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APPROXIMATION OF THE LOWER OPERATOR IN NONLINEAR DIFFERENTIAL GAMES WITH NON-FIXED TIME

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Approximate properties of the lower operator in nonlinear differential games with non-fixed time are studied.

Abstract

The generalization of the Pontryagin's second direct method [1–2] for nonlinear pursuit games led to the construction described by the operator $\widetilde{T}^t$, which is introduced in [3]. Operator's construction in nonlinear differential games was developed in [4 - 18]. In particular, lower analogue of the operator $\widetilde{T}^t$ and its applications to study of qualitative structure of phase space of differential games of pursuit-evading were suggested [9].Problems of approximation and simplified schemes for construction of operator $\widetilde{T}^t$ were studied in [7,10,13]. For the symmetry, $\widetilde{T}_t$ will be denoted the lower analogue of the operator $\widetilde{T}^t$

In the present article we study approximation properties of the lower operator $\widetilde{T}_t$ for differential games of pursuit with non-fixed time.

Let us consider the differential game

$$\dot{z} = f(z, u, v), \tag{1}$$

where $z \in \mathbb{R}^d, u \in P, v \in Q, f : \mathbb{R}^d \times P \times Q \longrightarrow \mathbb{R}^d$, P and Q are convex compact subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Along with the system (1) we also fix the set of M, $M \subset \mathbb{R}^d$, which is called terminal set.

We suppose that further the function f holds the following conditions.

A. function $f : \mathbb{R}^d \times P \times Q \longrightarrow \mathbb{R}^d$ is continuous and is locally Lipschitz type by z (i.e.the function f satisfies the Lipschitz condition on every compact set $D \subset \mathbb{R}^d$ with the the constant L_D, depending on compact D).

B. There is the constant $C \leq 0$ such that for all $z \in \mathbb{R}^d, u \in P, v \in Q$ the inequality

$$| z \cdot f(z, u, v) | \leq C(1+ | z |^2)$$

holds.

C. The set $f(z,u,Q)$ is convex for all $z \in \mathbb{R}^d, u \in P$.

Let $X[\Delta]$ denote the set of all measurable functions $a(\cdot) : \Delta \to X$. In the case of $\Delta = [\alpha, \beta]$, we simply write $X[\alpha, \beta]$. We call every function $u(\cdot) \in P[\alpha, \beta]$ (respectively $v(\cdot) \in Q[\alpha, \beta]$) as admissible control of pursuer(respectively evader).

We denote by $z(t, u(\cdot), v(\cdot), \xi)$ solution of equation (1), which corresponds to admissible controls $u(t)$, $v(t)$ and initial point ξ.

Definition 1. Operator T_ε associates every set $A \subset \mathbb{R}^d$ with the set $T_\varepsilon A$ of all points $\xi \subset \mathbb{R}^d$, such that there is admissible control pursuer $u(\cdot) \in P[0,\varepsilon]$ for any admissible control of evader $v(\cdot) \in Q[0,\varepsilon]$ the corresponding trajectory $z(t, u(\cdot), v(\cdot), \xi)$ with the beginning at the point $\xi \subset \mathbb{R}^d$ hits $A \subset \mathbb{R}^d$ in time not greater than ε, i.e. $z(t_*) \in A$ for of certain $t_* \in [0,\varepsilon]$.

By means of operations of association and intersection we can write the operator T_ε as follows:

$$T_\varepsilon A = \bigcup_{u(\cdot)\in P[0,\varepsilon]} \bigcap_{v(\cdot)\in Q[0,\varepsilon]} \bigcup_{t_*\in[0,\varepsilon]} \left[\xi \subset \mathbb{R}^d \mid z(t_*, u(\cdot), v(\cdot), \xi) \in A\right].$$

Let $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n = t\}$ be partition of segment $[0,t]$ and $\delta_i = \tau_i - \tau_{i-1}$, $\mid \omega \mid = t$. We assume

$$T_\omega M = T_{\delta_1} T_{\delta_2} T_{\delta_3} ... T_{\delta_n} M,$$

where $\delta_i = \tau_i - \tau_{i-1}, \quad i = 1, 2, ...n$.

Definition 2. $\tilde{T}_t = \bigcup\limits_{|\omega|=t} T_\omega M$.

The operator $\tilde{T}_t$ is called the lower operator of nonlinear differential games pursuit with non-fixed time.

In what follows, we shall assume that the boundary of M (∂M) is compact. We denote by D_* the set of all points of $\xi \in \mathbb{R}^d$, of which it is possible to achieve the set ∂M (the boundary of M) at the appropriate admissible controls $u(\cdot)$ and $v(\cdot)$ for a time not exceeding θ. Let $D = D_* + H$ and constants is the quantity that can depend only on the function f, sets P, Q, D and we shall suppose $t \leq \theta$. Condition B guarantees boundedness of the set D [14]. We assume

$K = \max\{| f(z, u, v) | z \in D, u \in P, v \in Q\}$ and L_1 is the constant Lipshitz of f on the set D.

Let operator $\overline{T}_\varepsilon$ differs from the operator T_ε in that in Definition 1 only constant controls $u(\cdot) = u \in P$ are taken instead of arbitrary admissible controls $u(\cdot) \in P[0, \varepsilon]$.

Let $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n = t\}$ be partition of segment $[0, t]$.

$$\overline{T}_\omega M = \overline{T}_{\delta_1}\overline{T}_{\delta_2}...\overline{T}_{\delta_n},$$

where $\delta_i = \tau_i - \tau_{i-1}, \quad i = 1, 2, ...n.$

Definition 3. $\overline{T}_t M = \bigcup\limits_{|\omega|=t} \overline{T}_\omega M.$

For completeness, we present some well-known properties of the operator $\widetilde{T}_t$.

Theorem 1 [15]. If M is an open subset of $\mathbb{R}^d$, then

$$\widetilde{T}_t M = \overline{T}_t M$$

.

We note that for arbitrary family A_α the following inclusion

$$\bigcup_\alpha \overline{T}_\varepsilon A_\alpha \subset \overline{T}_\varepsilon \bigcup_\alpha A_\alpha$$

is valid.

Lemma 1 [10]. Let $A_\alpha \subset \mathbb{R}^d$ non-decreasing direction of open sets. Then following equality holds

$$\bigcup_\alpha \overline{T}_\varepsilon A_\alpha = \overline{T}_\varepsilon \bigcup_\alpha A_\alpha.$$

Lemma 2 [10]. Let ω_k be infinitely reducing sequence of partitions of the segment $[0, t]$ i.e. $\omega_k \subset \omega_{k+1}$, $| \omega_k |= t, \max | \tau_i^k - \tau_{i-1}^k |\to 0$ for $k \to \infty$. Then the following equality holds

$$\overline{T}_t M = \bigcup_{k\geq 1} \overline{T}_{\omega_k} M$$

for open set $M \subset \mathbb{R}^d$.

A simplified schemes for constructing of alternating integral were proposed in [10,13].

For nonlinear differential games the problem of working out a simplified schemes for the construction of the operator $\widetilde{T}_t M$ is relevant.

Consider the following operator

$$\Theta_\varepsilon B = \bigcup_{u\in P}\bigcap_{v\in Q}\bigcup_{0\le t_*\le\varepsilon}\{\xi\in R^d \mid z(\varepsilon,u,v,\xi)=\xi+t_* f(\xi,u,v)\in B.\}$$

The definition of the operator $\widetilde{\Theta}_t$ is similar to the definition of the operator $\widetilde{T}_t$.

In the present article we consider the problem of approximation of the operator $\tilde{T}_t$ by means of iteration of operator Θ_ε and its application to the problem of pursuit.

Lemma 3. There is a positive number L such that the following inclusions

$$\overline{T}_\varepsilon(A\underline{*}2L\varepsilon^2 H)\subset\Theta_\varepsilon(A\underline{\star}\, L\varepsilon^2 H)\subset\overline{T}_\varepsilon A \qquad (2)$$

hold.

Proof. The first we prove the left-side of the inclusion (2). Let $\xi\in\overline{T}_\varepsilon(A\underline{*}2L\varepsilon^2 H)$. Then, there exists an admissible control of the pursuer $u\in P$ such that for any admissible control evader $v(\cdot)\in Q[0,\varepsilon]$, there is $t_*\in[0,\varepsilon]$ for trajectory $z(t_*,u,v(\cdot),\xi)$ corresponding to controls $u\in P$, $v(\cdot)\in Q$ and the initial point $\xi\in\mathbb{R}^d$ the following inclusion $z(t_*,u,v(\cdot),\xi)\in A\underline{*}2L\varepsilon^2 H$ holds. i.e.

$$z(t_*,u,v(\cdot),\xi)=\xi+\int_0^{t_*} f(z(t),u,v(t)dt+2L\varepsilon^2 H\in A. \qquad (3)$$

By virtue of the condition A for arbitrary controls $u\in P$, $v(\cdot)\in Q$ and the initial point $\xi\in\mathbb{R}^d$ we have the relation

$$\mid f(z(t),u,v(t))-f(\xi,u,v(t))\mid\le L_1\mid z(t)-\xi\mid. \qquad (4)$$

On the other hand,

$$\mid z(t,u,v(\cdot),\xi)-\xi\mid\le K\varepsilon, t\in[0,\varepsilon].$$

Hence, using the inequality (4), we obtain

$$\mid f(z(t),u,v(t))-f(\xi,u,v(t))\mid\le L\varepsilon, \qquad (5)$$

where $L=L_1K$.

Now we prove that for any $v(\cdot) \in Q[0, \varepsilon]$ there is a constant control $v \in Q$ for which the equality

$$\xi + t_* f(\xi, u, v) = \xi + \int_0^{t_*} f(\xi, u, v(t))dt \tag{6}$$

is fulfilled.

By virtue of the condition C, the set $f(\xi, u, Q)$ is convex for any $u \in P$. Therefore

$$\int_0^{t_*} f(\xi, u, v(t)dt \in t_* f(\xi, u, Q).$$

It follows that there is a $v \in Q$ such that

$$\int_0^{t_*} f(\xi, u, v(t))dt = t_* f(\xi, u, v).$$

Therefore, for any $v(\cdot) \in Q[0, \varepsilon]$ there is a constant control $v \in Q$ for which equality

$$\xi + \int_0^{t_*} f(\xi, u, v(t))dt = \xi + t_* f(\xi, u, v)$$

holds.

Applying inequality (5) to the right side of equality (6) we have

$$\xi + t_* f(\xi, u, v) \in \xi + \int_0^{t_*} f(z(t), u, v(t))dt + L\varepsilon^2 H.$$

Hence, using the condition (3) we obtain

$$\xi + t_* f(\xi, u, v) + L\varepsilon^2 H \subset \xi + \int_0^{t\ ast} f(z(t), u, v(t))dt + 2L\varepsilon^2 H \subset A.$$

Consequently,

$$\xi \in \Theta_\varepsilon(A \underline{*} L\varepsilon^2 H).$$

Similarly, the right side of the turn proved (2).

Lemma 4. The following inclusions

$$\Theta_\varepsilon(A \underline{*} L\delta^2(1 + L_1\varepsilon)H) + L\delta^2 H \subset \Theta_\varepsilon A \tag{7}$$

,

$$\overline{T}_\varepsilon(A\underline{*}L\delta^2(1+L_1\varepsilon)H)+L\delta^2 H\subset\ \overline{T}_\varepsilon A \tag{8}$$

hold. Proof. Let η be an arbitrary element from the left part of the inclusion (7). Then there is $\xi\in\ \Theta_\varepsilon(A\underline{*}L\delta^2(1+L_1\varepsilon)H)$ such that

$$\mid \eta-\xi \mid\leq L\delta^2. \tag{9}$$

By virtue of condition A, we have

$$\mid f(\xi,u,v)-f(\eta,u,v)\mid\leq L_1\mid \eta-\xi\mid .$$

From inequality (9) we get

$$\mid f(\xi,u,v)-f(\eta,u,v)\mid\leq L_1L\delta^2. \tag{10}$$

Consider the sum $\eta+t_*f(\eta,u,v)$. Using inequality (9) and (10) we have

$$\eta+t_*f(\eta,u,v)\in\xi+L\delta^2H+t_*(f(\eta,u,v)+L_1L\delta^2H)\subset\xi+t_*f(\xi,u,v)+L\delta^2(1+L_1\varepsilon).$$

Now, considering that $\xi\in\Theta_\varepsilon(A\underline{*}L\delta^2(1+L_1\varepsilon)H)$ we come to the inclusion $\eta+t_*f(\eta,u,v)\in A$. this implies $\eta\in\Theta_\varepsilon(M)$. This was to be proved. Similarly, the inclusion (8) will be proved. Lemma 4 is proved.

Further, we consider only uniform partitions of the segments [0, t]. Let $\omega_n=\{0,\varepsilon,2\varepsilon,...,n\varepsilon=t\}$, where $\varepsilon=\frac{t}{n}$. Let $\Gamma(n,\varepsilon)=L\varepsilon^2\sum_{k=1}^{n}(1+L_1\varepsilon)^{k-1}$. We assume

$$\Theta_{2\varepsilon}A=\Theta_\varepsilon\Theta_\varepsilon A,\Theta_{k\varepsilon}A=\Theta_\varepsilon\Theta_{(k-1)\varepsilon}A,\quad \Theta_{\omega_n}A=\Theta_{n\varepsilon}A.$$

Note that the notation $\overline{T}_{k\varepsilon}$ is entered in the same way as $\Theta_{k\varepsilon}$

Theorem 2. The following inclusions

$$\overline{T}_{\omega_n}(M\underline{*}2\Gamma(n,\varepsilon)H)\subset\Theta_{\omega_n}(M\underline{*}\Gamma(n,\varepsilon)H)\subset\overline{T}_{\omega_n}(M) \tag{11}$$

hold.

Proof. We prove the right side of inclusions (11). Let $\omega_n=\{0,\varepsilon,\quad 2\varepsilon=t\}$, where $\varepsilon=\frac{t}{2}$. From Lemma 3 it follows that

$$\Theta_\varepsilon(M\underline{*}L\varepsilon^2H)\subset\overline{T}_\varepsilon M.$$

Now using the inclusion (7) we have

$$\Theta_{2\varepsilon}(M\underline{*}\Gamma(2,\varepsilon)H) \subset \Theta_{\varepsilon}(\Theta_{\varepsilon}(M\underline{*}\Gamma(1,\varepsilon)H))\underline{*}L\varepsilon^2 H).$$

Applying Lemma 3 to the right-hand side of this inclusion, we arrive at the following relation

$$\Theta_{2\varepsilon}(M\underline{*}\Gamma(2,\varepsilon)H) \subset \overline{T}_{\varepsilon}(\Theta_{\varepsilon}(M\underline{*}L\varepsilon^2 H)).$$

Using again Lemma 3, we obtain

$$\Theta_{2\varepsilon}(M\underline{*}\Gamma(2,\varepsilon)H) \subset \overline{T}_{\varepsilon}\overline{T}_{\varepsilon}M = \overline{T}_{2\varepsilon}M.$$

Suppose

$$\Theta_{p\varepsilon}(M\underline{*}\Gamma(p,\varepsilon)H) \subset \overline{T}_{p\varepsilon}M. \tag{12}$$

We shall prove the validity of the following relation

$$\Theta_{(p+1)\varepsilon}(M\underline{*}\Gamma(p+1,\varepsilon)H) \subset \overline{T}_{(p+1)\varepsilon}M. \tag{13}$$

Let us consider the set

$$\Theta_{(p+1)\varepsilon}(M\underline{*}\Gamma(p+1,\varepsilon)H) = \Theta_{\varepsilon}\Theta_{p\varepsilon}(M\underline{*}\Gamma(p,\varepsilon)H)\underline{*}L\varepsilon^2(1+L_1\varepsilon)^p H)$$

Applying Lemma 4 to the right side of this inclusion p-times we have

$$\Theta_{(p+1)\varepsilon}(M\underline{*}\Gamma(p+1,\varepsilon)H) \subset \Theta_{\varepsilon}((\Theta_{p\varepsilon}(M\underline{*}\Gamma(p,\varepsilon)H))\underline{*}L\varepsilon^2 H).$$

By virtue of Lemma 3 one obtains

$$\Theta_{(p+1)\varepsilon}(M\underline{*}\Gamma(p+1,\varepsilon)H) \subset \ \overline{T}_{\varepsilon}\Theta_{p\varepsilon}(M\underline{*}\Gamma(p,\varepsilon)H).$$

Now due to the inclusion (12) we have

$$\Theta_{(p+1)\varepsilon}(M\underline{*}\Gamma(p+1,\varepsilon)H) \subset \ \overline{T}_{\varepsilon}\overline{T}_{p\varepsilon}M = \overline{T}_{(p+1)\varepsilon}.$$

This implies the inclusion

$$\Theta_{n\varepsilon}(M\underline{*}\Gamma(n,\varepsilon)H) \subset \overline{T}_{n\varepsilon}M,$$

is valid for any $n \in \mathbb{N}$. Consequently, $\Theta_{\omega_n}(M \underline{*} \Gamma(n,\varepsilon)H) \subset \overline{T}_{\omega_n} M$. Similarly of that ,the left side of the inclusion (11) will established. Theorem 2 is proved.

Theorem 3. The following equality holds

$$\overline{T}_t M = \bigcup_{\delta>0} \Theta_t(M \underline{*} \delta H),$$

for open M, $M \subset \mathbb{R}^d$.

Proof. Consider the quantity $\Gamma(\varepsilon) = L\varepsilon^2 \sum_{k=1}^{n} (1 + L_1\varepsilon)^k$. It is not difficult to see that $\Gamma(\varepsilon) \leq \varepsilon L(e^{L_1\theta} - 1)$. We choose ε such that $\Gamma(\varepsilon) \leq \varepsilon L(e^{L\theta} - 1) < \delta$, i.e. $\varepsilon < \frac{\delta}{L(e^{L_1\theta}-1)}$. By virtue of this, inclusion (11) implies

$$\overline{T}_{\omega_n}(M \underline{ast} 2\delta H) \subset \Theta_{\omega_n}(M \underline{*} \delta H) \subset \overline{T}_{\omega_n}(M).$$

Passing to the union over all ω_n in these relations by term, we obtain

$$\overline{T}_t(M \underline{*} 2\delta H) \subset \Theta_t(M \underline{*} \delta H) \subset \overline{T}_t M.$$

Turning to the union over all $\delta > 0$ in these inclusions, we arrive to the following inclusions

$$\bigcup_{\delta>0} \overline{T}_t(M \underline{*} 2\delta H) \subset \bigcup_{delta>0} \Theta_t(M \underline{*} \delta H) \subset \overline{T}_t M.$$

It follows, by Lemmas 1 and 2, we have

$$\overline{T}_t M = \bigcup_{\delta>0} \Theta_t(M \underline{*} \delta H).$$

Theorem 3 is proved.

Theorems 1 and Theorems 3 imply

Corollary. The following equality holds

$$T_t M = \bigcup_{\delta>0} \Theta_t(M \underline{*} \delta H),$$

for open M, $M \subset \mathbb{R}^d$.

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