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New Iterative Method with Application

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Abstract

In this paper, we consider iterative methods to find a simple root of a nonlinear equation

$f(x) = 0$, where $f : D\in R\to R$ for an open interval D is a scalar function.

Keywords: Newtons method ; Third-order convergence; Non-linear equations; Root-finding; Iterative method

Introduction

One well-known 1-step iterative zero finding method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2,... \tag{1}$$

To derive (1), we approximate the given function f at $x = x_n$ by a linear function y of the form

$y(x) = a(x - x_n)+b$. Then the requirement that both f and y, and their first derivative, agree at $x = x_n$

leads to

$$y(x) = f'(x_n)(x - x_n) + f(x_n) \tag{2}$$

Finally, solving $y(x_{n+1}) = 0$ for x_{n+1} yields (1). Since (2) is the equation of the line tangent to f at $x = x_n$, it is clear that Newton's method applied to f may be interpreted as a sequence of tangent lines with zeros converging to the zero of the function. (See Figure 1.1)

Newton's method is a quadratically converging ($p = 2$) zero-finding algorithm. It requires two function evaluations (r = 2) and it's *E.I.* = 1.4142 .

Using Taylor's theorem, we can obtain its error equation . Assuming that α is the zero of f . i. e. $f(\alpha)= 0$,

$f'(\alpha) \neq 0$ and $e_n = x_n - \alpha$;

$$e_{n+1} = x_{n+1} - \alpha$$

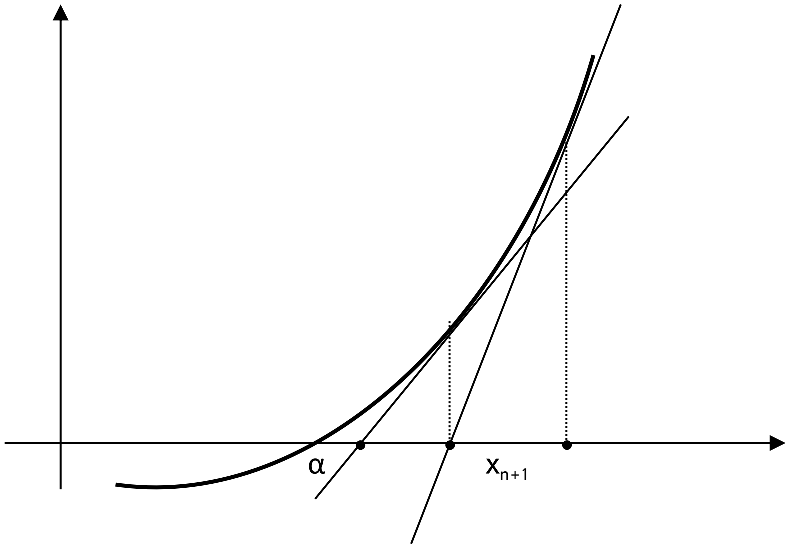
$$= x_n - \frac{f(x_n)}{f'(x_n)} - \alpha = e_n - \frac{f(x_n)}{f'(x_n)}$$

$$= e_n - \frac{1}{f'(x_n)}\left[f(\alpha) + f'(x_n)(x_n - \alpha) + \frac{f''(\xi)}{2}(x_n - \alpha)^2 \right]$$

$$= e_n - \frac{1}{f'(x_n)}\left[f'(x_n)e_n + \frac{f''(\xi)}{2}e_n^2 \right]$$

$$= e_n - e_n - \frac{f''(\xi)}{2f'(x_n)}e_n^2$$

$$= -\frac{f''(\xi)}{2f'(x_n)}e_n^2$$, where ξ is between x_n and α



Figure（1.1）: A geometric interpretation of Newton's Method

Recently, some modified Newton methods with cubic and fourth order convergence have been developed in [1-7]. Here, we will obtain a new modification of Newtons method.

Materials and Methods

Noor [6] proposed new fourth order method defined by

$$x_{n+1}=y_n-\frac{f(y_n)}{f'(x_n)}-\left(\frac{f(y_n)}{f'(x_n)}\right)\left(1-\frac{f'(y_n)}{f'(x_n)}\right)-\frac{1}{2}\left(\frac{f(y_n)}{f'(x_n)}\right)^2\frac{f''(y_n)}{f'(x_n)}, \tag{3}$$

Where $y_n=x_n-f(x_n)/f'(x_n)$.

It is clear that to implement (3,2), one has to evaluate the second derivative of the function. This can create some problems. In order to overcome this drawback, several technique have been developed [2–5].

In [8],a second-derivative-free method is obtained through approximating the second derivative $f''(y_n)$ in by (3).

$$f''(y_n)\cong\frac{f'(y_n)-f'(x_n)}{y_n-x_n}. \tag{4}$$

In a recent paper, Noor and Khan [8] have used the same approximation of the second derivative (4) in (3) to suggest the following Iterative methods

$$x_{n+1}=y_n-\frac{2f(y_n)}{f'(x_n)}+\left(\frac{f(y_n)f'(y_n)}{f'^2(x_n)}\right)+\left(\frac{f'(y_n)-f'(x_n)}{2f(x_n)}\right)\left(\frac{f(y_n)}{f'(x_n)}\right)^2. \tag{5}$$

The following approximations of $f'(y_n)$ are obtained in [10]

$$f'(y_n) \approx \frac{f'(x_n)f(x_n)^2}{f(x_n)^2 + 2f(x_n)f(y_n) + f(y_n)^2}, \tag{6}$$

$$f'(y_n) \approx \frac{f'(x_n)(f(x_n) + f(y_n)(\beta - 2))}{f(x_n) + \beta f(y_n)}, \tag{7}$$

where $\beta \in R$. We then apply the approximations (6) and (7) to the method (5). Now, Combining (6) and (5), we get the new iterative method

$$x_{n+1} = y_n - \left(\frac{2f(y_n)}{f'(x_n)}\right) + \left(\frac{f(x_n)^2 f(y_n)}{f'(x_n)(f(x_n)+f(y_n))^2}\right) + \left(-\frac{f(y_n)f'(x_n)(2f(x_n)+f(y_n))}{2f(x_n)(f(x_n)+f(y_n))^2}\right)\left(\frac{f(y_n)}{f'(x_n)}\right)^2. \tag{8}$$

Using (7) in (5), we get a new family of iterative method

$$x_{n+1} = y_n - \left(\frac{2f(y_n)}{f'(x_n)}\right) + \left(\frac{f(y_n)(f(x_n)+f(y_n)(\beta-2))}{f'(x_n)(f(x_n)+\beta f(y_n))}\right) + \left(-\frac{f'(x_n)f(y_n)}{f(x_n)(f(x_n)+\beta f(y_n))}\right)\left(\frac{f(y_n)}{f'(x_n)}\right)^2. \tag{9}$$

However, for the following iteration scheme

$$x_{n+1} = x_n - \left(\frac{f(x_n)+\beta f(y_n)}{f(x_n)+\gamma f(y_n)}\right)\left(\frac{f(x_n)}{f'(x_n)}\right). \tag{10}$$

where ・ and ・ are parameters to be determined .

In fact, ・・ ・1 and ・・・ 2 the well-known Traub-Ostrowski method (TOM) [10] is obtained.

$$x_{n+1} = x_n - \left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right)\left(\frac{f(x_n)}{f'(x_n)}\right). \tag{11}$$

If we choose ・・2 and ・・1, we get the third order method.

$$x_{n+1} = x_n - \left(\frac{f(x_n)+2f(y_n)}{f(x_n)+f(y_n)}\right)\left(\frac{f(x_n)}{f'(x_n)}\right). \tag{12}$$

which introduced by Chun in [11]

If ・・ 0 and ・・・1, then we obtain a third order method

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f(x_n)-f(y_n)}\right)\left(\frac{f(x_n)}{f'(x_n)}\right). \tag{13}$$

which was introduced by Xiaojian in [12].

If ・・・ 3 and ・・・ 4, then we obtain a new third-order method

$$x_{n+1} = x_n - \left(\frac{f(x_n) - 3f(y_n)}{f(x_n) - 4f(y_n)} \right) \left(\frac{f(x_n)}{f'(x_n)} \right). \tag{14}$$

If · · 1 and · · 0, then we obtain a new third-order method

$$x_{n+1} = x_n - \left(\frac{f(x_n) + f(y_n)}{f'(x_n)} \right). \tag{15}$$

Results and Discussion

All computations were done using the Mathematica package using 64 digit floating point arithmetic's.

We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: $|x_{n+1} - x_n| \prec \varepsilon$ and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. We used the fixed stopping criterion $\epsilon = 10^{-15}$.

We employ the present methods to solve some nonlinear equations, which not only illustrate the methods practically but also serve to check the validity of theoretical results we have derived.

$$f_1(x) = x^3 + 4x^2 - 10, \alpha = 1.365230013414096845760806829 0,$$
$$f_2(x) = \sin^2 x - x^2 + 1, \alpha = 1.404491648215341,$$
$$f_3(x) = x^2 - e^x - 3x + 2, \alpha = 0.2575302854398608,$$
$$f_4(x) = \cos x - x, \alpha = 0.7390851332151606416553120876 7,$$
$$f_5(x) = (x-1)^3 - 2, \alpha = 2.2\ 599210498948731647672106073,$$
$$f_6(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \alpha = \text{-}1.207647827130919,$$
$$f_7(x) = (x+2)e^x - 1, \alpha = \text{-}0.4428544010023885831413280000 0,$$
$$f_8(x) = e^{(x^2+7x-30)} - 1, \alpha = 3.$$

Displayed in **Table 1** the number of iterations to approximate the zero (N) and the number of function evaluations (TNFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative.

We present some numerical test results for various iterative schemes in **Table 1**.

Compared with the Newton method (NM), the method in (5)(NOR), the new methods in(9)(MNR1), and as an example of (9) we take β = 0 (MNR2), and β = 1 (MNR3).

We present some numerical test results for various iterative schemes in Table 2. Compared with the Newton method (NM), the method of Chun (13)(CM), Xiaojian (XM)(14). and the methods (14)(OM1) and (15) (OM2). The test results in Table 2 show that for most of the functions we tested, the methods introduced in the present presentation have at least equal performance compared to the other third-order method, and can also compete with Newtons method.

Table 1. Comparison of various fourth order schemes and Newtons method

	N					TNFE				
	NM	NOR	MNR1	MNR2	MNR3	NM	NOR	MNR1	MNR2	MNR3
$f_1, x_0 = 1.0$	5	3	3	3	3	10	12	9	9	9
$f_1, x_0 = -0.3$	53	38	22	56	17	106	152	66	168	51
$f_2, x_0 = 2.0$	5	3	3	4	3	10	12	9	12	9
$f_2, x_0 = 3.0$	6	3	4	4	4	12	12	12	12	12
$f_3, x_0 = 1.0$	4	2	2	2	2	8	8	6	6	6
$f_3, x_0 = 2.0$	5	3	3	3	3	10	12	9	9	9
$f_4, x_0 = 1.0$	4	2	2	2	2	8	8	6	6	6
$f_4, x_0 = 1.7$	4	3	3	3	3	8	12	9	9	9
$f_5, x_0 = 0.0$	NC	4	3	4	3	-	16	9	12	9
$f_5, x_0 = -1.0$	12	5	6	5	6	24	20	18	15	18
$f_6, x_0 = -1.0$	5	3	4	3	4	10	12	12	9	12
$f_6, x_0 = -2.0$	8	5	5	5	5	16	20	15	15	15
$f_7, x_0 = 2.0$	8	5	5	5	5	16	20	15	15	15
$f_7, x_0 = -5.0$	4	2	2	2	2	8	8	6	6	6

Table 2. Comparison of various cubically convergent iterative schemes and Newtons method

	N					TNFE				
	NM	CM	XM	OM1	OM2	NM	CM	XM	OM1	OM2
$f_1, x_0 = 3.0$	6	4	4	4	4	12	12	12	12	12
$f_2, x_0 = 3.0$	6	4	4	4	4	12	12	12	12	12
$f_3, x_0 = 1.0$	8	3	3	3	3	24	9	9	9	9
$f_4, x_0 = 3.0$	7	3	4	4	3	14	12	12	12	9
$f_5, x_0 = 2.5$	5	3	3	3	3	10	9	9	9	9
$f_6, x_0 = -1.0$	5	5	3	4	4	10	15	9	12	12
$f_7, x_0 = -1.0$	6	NC	4	4	6	12	NC	12	12	18

Conclusions

We have proposed two families of iterative methods for solving nonlinear equations. Numerical results show that the number of iterations of the new method are always less than that of the classical Newtons method and can be compared with other methods.

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