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THE HERMITE–HADAMARD INEQUALITY ON HYPERCUBOID

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Abstract. Given any $\mathbf{a} := (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} := (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$. The $\mathbf{n}$-fold convex function defined on $[\mathbf{a}, \mathbf{b}]$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $\mathbf{a} < \mathbf{b}$ is a convex function in each variable separately. In this work we prove an inequality of Hermite-Hadamard type for $\mathbf{n}$-fold convex functions. Namely, we establish the inequality

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \le \frac{1}{\mathbf{b}-\mathbf{a}} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, d\mathbf{x} \le \frac{1}{2^n} \sum_{\mathbf{c}} f(\mathbf{c}),$$

where $\sum_{\mathbf{c}} f(\mathbf{c}) := \sum_{\substack{c_i \in \{a_i, b_i\} \\ 1 \le i \le n}} f(c_1, c_2, \ldots, c_n)$. Some other related result are given.

1. Introduction

The classical Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(t)\, dt \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds for all convex functions defined on a real interval $[a, b]$.

Along the past thirty years, several authors give an attention for various kind of this inequality and related type inequalities. Indeed the history of (1.1) is very long to summarize in one or two paragraph, however, we can simply say without any worry, the real work over all these thirty years started in 1992 by Dragomir [5]. In literature, the referenced work [5] was considered as base to study and investigate (1.1) by many other authors later.

A progressive work make many interested authors to generalize (1.1) and establish a number of formulation in various forms. In sequence of papers, Dragomir proved various inequalities of Hermite-Hadamard type for several assumption for the functions involved; e.g., convex mappings defined on a disk in the plane and convex mappings defined on a ball in the space . For a comprehensive work regarding (1.1) the reader may refer to [5].

In 2006, de la Cal and Cárcamo [3] studied the Hermite-Hadamard type for convex functions on n-dimensional convex bodies by translating the problem into

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of finding appropriate majorants of the involved random vector for the usual convex order. Two main results was obtained in [3] the first one regarding mappings defined on polytopes in $\mathbb{R}^n$, while the second result discussed (1.1) for symmetric random vectors taking values in a closed ball for a given (but arbitrary) norm on $\mathbb{R}^n$, (see also [4]). In 2008, a formulation on simplicies was presented; the key idea of the presented approach was passed through a volume type formula and its higher dimensional generalization. In 2009, by using of a stochastic approach, de la Cal *et. al.* established a multidimensional version of the classical Hermite-Hadamard inequalities which holds for convex functions on general convex bodies. In 2012, Yang [13] proved an extension of (1.1) for functions defined on a convex subsets of $\mathbb{R}^3$, indeed the author introduced a version of (1.1) for function f defined on an annulus domain. Recently, Moslehian [11] introduced several matrix and operator inequalities of Hermite-Hadamard type and presented some operator inequalities of Hermite-Hadamard type in which the classical convexity was used instead of the operator convexity. For more details, generalization and counterparts the reader may refer to [1]–[13] and the references therein.

Let us consider the bi-dimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex in Δ if

$$f\left(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w\right) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

holds for all $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

Dragomir [6] established a new concept of convexity which is called the co–ordinated convex function, as follows:
A function $f : \Delta \to \mathbb{R}$ is convex in Δ is called co–ordinated convex on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$, are convex for all $y \in [c,d]$ and $x \in [a,b]$.

In [6], Dragomir established the following similar inequality of Hadamard's type for co–ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.1. *Suppose that $f : \Delta \to \mathbb{R}$ is co–ordinated convex on Δ. Then one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy dx \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \end{aligned} \tag{1.2}$$

The above inequalities are sharp.

In [1], Alomari proved the weighted version of (1.2) which is known as Fejér inequality, as follows:

Theorem 1.2. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co–ordinated convex function, Then the double inequality*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{\int_a^b\int_c^d f(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx} \le \frac{f(a,c)+f(c,d)+f(b,c)+f(b,d)}{4} \tag{1.3}$$

holds, where $p:[a,b]\times[c,d]\to\mathbb{R}$ is positive, integrable, and symmetric about $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$. The above inequalities are sharp.

In this work, a new inequality of Hermite-Hadamard type on hypercuboid is proved.

2. n-fold convex functions

Given any $\mathbf{a}:=(a_1,a_2,\ldots,a_n)$ and $\mathbf{b}:=(b_1,b_2,\ldots,b_n)$ in $\mathbb{R}^n$, we define

$$\mathbf{a}\le\mathbf{b}\Longleftrightarrow a_i\le b_i,\forall i,\ 1\le i\le n.$$

Clearly, this is a partial order on $\mathbb{R}^n$, and it may be called the **product order** or the **componentwise order** on $\mathbb{R}^n$. If $n>1$, then the product order on $\mathbb{R}^n$ is not a total order; for example, if $\mathbf{x}:=(1,0,0,\cdots,0)$ and $\mathbf{y}:=(0,1,0,\cdots,0)$, then neither $x\le y$ nor $y\le x$.

Let

$$I_{\mathbf{a},\mathbf{b}}:=\prod_{i=1}^{n}I_{a_i,b_i}=I_{a_1,b_1}\times\cdots\times I_{a_n,b_n}.$$

A subset I of $\mathbb{R}^n$ is said to be an n-interval if $I_{\mathbf{a},\mathbf{b}}\subseteq I$ for every $\mathbf{a},\mathbf{b}\in I$. For example, if $I_1,\ldots,I_n$ are intervals in $\mathbb{R}$, then $I_1\times\cdots\times I_n$ is an n-interval. Furthermore, an n-interval of the form $I_1\times\cdots\times I_n$, where each of the $I_1\times\cdots\times I_n$ is a closed and bounded interval in $\mathbb{R}$, is called a **hypercuboid** in $\mathbb{R}^n$.

Throughout this paper, we will consider, for all $a_i,b_i\in\mathbb{R}$, $[\mathbf{a},\mathbf{b}]:=\prod\limits_{1\le i\le n}[a_i,b_i]$, and $\mathbf{c}=(c_1,c_2,\ldots,c_n)$, $c_i\in\{a_i,b_i\}$, $1\le i\le n$. Also, for $x_i,y_i\in[a_i,b_i]$ and $t_i\in[0,1]$, define

$$\mathbf{tx}=(t_1x_1,t_2x_2,\ldots,t_nx_n),$$

and

$$(\mathbf{1}-\mathbf{t})\,\mathbf{y}=((1-t_1)\,y_1,(1-t_2)\,y_2,\ldots,(1-t_n)\,y_n).$$

Let $f:[\mathbf{a},\mathbf{b}]\subseteq\mathbb{R}^n\to\mathbb{R}$, for the vector $\mathbf{c}$, we define

$$\sum_{\mathbf{c}}f(\mathbf{c}):=\sum_{\substack{c_i\in\{a_i,b_i\}\\1\le i\le n}}f(c_1,c_2,\ldots,c_n),$$

for all possible choices of $c_i\in\{a_i,b_i\}$, $(i=1,2,\cdots,n)$.

Definition 2.1. A subset $\mathbb{D}\subseteq\mathbb{R}^n$ is said to be **n**-fold convex if and only if whenever $\mathbf{x},\mathbf{y}\in\mathbb{D}$ then $\mathbf{tx}+(\mathbf{1}-\mathbf{t})\,\mathbf{y}\in\mathbb{D}$.

Corollary 2.2. *Every convex subset of $\mathbb{R}^n$ is an $\mathbf{n}$-fold convex, and the converse is not true in general.*

Proof. Follows directly from the definition. □

There is a subset $\mathbb{D} \subseteq \mathbb{R}^n$ which is $\mathbf{n}$-fold convex but is not convex. For example, consider $\mathbb{D} \subset \mathbb{R}^2$, in the Figure (1). On the other hand, there is a subset $\mathbb{D} \subseteq \mathbb{R}^n$

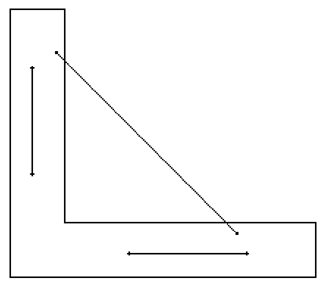

FIGURE 1. **2**-fold convex set which is not convex.

which is not convex nor $\mathbf{n}$-fold, see the Figure (2).

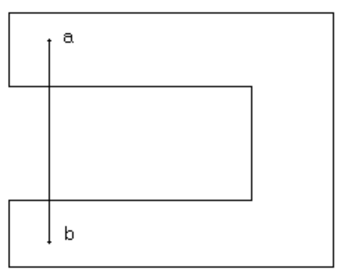

FIGURE 2. A non convex set nor **2**-fold convex.

Definition 2.3. A function $f : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ is said to be $\mathbf{n}$-fold convex or convex on the coordinates if and only if the inequality

$$f(\mathbf{t}\mathbf{x} + (\mathbf{1} - \mathbf{t})\,\mathbf{y}) \le \left(\prod_{1\le i\le n} p_i\right) \sum_{\mathbf{c}} f(\mathbf{c}), \tag{2.1}$$

holds, for all $\mathbf{x}, \mathbf{y} \in [\mathbf{a}, \mathbf{b}]$ and $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$, where,

$$p_i = \begin{cases} t_i, & \text{if} \quad c_i = a_i \\ 1 - t_i, & \text{if} \quad c_i = b_i \end{cases} \tag{2.2}$$

for all $1 \le i \le n$. Equivalently, f is said to be $\mathbf{n}$-fold convex on $[\mathbf{a}, \mathbf{b}]$ iff f is convex in each coordinate separately on $[a_i, b_i]$ for all $i = 1, 2, \cdots, n$. On the other hand, f is called $\mathbf{n}$-fold concave if the inequality (2.1) is reversed.

Corollary 2.4. *Every convex function defined on* $[\mathbf{a},\mathbf{b}]\subseteq\mathbb{R}^n$ *is* $\mathbf{n}$*-fold convex, and the converse is not true in general.*

Proof. Consider $f:[\mathbf{a},\mathbf{b}]\to\mathbb{R}$ be an $\mathbf{n}$-fold convex function. We carry out our proof using induction.

Let

$$P(n): f\left(\mathbf{t}\mathbf{x}+(\mathbf{1}-\mathbf{t})\,\mathbf{y}\right)\le\left(\prod_{1\le i\le n}p_i\right)\sum_{\mathbf{c}}f\left(\mathbf{c}\right),\quad n\in\mathbb{N} \tag{2.3}$$

holds, for all $\mathbf{x},\mathbf{y}\in[\mathbf{a},\mathbf{b}]$ and $\mathbf{t}\in[\mathbf{0},\mathbf{1}]$, where,

$$\sum_{\mathbf{c}}f\left(\mathbf{c}\right)=\sum_{\substack{c_i\in\{x_i,y_i\}\\1\le i\le n}}f\left(c_1,c_2,\ldots,c_n\right),$$

for all possible choices of $c_i\in\{x_i,y_i\}$, and

$$p_i=\begin{cases}t_i, & \text{if}\quad c_i=x_i\\ 1-t_i, & \text{if}\quad c_i=y_i\end{cases}$$

for all $1\le i\le n$.

For $n=2$, let $[\mathbf{a},\mathbf{b}]=[a_1,b_1]\times[a_2,b_2]$, for each pair $\mathbf{x},\mathbf{y}\in[\mathbf{a},\mathbf{b}]$; $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$, since f is convex on $[\mathbf{a},\mathbf{b}]$, then

$$\begin{aligned}P(2):&f\left(t_1x_1+(1-t_1)\,y_1,z\right)\\&\le t_1f\left(x_1,z\right)+(1-t_1)\,f\left(y_1,z\right)\\&=t_1f\left(x_1,t_2x_2+(1-t_2)\,y_2\right)+(1-t_1)\,f\left(y_1,t_2x_2+(1-t_2)\,y_2\right)\\&\le t_1t_2f\left(x_1,x_2\right)+t_1\left(1-t_2\right)f\left(x_1,y_2\right)+(1-t_1)\,t_2f\left(y_1,x_2\right)\\&\qquad+(1-t_1)\,(1-t_2)\,f\left(y_1,y_2\right)\\&=\left(\prod_{i=1}^{2}p_i\right)\sum_{\substack{c_i\in\{x_i,y_i\}\\1\le i\le 2}}f\left(\mathbf{c}\right),\end{aligned} \tag{2.4}$$

where $\mathbf{c}=(c_1,c_2)$, which mean that f is $\mathbf{2}$-fold convex on $[\mathbf{a},\mathbf{b}]$.

For $n=k$, assume that $P(k)$ holds, and let $[\mathbf{a},\mathbf{b}]=\prod_{i=1}^{k}[a_i,b_i]$, $\mathbf{x}=(x_1,x_2,\cdots,x_k)$ and $\mathbf{y}=(y_1,y_2,\cdots,y_k)$, since f is $\mathbf{k}$-fold convex on $[\mathbf{a},\mathbf{b}]$, then

$$f\left(\mathbf{t}\mathbf{x}+(1-\mathbf{t})\,\mathbf{y}\right)\le\left(\prod_{i=1}^{k}p_i\right)\sum_{\substack{c_i\in\{x_i,y_i\}\\1\le i\le k}}f\left(\mathbf{c}\right) \tag{2.5}$$

for all $t\in[0,1]$ and $\mathbf{x},\mathbf{y}\in[\mathbf{a},\mathbf{b}]$, where

$$p_i=\begin{cases}t_i, & \text{if}\quad c_i=x_i\\ 1-t_i, & \text{if}\quad c_i=y_i\end{cases}$$

for all $i=1,2,\cdots,k$.

It remains to show that $P(n)$ holds when $n=k+1$, therefore

$$\begin{aligned}
P(k+1):& f\left(t_1x_1+(1-t_1)\,y_1,\ldots,t_{k+1}x_{k+1}+(1-t_{k+1})\,y_{k+1}\right)\\
&= f\left(t_1x_1+(1-t_1)\,y_1,\ldots,t_kx_k+(1-t_k)\,y_k,t_{k+1}x_{k+1}+(1-t_{k+1})\,y_{k+1}\right)\\
&=\left(\prod_{i=1}^{k}p_i\right)\sum_{\substack{c_i\in\{x_i,y_i\}\\1\le i\le k}} f\left(\mathbf{c},t_{k+1}x_{k+1}+(1-t_{k+1})\,y_{k+1}\right)\\
&\le t_{k+1}\left(\prod_{i=1}^{k}p_i\right)\sum_{\substack{c_i\in\{x_i,y_i\}\\1\le i\le k}} f\left(\mathbf{c},x_{k+1}\right)\\
&\qquad+(1-t_{k+1})\left(\prod_{i=1}^{k}p_i\right)\sum_{\substack{c_i\in\{x_i,y_i\}\\1\le i\le k}} f\left(\mathbf{c},y_{k+1}\right),\qquad \text{(follows from (2.5))}\\
&=\left(\prod_{i=1}^{k+1}p_i\right)\sum_{\substack{c_i\in\{x_i,y_i\}\\1\le i\le k+1}} f\left(\mathbf{c}\right)
\end{aligned}$$

where,

$$p_i=\begin{cases} t_i, & \text{if}\quad c_i=x_i\\ 1-t_i, & \text{if}\quad c_i=y_i\end{cases}$$

for all $i=1,2,\cdots,k+1$. Hence, by mathematical induction, $P(n)$ holds for all $n\in\mathbb{N}$. On the other hand, the function $f:[0,1]^2\to[0,\infty)$, $f(x,y)=y$ is **2**-fold convex on $[0,1]^2$ but is not convex. The reverse of (2.1) follows directly by replacing f by $-f$, and thus the proof is completely established. □

The following Jensen's type inequality holds:

Theorem 2.5. *Let $f:[\mathbf{a},\mathbf{b}]\to\mathbb{R}$ be $\mathbf{k}$-fold convex. Let $x_i^{(r)}$ be a finite sequence of real numbers, for all $i,r=1,2,\cdots,k$, and consider $\mathbf{x}=\left(x_i^{(1)},x_i^{(2)},\ldots,x_i^{(k)}\right)$, $\alpha=\left(\alpha_i^{(1)},\alpha_i^{(2)},\ldots,\alpha_i^{(k)}\right)$, with $\sum\alpha=\mathbf{1}$, i.e., $\sum\limits_{i=1}^{k}\alpha_i^{(r)}=1$, for all $r=1,2,\cdots,k$. Then the inequality*

$$f\left(\sum\alpha\mathbf{x}\right)\le\left(\prod_{1\le i\le k}\gamma_i\right)\cdot\sum_{\mathbf{c}}f\left(\mathbf{c}\right)\tag{2.6}$$

holds, where

$$\sum\alpha\mathbf{x}:=\left(\sum_{i=1}^{k}\alpha_i^{(1)}x_i^{(1)},\sum_{i=1}^{k}\alpha_i^{(2)}x_i^{(2)},\ldots,\sum_{i=1}^{k}\alpha_i^{(k)}x_i^{(k)}\right),$$

$\mathbf{c} := (c_1, c_2, \ldots, c_k), c_i \in \left\{x_i^{(j)}\right\}_{j=1}^{k}$ *and*

$$\gamma_i = \begin{cases} \alpha_i^{(1)}, & c_i = x_i^{(1)} \\ \alpha_i^{(2)}, & c_i = x_i^{(2)} \\ \vdots & \\ \alpha_i^{(k)}, & c_i = x_i^{(k)} \end{cases}$$

If f is **n***-fold concave then the inequality (2.6) is reversed.*

Proof. Use the definition of **n**-fold convex and apply the classical Jensen's inequality for convex function of one variable in each variable. □

The following Hermite-Hadamard inequality holds:

Theorem 2.6. *Let $f : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ be* **n***-fold convex. Then the inequality*

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \leq \frac{1}{\mathbf{b}-\mathbf{a}} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, d\mathbf{x} \leq \frac{1}{2^n} \sum_{\mathbf{c}} f(\mathbf{c}), \tag{2.7}$$

holds, where

$$\sum_{\mathbf{c}} f(\mathbf{c}) := \sum_{\substack{c_i \in \{a_i, b_i\} \\ 1 \leq i \leq n}} f(c_1, c_2, \ldots, c_n).$$

The inequality (2.7) is sharp. If f is **n***-fold concave then the inequality (2.7) is reversed.*

Proof. Since f is **n**-fold convex on $[\mathbf{a}, \mathbf{b}]$, then for all $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$, we have

$$f(\mathbf{ta} + (\mathbf{1}-\mathbf{t})\mathbf{b}) \leq \sum_{\mathbf{c}} \left(\prod_{1 \leq i \leq n} p_i\right) f(\mathbf{c}). \tag{2.8}$$

Integrating (2.8) with respect to $\mathbf{t}$ on $[\mathbf{0}, \mathbf{1}]$ we get

$$\begin{aligned} \int_{\mathbf{0}}^{\mathbf{1}} f(\mathbf{ta} + (\mathbf{1}-\mathbf{t})\mathbf{b})\, d\mathbf{t} &\leq \int_{\mathbf{0}}^{\mathbf{1}} \left(\sum_{\mathbf{c}} \left(\prod_{1 \leq i \leq n} p_i\right) f(\mathbf{c})\right) d\mathbf{t} \\ &= \left(\sum_{\mathbf{c}} f(\mathbf{c})\right) \int_{\mathbf{0}}^{\mathbf{1}} \left(\prod_{1 \leq i \leq n} p_i\right) d\mathbf{t} \\ &= \frac{1}{2^n} \sum_{\mathbf{c}} f(\mathbf{c}) \end{aligned} \tag{2.9}$$

where, p_i is defined in (2.2).

On the other hand, again since f is **n**-fold convex on $[\mathbf{a}, \mathbf{b}]$, then for $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$, we have

$$\begin{aligned} f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) &= f\left(\frac{\mathbf{ta} + (\mathbf{1}-\mathbf{t})\mathbf{b}}{2} + \frac{\mathbf{tb} + (\mathbf{1}-\mathbf{t})\mathbf{a}}{2}\right) \\ &\leq \frac{1}{2}\left[f(\mathbf{ta} + (\mathbf{1}-\mathbf{t})\mathbf{b}) + f(\mathbf{tb} + (\mathbf{1}-\mathbf{t})\mathbf{a})\right]. \end{aligned} \tag{2.10}$$

Integrating inequality (2.10) with respect to $\mathbf{t}$ on $[\mathbf{0},\mathbf{1}]$ we get

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \le \frac{1}{2}\int_{\mathbf{0}}^{\mathbf{1}} \left[f\left(\mathbf{ta}+(\mathbf{1}-\mathbf{t})\,\mathbf{b}\right)+f\left(\mathbf{tb}+(\mathbf{1}-\mathbf{t})\,\mathbf{a}\right)\right] d\mathbf{t}$$
$$= \frac{1}{2}\int_{\mathbf{0}}^{\mathbf{1}} f\left(\mathbf{ta}+(\mathbf{1}-\mathbf{t})\,\mathbf{b}\right) d\mathbf{t} + \frac{1}{2}\int_{\mathbf{0}}^{\mathbf{1}} f\left(\mathbf{tb}+(\mathbf{1}-\mathbf{t})\,\mathbf{a}\right) d\mathbf{t}. \quad (2.11)$$

By putting $\mathbf{1}-\mathbf{t}=\mathbf{s}$, in the second integral on the right-hand side of (2.11), we get

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \le \frac{1}{2}\int_{\mathbf{0}}^{\mathbf{1}} f\left(\mathbf{ta}+(\mathbf{1}-\mathbf{t})\,\mathbf{b}\right) d\mathbf{t} + \frac{1}{2}\int_{\mathbf{0}}^{\mathbf{1}} f\left(\mathbf{tb}+(\mathbf{1}-\mathbf{t})\,\mathbf{a}\right) d\mathbf{t}$$
$$= \int_{\mathbf{0}}^{\mathbf{1}} f\left(\mathbf{ta}+(\mathbf{1}-\mathbf{t})\,\mathbf{b}\right) d\mathbf{t}. \quad (2.12)$$

From (2.9) and (2.12), we get

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \le \int_{\mathbf{0}}^{\mathbf{1}} f\left(\mathbf{ta}+(\mathbf{1}-\mathbf{t})\,\mathbf{b}\right) d\mathbf{t} \le \frac{1}{2^n}\sum_{\mathbf{c}} f\left(\mathbf{c}\right). \quad (2.13)$$

By putting $\mathbf{ta}+(\mathbf{1}-\mathbf{t})\,\mathbf{b}=\mathbf{x}$ in the integral involved in (2.13), it is easy to observe that

$$\int_{\mathbf{0}}^{\mathbf{1}} f\left(\mathbf{ta}+(\mathbf{1}-\mathbf{t})\,\mathbf{b}\right) d\mathbf{t} = \frac{1}{\mathbf{b}-\mathbf{a}}\int_{\mathbf{a}}^{\mathbf{b}} f\left(\mathbf{x}\right) d\mathbf{x}. \quad (2.14)$$

which proves the inequality (2.7). The sharpness follows by taking the function $f\left(\mathbf{x}\right)=\prod\limits_{i=1,\ldots,n} x_i$. If f is $\mathbf{n}$-fold concave, replacing $-f$ instead of f in (2.7) we get the required result. □

Next, we consider a weighted version of (2.7) which is known as Fejér inequality, before that we need the following preliminary lemma:

Lemma 2.7. *Let $f:[\mathbf{a},\mathbf{b}]\to\mathbb{R}$ be $\mathbf{n}$-fold convex function. Let $\mathbf{x_1}=\left(x_1^{(1)},\ldots,x_1^{(n)}\right)$, $\mathbf{x}_2=\left(x_2^{(1)},\ldots,x_2^{(n)}\right)$, $\mathbf{y_1}=\left(y_1^{(1)},\ldots,y_1^{(n)}\right)$, $\mathbf{y_2}=\left(y_2^{(1)},\ldots,y_2^{(n)}\right)$ be any vectors in $[\mathbf{a},\mathbf{b}]$ such that $\mathbf{a}\le\mathbf{y_1}\le\mathbf{x_1}\le\mathbf{x_2}\le\mathbf{y_2}\le\mathbf{b}$, with $\mathbf{x_1}+\mathbf{x_2}=\mathbf{y_1}+\mathbf{y_2}$. Then, for the convex partial mappings $f_i:[a_i,b_i]\to\mathbb{R}$, $f_i\left(t_i\right)=f\left(z_1,\ldots,z_{i-1},t_i,z_{i+1},\ldots,z_n\right)$, for all fixed $z_j\in[a_j,b_j]$ $(j=1,2,\cdots,n)$ with $j\ne i$. the following inequality holds:*

$$f\left(z_1,\ldots,z_{i-1},x_1^{(i)},z_{i+1},\ldots,z_n\right)+f\left(z_1,\ldots,z_{i-1},x_2^{(i)},z_{i+1},\ldots,z_n\right)$$
$$\le f\left(z_1,\ldots,z_{i-1},y_1^{(i)},z_{i+1},\ldots,z_n\right)+f\left(z_1,\ldots,z_{i-1},y_2^{(i)},z_{i+1},\ldots,z_n\right) \quad (2.15)$$

Proof. Consider $f_i:[a_i,b_i]\to\mathbb{R}$, $f_i\left(t_i\right)=f\left(z_1,\ldots,z_{i-1},t_i,z_{i+1},\ldots,z_n\right)$, for all fixed $z_j\in[a_j,b_j]$ $(j=1,2,\cdots,n)$ with $j\ne i$. If $\mathbf{y}_1=\mathbf{y}_2$ then we are done. Suppose $\mathbf{y}_1\ne\mathbf{y}_2$ and write

$$x_1^{(i)}=\frac{y_2^{(i)}-x_1^{(i)}}{y_2^{(i)}-y_1^{(i)}}y_1^{(i)}+\frac{x_1^{(i)}-y_1^{(i)}}{y_2^{(i)}-y_1^{(i)}}y_2^{(i)},$$

and

$$x_2^{(i)} = \frac{y_2^{(i)} - x_2^{(i)}}{y_2^{(i)} - y_1^{(i)}} y_1^{(i)} + \frac{x_2^{(i)} - y_1^{(i)}}{y_2^{(i)} - y_1^{(i)}} y_2^{(i)},$$

for all $i = 1, 2, \cdots, n$.

Since f_i is convex on $[a_i, b_i]$, we have

$$\begin{aligned}
&f\left(z_1, \ldots, z_{i-1}, x_1^{(i)}, z_{i+1}, \ldots, z_n\right) + f\left(z_1, \ldots, z_{i-1}, x_2^{(i)}, z_{i+1}, \ldots, z_n\right) \\
&\le \frac{y_2^{(i)} - x_1^{(i)}}{y_2^{(i)} - y_1^{(i)}} f\left(z_1, \ldots, z_{i-1}, y_1^{(i)}, z_{i+1}, \ldots, z_n\right) \\
&\qquad\qquad + \frac{x_1^{(i)} - y_1^{(i)}}{y_2^{(i)} - y_1^{(i)}} f\left(z_1, \ldots, z_{i-1}, y_2^{(i)}, z_{i+1}, \ldots, z_n\right) \\
&\qquad + \frac{y_2^{(i)} - x_2^{(i)}}{y_2^{(i)} - y_1^{(i)}} f\left(z_1, \ldots, z_{i-1}, y_1^{(i)}, z_{i+1}, \ldots, z_n\right) \\
&\qquad\qquad + \frac{x_2^{(i)} - y_1^{(i)}}{y_2^{(i)} - y_1^{(i)}} f\left(z_1, \ldots, z_{i-1}, y_2^{(i)}, z_{i+1}, \ldots, z_n\right) \\
&= \frac{2y_2^{(i)} - \left(x_1^{(i)} + x_2^{(i)}\right)}{y_2^{(i)} - y_1^{(i)}} f\left(z_1, \ldots, z_{i-1}, y_1^{(i)}, z_{i+1}, \ldots, z_n\right) \\
&\qquad + \frac{\left(x_1^{(i)} + x_2^{(i)}\right) - 2y_1^{(i)}}{y_2^{(i)} - y_1^{(i)}} f\left(z_1, \ldots, z_{i-1}, y_2^{(i)}, z_{i+1}, \ldots, z_n\right) \\
&= f\left(z_1, \ldots, z_{i-1}, y_1^{(i)}, z_{i+1}, \ldots, z_n\right) + f\left(z_1, \ldots, z_{i-1}, y_2^{(i)}, z_{i+1}, \ldots, z_n\right),
\end{aligned}$$

for all $i = 1, 2, \cdots, n$, which shows that (2.15) holds. □

A Fejěr type inequality may be stated as follows:

Theorem 2.8. *Let $f : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ be $\mathbf{n}$-fold convex. Then the double inequality*

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \le \frac{\int_{\mathbf{a}}^{\mathbf{b}} p(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}}{\int_{\mathbf{a}}^{\mathbf{b}} p(\mathbf{x}) d\mathbf{x}} \le \frac{1}{2^n} \sum_{\mathbf{c}} f(\mathbf{c}) \tag{2.16}$$

holds, where $p : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ is positive, integrable, and symmetric about $x_i = \frac{a_i+b_i}{2}$ for all $i = 1, 2, \cdots, n$. The above inequalities are sharp.

Proof. Since p is positive, integrable, and symmetric about $x_i = \frac{a_i+b_i}{2}$ for all $i = 1, 2, \cdots, n$. Then, by Lemma 2.7 one has:

$$\begin{aligned}
f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)\int\limits_{\mathbf{a}}^{\mathbf{b}} p(\mathbf{x})\,d\mathbf{x} &= \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) p(\mathbf{x})\,d\mathbf{x} \\
&\qquad + \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) p(\mathbf{a}+\mathbf{b}-\mathbf{x})\,d\mathbf{x} \\
&= \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} \left[f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) + f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)\right] p(\mathbf{x})\,d\mathbf{x} \\
&\le \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} \left[f(\mathbf{x}) + f(\mathbf{a}+\mathbf{b}-\mathbf{x})\right] p(\mathbf{x})\,d\mathbf{x} \\
&= \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} f(\mathbf{x})\,p(\mathbf{x})\,d\mathbf{x} + \int_{\frac{\mathbf{a}+\mathbf{b}}{2}}^{\mathbf{b}} f(\mathbf{x})\,p(\mathbf{x})\,d\mathbf{x} \\
&= \int_{\mathbf{a}}^{\mathbf{b}} p(\mathbf{x})\,f(\mathbf{x})\,d\mathbf{x}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2^n}\sum_{\mathbf{c}} f(\mathbf{c}) \int_{\mathbf{a}}^{\mathbf{b}} p(\mathbf{x})\,d\mathbf{x} \\
&= \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} \left[\frac{1}{2^n}\sum_{\mathbf{c}} f(\mathbf{c})\right] p(\mathbf{x})d\mathbf{x} + \int_{\frac{\mathbf{a}+\mathbf{b}}{2}}^{\mathbf{b}} \left[\frac{1}{2^n}\sum_{\mathbf{c}} f(\mathbf{c})\right] p(\mathbf{a}+\mathbf{b}-\mathbf{x})d\mathbf{x} \\
&= \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} \left[\frac{1}{2^n}\sum_{\mathbf{c}} f(\mathbf{c})\right] p(\mathbf{x})d\mathbf{x} \\
&\ge \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} \left[f(\mathbf{x}) + f(\mathbf{a}+\mathbf{b}-\mathbf{x})\right] p(\mathbf{x})\,d\mathbf{x} \\
&= \int_{\mathbf{a}}^{\frac{\mathbf{a}+\mathbf{b}}{2}} p(\mathbf{x})\,f(\mathbf{x})d\mathbf{x} + \int_{\frac{\mathbf{a}+\mathbf{b}}{2}}^{\mathbf{b}} p(\mathbf{x})\,f(\mathbf{x})d\mathbf{x} \\
&= \int_{\mathbf{a}}^{\mathbf{b}} p(\mathbf{x})\,f(\mathbf{x})\,d\mathbf{x},
\end{aligned}$$

which proves (2.16). To prove the sharpness in (2.16), take $p(\mathbf{x}) = 1$, then the inequality (2.16) is reduced to the double inequality (2.7), and therefore if we choose $f(\underset{\sim}{x}) = \prod\limits_{i=1,\dots,n} x_i$, in (2.16), then the equality holds, which shows that (2.16) is sharp, and thus the proof is completely finished. □

3. A Matrix version of H.–H. Inequality

A matrix function, $f(A)$, or function of a matrix can have several different meanings. It can be an operation on a matrix producing a scalar, such as $tra(A)$ and $\det(A)$; it can be a mapping from a matrix space to a matrix space, like $f(A) = A^2$; it can also be entrywise operations on the matrix, for instance, $g(A) = (a_{ij})^2$.

A natural generalization of the classical Hermite–Hadamard inequality (1.1) to Hermitian matrices could be the double inequality

$$f\left(\frac{A+B}{2}\right) \le \int_0^1 f\left(tA + (1-t)B\right)dt \le \frac{f(A)+f(B)}{2} \tag{3.1}$$

which is however not true, in general as shown recently in [11].

Moslehian [11] introduced several matrix and operator inequalities of Hermite-Hadamard type and he presented some operator inequalities of Hermite-Hadamard type in which the classical convexity was used instead of the operator convexity.

In this section, we introduce a matrix version of Hermite–Hadamard inequality for function of a matrix producing a scalar.

Let $\mathcal{M}_{n\times n}(\mathbb{R})$ be the set of all real $(n\times n)$–matrices with real entries, given a function $f:\mathcal{M}_{n\times n}(\mathbb{R})\to\mathbb{R}$ and $A,B\in\mathcal{M}_{n\times n}(\mathbb{R})$. Clearly, each square n-matrix is just a point in $\mathbb{R}^{n^2}$. For example a 2×2-matrix is just a point in $\mathbb{R}^4$; i.e., it has four real coordinates; e.g., the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is just the vector $(1,2,3,4)$. At first this may seem an oversimplification because it ignores the matrix product. Thus we define $\mathcal{M}_{2\times 2}(\mathbb{R})$ to be in $\mathbb{R}^4$ with the following product defined in it

$$(a,b,c,d)\,(u,v,x,y) = (au+bx, av+by, cu+dx, cv+dy)$$

which is just the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u & v \\ x & y \end{pmatrix}$$

written as a vector in $\mathbb{R}^4$. Finally, the integration limits A,B are just vectors in $\mathbb{R}^{n^2}$ (with n = 2 in our case). Thus the integral is really multiple integral.

To state our result we need to understand the following terminologies:

$$X = (x_{ij})_{n\times n} = (x_{11},\dots,x_{1n},x_{21},\dots,x_{2n},\dots,x_{n1},\dots,x_{nn})$$
$$A \le B \Leftrightarrow a_{ij} \le b_{ij}, \forall i,j = 1,\cdots,n.$$

Given a matrix function of real variables $F:\mathcal{M}_{n\times n}(\mathbb{R})\to\mathbb{R}$. For a matrix $C = (c_{ij})_{n\times n}$, we define

$$\sum_C F(C) := \sum_{\substack{c_{ij}\in\{a_{ij},b_{ij}\}\\ 1\le i,j\le n}} F\left((c_{ij})_{n\times n}\right).$$

for all possible choices of $c_{ij}\in\{a_{ij},b_{ij}\}$.

We define the matrix-interval to be $[A,B]=\prod_{j=1}^{n}\prod_{i=1}^{n}[a_{ij},b_{ij}]$, with length to be $B-A=\prod_{j=1}^{n}\prod_{i=1}^{n}(b_{ij}-a_{ij})$. Depending on this, we understand $\int_A^B f(X)\,dX$ to be:

$$\int_A^B f(X)\,dX=\int_{a_{nn}}^{b_{nn}}\cdots\int_{a_{11}}^{b_{11}} f(x_{11},\ldots,x_{nn})\,dx_{11}\cdots dx_{nn}.$$

Next result illustrate a matrix version of H.–H. inequality for function of a matrix producing a scalar:

Theorem 3.1. *Let $A,B\in\mathcal{M}_{n\times n}(\mathbb{R})$ with $A<B$. Let $f:[A,B]\to\mathbb{R}$ be $\mathbf{n}^2$-fold convex. Then the inequality*

$$f\left(\frac{A+B}{2}\right)\le\frac{1}{B-A}\int_A^B f(X)\,dX\le\frac{1}{2^{n^2}}\sum_{\substack{C=(c_{ij})_{n\times n}\\ c_{ij}\in\{a_{ij},b_{ij}\}\\ 1\le i,j\le n}} f(C). \qquad (3.2)$$

holds, where

$$\sum_C f(C):=\sum_{\substack{C=(c_{ij})_{n\times n}\\ c_{ij}\in\{a_{ij},b_{ij}\}\\ 1\le i,j\le n}} f(c_{11},c_{12},\cdots,,c_{nn}).$$

The inequality (3.2) is sharp. If f is $\mathbf{n}^2$-fold concave then the inequality (3.2) is reversed.

Proof. The proof follows directly from Theorem 2.6. □

Remark 3.2. A Jensen's type inequality for matrix functions used above; may be deduced in a similar manner as in Theorem 2.5.

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