorean triangle**. The 3-4-5 Pythagorean triangle, when merged with equivalent sized $1:2:\sqrt{5}$ triangle, having their **common hypotenuse**, as in Figure 5, or as in quadrilateral ABQP in Figures 1, 2 and 3, reveals the Pi:Phi correlation, thus further ratifying the geometric intimacy between the two triangles.

$$\frac{\text{Area of } 1:2:\sqrt{5} \text{ triangle}}{\text{Area of equivalent sized Pythagorean Triangle}} = \frac{5}{6} \approx \frac{\phi^2}{\pi}$$

this gives the Pi:Phi correlation as **$5\pi \approx 6\phi^2$** , which is precise up to four decimal places.

Thus, remarkably, the ratio between areas of 3-4-5 triangle and $1:2:\sqrt{5}$ triangle, having hypotenuse of same length, equals the ratio between π and ϕ^2 , which further authenticates the complementary relationship between two triangles.

More importantly, just as the side lengths of $1:2:\sqrt{5}$ triangle provide the fractional expression for the Golden Ratio as $\frac{1+\sqrt{5}}{2}$, they can also approximate the value of Pi as;

$$(\sin 63.435^\circ + \sin 26.565^\circ) + (\sin 63.435^\circ + \sin 26.565^\circ)^2 \approx \pi$$

and it gives the exactly same value as the above equation **$5\pi \approx 6\phi^2$**

In other words, $\frac{5}{6} [(\sin 63.435^\circ + \sin 26.565^\circ) + (\sin 63.435^\circ + \sin 26.565^\circ)^2] = \phi^2$ precisely.

The Incircle and the $\pi:\phi$ Special Relationship:

The above equation **$5\pi \approx 6\phi^2$** is impressively precise up to four decimal places. However, the unique and magnificent geometry of $1:2:\sqrt{5}$ triangle enables it to reveal **very precise and entirely accurate Pi:Phi**

correlation. If the incircle is incorporated into the 1:2:√5 triangle, the following extremely precise relation is observed between the area of 1:2:√5 triangle and the area of its incircle;

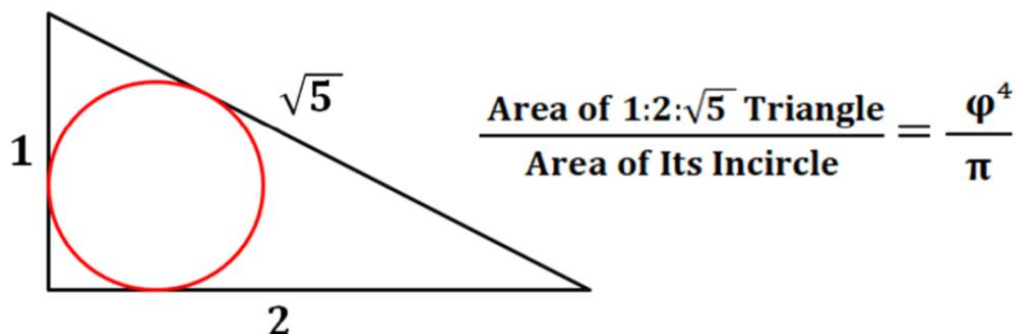


Figure 9: Incircle of 1:2:√5 triangle and Special π:φ Relationship

Here, $\frac{\text{Area of } 1:2:\sqrt{5} \text{ triangle}}{\text{Area of its Incircle}} = \frac{\phi^4}{\pi}$

and this is **the ultimate geometric substantiation of φ**, and this is **the extremely precise Pi:Phi correlation, based upon classical geometry**, as revealed by the unique 1:2:√5 triangle.

Moreover, this “1:2:√5 triangle specific” precise relationship is also observed with a couple of 1:2:√5 triangles, having one of their catheti in common. Two equal sized 1:2:√5 triangles, having **common longer cathetus AD**, show the following precise relation;

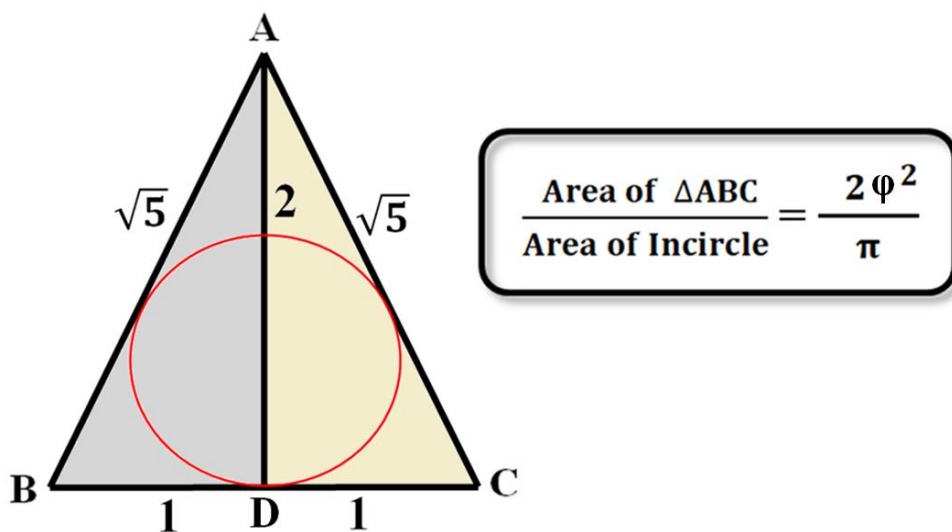


Figure 10: Two 1:2:√5 triangles, with Common Longer Leg AD

Here, $\frac{\text{Area of combined Triangle } ABC}{\text{Area of Incircle}} = \frac{2\phi^2}{\pi}$

Similarly, if two 1:2:√5 triangles have their **common shorter cathetus**, the ratio of the area of combined triangle so formed, with the area of its Incircle is precisely: $\frac{\text{Area of combined Triangle}}{\text{Area of Incircle}} = \frac{\phi^6}{2\pi}$

Finally, and most surprisingly, this "1:2:√5 triangle specific precision" is also observed in **the merger of 1:2:√5 triangle -Pythagorean triangle**, which is the paramount validation of the geometric intimacy between these two triangles.

As shown below in Figure 11, the Pythagorean triangle ACD and the 1:2:√5 triangle ABD share their longer cathetus AD, and the precise relation observed as;

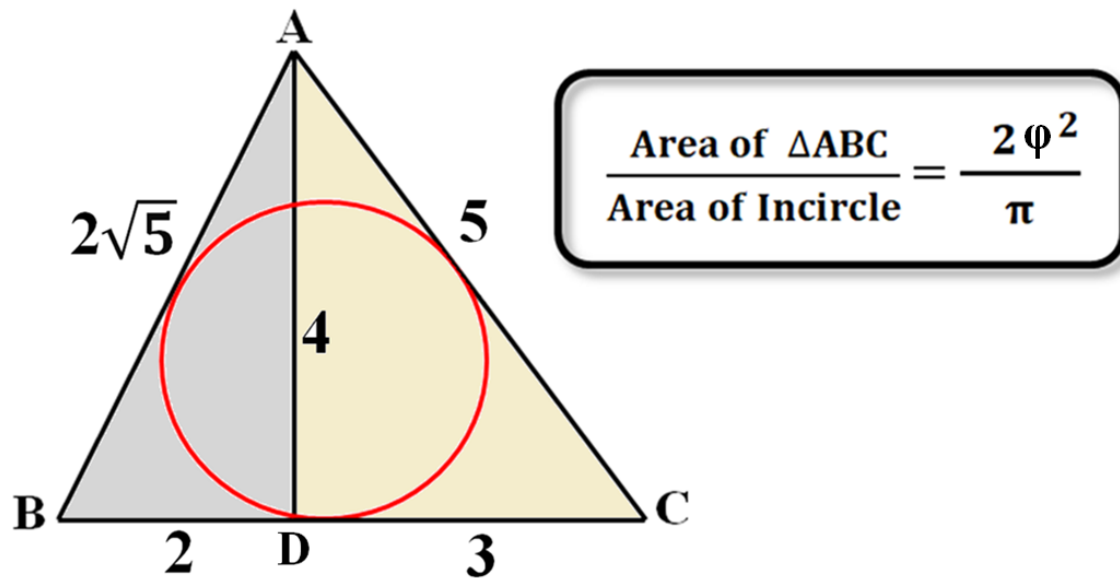


Figure 11: 1:2:√5 triangle and 3-4-5 Triangle, with Common Longer Leg AD; and The Incircle

In above figure,
$$\frac{\text{Area of combined Triangle ABC}}{\text{Area of Incircle}} = \frac{2\phi^2}{\pi}$$

Also, 1:2:√5 triangle and Pythagorean triple, having **common shorter cathetus** as illustrate below in Figure 12, reveal similar kind of precise relation;

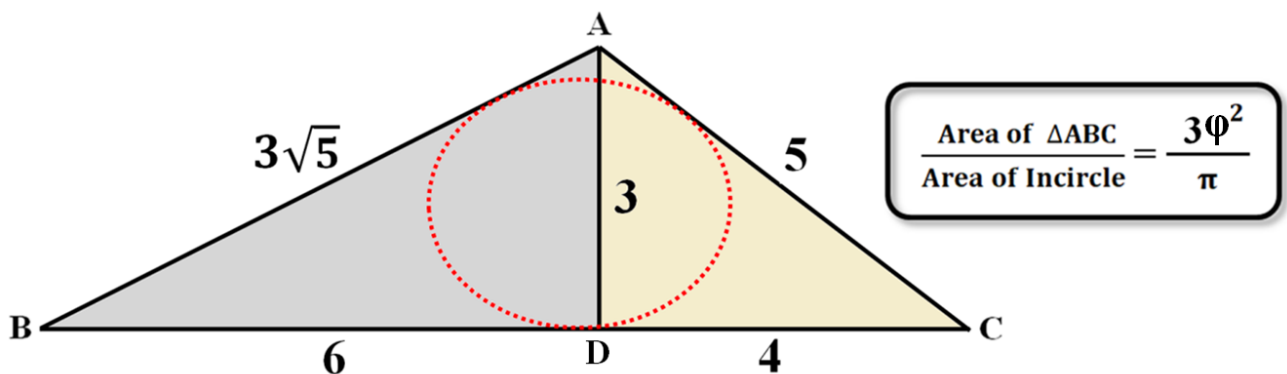


Figure 12: 1:2:√5 triangle ABD and 3-4-5 Triangle ACD, with Common Shorter Leg AD

In above diagram;
$$\frac{\text{Area of combined Triangle ABC}}{\text{Area of Incircle}} = \frac{3\phi^2}{\pi}$$

This finally corroborates the classical geometric relationship between $1:2:\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple. And, all these peculiar ratios impart the extremely precise geometric substantiation of $\pi : \varphi$ correlation.

Conclusion

This paper unveiled the hidden link in geometry, namely, the precise and perfect complementary relationship between the $1:2:\sqrt{5}$ triangle and the 3-4-5 Pythagorean triangle. The confluence of the $1:2:\sqrt{5}$ triangle with 3-4-5 Pythagorean triple engendered the very origin of golden ratio in geometry.

Also, the Incircles are incorporated in various geometric formations. With advent of circles, naturally, the π entered into picture, and then, by sheer geometric measurements, the $1:2:\sqrt{5}$ triangle, alone as well as in collaboration with the Pythagorean triple, revealed the "extremely precise Special Relationships". The ratio of the area of $1:2:\sqrt{5}$ triangle to the area of the Incircle therein, is found to be the geometric expression of extremely precise $\pi:\varphi$ correlation. Similar precise $\pi:\varphi$ Correlation is also observed with the merger of $1:2:\sqrt{5}$ triangle with the 3-4-5 Pythagorean triangle, which is the paramount validation of the geometric intimacy between these two triangles.

Hence, the intimate and classical geometric relationship between $1:2:\sqrt{5}$ triangle and 3-4-5 Pythagorean triple not only provided for the ultimate geometric substantiation of Golden Ratio, but it also revealed the Golden Link in Geometry; that is the unique $\pi:\varphi$ correlation, with an unprecedented level of precision, and which is firmly premised upon the classical geometric principles.