ON POMPEIU–ČEBYŠEV TYPE INEQUALITIES FOR POSITIVE LINEAR MAPS OF SELFADJOINT OPERATORS IN INNER PRODUCT SPACES

MOHAMMAD W. ALOMARI

ABSTRACT. In this work, generalizations of some inequalities for continuous h-synchronous (h-asynchronous) functions of linear bounded selfadjoint operators under positive linear maps in Hilbert spaces are proved.

1. INTRODUCTION

Let $\mathcal{B}(H)$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$ with the identity operator 1_H in $\mathcal{B}(H)$. Let $A\in\mathcal{B}(H)$ be a selfadjoint linear operator on $(H;\langle\cdot,\cdot\rangle)$. Let $C(\mathrm{sp}(A))$ be the set of all continuous functions defined on the spectrum of A $(\mathrm{sp}(A))$ and let $C^*(A)$ be the C^*-algebra generated by A and the identity operator 1_H.

Let us define the map $\mathcal{G}: C(\mathrm{sp}(A))\to C^*(A)$ with the following properties ([5], p.3):

(1) $\mathcal{G}(\alpha f+\beta g)=\alpha\mathcal{G}(f)+\beta\mathcal{G}(g)$, for all scalars α,β.
(2) $\mathcal{G}(fg)=\mathcal{G}(f)\mathcal{G}(g)$ and $\mathcal{G}\left(\overline{f}\right)=\mathcal{G}(f)^*$; where $\overline{f}$ denotes to the conjugate of f and $\mathcal{G}(f)^*$ denotes to the Hermitian of $\mathcal{G}(f)$.
(3) $\|\mathcal{G}(f)\|=\|f\|=\sup\limits_{t\in\mathrm{sp}(A)}|f(t)|$.
(4) $\mathcal{G}(f_0)=1_H$ and $\mathcal{G}(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$ for all $t\in\mathrm{sp}(A)$.

Accordingly, we define the continuous functional calculus for a selfadjoint operator A by

$$f(A)=\mathcal{G}(f)\text{ for all } f\in C(\mathrm{sp}(A)).$$

If both f and g are real valued functions on $\mathrm{sp}(A)$ then the following important property holds:

$$f(t)\geq g(t)\text{ for all } t\in\mathrm{sp}(A)\ \text{ implies } f(A)\geq g(A), \tag{1.1}$$

in the operator order of $\mathcal{B}(H)$.

In [1] and formally in [2], the author of this paper generalized the concept of monotonicity as follows:

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Definition 1.1. A real valued function f defined on $[a,b]$ is said to be increasing (decreasing) with respect to a positive function $h:[a,b]\to\mathbb{R}_+$ or simply h-increasing (h-decreasing) if and only if

$$h(x)\,f(t)-h(t)\,f(x)\geq(\leq)\;0,$$

whenever $t\geq x$ for every $x,t\in[a,b]$. In special case if $h(x)=1$ we refer to the original monotonicity. Accordingly, for $0<a<b$ we say that f is t^r-increasing (t^r-decreasing) for $r\in\mathbb{R}$ if and only if

$$x\leq t\Longrightarrow x^r f(t)-t^r f(x)\geq(\leq)\;0$$

for every $x,t\in[a,b]$.

Example 1.2. Let $0<a<b$ and define $f:[a,b]\to\mathbb{R}$ given by

(1) $f(s)=1$, then f is t^r-decreasing for all $r>0$ and t^r-increasing for all $r<0$.
(2) $f(s)=s$, then f is t^r-decreasing for all $r>1$ and t^r-increasing for all $r<1$.
(3) $f(s)=s^{-1}$, then f is t^r-decreasing for all $r>-1$ and t^r-increasing for all $r<-1$.

Remark 1.3. Every h-increasing function is increasing. The converse need not be true. For more details see [2].

The concept of synchronization has a wide range of usage in several areas of mathematics. Simply, two functions $f,g:[a,b]\to\mathbb{R}$ are called synchronous (asynchronous) if and only if the inequality

$$(f(t)-f(x))\,(g(t)-g(x))\geq(\leq)\;0,$$

holds for all $x,t\in[a,b]$.

In [2], Alomari generalized the concept of synchronization of functions of real variables. Indeed, we have

Definition 1.4. The real valued functions $f,g:[a,b]\to\mathbb{R}$ are called synchronous (asynchronous) with respect to a non-negative function $h:[a,b]\to\mathbb{R}_+$ or simply h-synchronous (h-asynchronous) if and only if

$$(h(y)\,f(x)-h(x)\,f(y))\,(h(y)\,g(x)-h(x)\,g(y))\geq(\leq)\;0 \tag{1.2}$$

for all $x,y\in[a,b]$.

In other words if both f and g are either h-increasing or h-decreasing then

$$(h(y)\,f(x)-h(x)\,f(y))\,(h(y)\,g(x)-h(x)\,g(y))\geq 0.$$

While, if one of the function is h-increasing and the other is h-decreasing then

$$(h(y)\,f(x)-h(x)\,f(y))\,(h(y)\,g(x)-h(x)\,g(y))\leq 0.$$

In special case if $h(x)=1$ we refer to the original synchronization. Accordingly, for $0<a<b$ we say that f and g are t^r-synchronous (t^r-asynchronous) for $r\in\mathbb{R}$ if and only if

$$(x^r f(t)-t^r f(x))\,(x^r g(t)-t^r g(x))\geq(\leq)\;0$$

for every $x,t\in[a,b]$.

Remark 1.5. In Definition (1.4), if $f = g$ then f and g are always h-synchronous regardless of h-monotonicity of f (or g). In other words, a function f is always h-synchronous with itself.

Example 1.6. Let $0 < a < b$ and define $f, g : [a, b] \to \mathbb{R}$ given by

(1) $f(s) = 1 = g(s)$, then f and g are t^r-synchronous for all $r \in \mathbb{R}$.
(2) $f(s) = 1$ and $g(s) = s$, then f is t^r-synchronous for all $r \in (-\infty, 0) \cup (1, \infty)$ and t^r-asynchronous for all $0 < r < 1$.
(3) $f(s) = 1$ and $g(s) = s^{-1}$, then f is t^r-synchronous for all $r \in (-\infty, -1) \cup (0, \infty)$ and t^r-asynchronous for all $-1 < r < 0$.
(4) $f(s) = s$ and $g(s) = s^{-1}$, then f is t^r-synchronous for all $r \in (-\infty, -1) \cup (1, \infty)$ and t^r-asynchronous for all $-1 < r < 1$.

In [3], Dragomir studied the Čebyšev functional

$$C(f, g; A, x) := \langle f(A) g(A) x, x \rangle - \langle g(A) x, x \rangle \langle f(A) x, x \rangle, \tag{1.3}$$

for any selfadjoint operator $A \in \mathcal{B}(H)$ and $x \in H$ with $\|x\| = 1$.

In [3], proved the following result concerning continuous synchronous (asynchronous) functions of selfadjoint linear operators in Hilbert spaces.

Theorem 1.7. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and synchronous (asynchronous) on $[\gamma, \Gamma]$, then*

$$\langle f(A) g(A) x, x \rangle \geq (\leq) \langle g(A) x, x \rangle \langle f(A) x, x \rangle \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$.

In [2], Alomari generalized Theorem 1.7 for continuous h-synchronous (h-asynchronous) functions of selfadjoint linear operators in Hilbert spaces by introduciing the Pompeiu–Čebyšev functional such as:

$$\mathcal{P}(f, g, h; A, x) := \left\langle h^2(A) x, x \right\rangle \langle f(A) g(A) x, x \rangle - \langle h(A) g(A) x, x \rangle \langle h(A) f(A) x, x \rangle \tag{1.5}$$

for $x \in H$ with $\|x\| = 1$. This naturally, generalizes the Čebyšev functional (1.3).

Moreover, he proved the following essential result:

Theorem 1.8. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}_+$ be a non-negative and continuous function. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both f and g are h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then*

$$\left\langle h^2(A) x, x \right\rangle \langle f(A) g(A) x, x \rangle \geq (\leq) \langle h(A) g(A) x, x \rangle \langle h(A) f(A) x, x \rangle \tag{1.6}$$

for any $x \in H$ with $\|x\| = 1$.

For more related results, we refer the reader to [4], [6] and [7].

In this work, some inequalities for continuous h-synchronous (h-asynchronous) functions of linear bounded selfadjoint operators under positive linear maps in Hilbert spaces of the Pompeiu–Čebyšev functional (1.5) are proved. The proof Techniques are similar to that ones used in [4].

2. Main results

Let us start with the following result regarding the positivity of $\mathcal{P}(f,g,h;A,x)$.

Theorem 2.1. *Let A be a selfadjoint operator with $\operatorname{sp}(A)\subset[\gamma,\Gamma]$ for some real numbers γ,Γ with $\gamma<\Gamma$. Let $h:[\gamma,\Gamma]\to\mathbb{R}_+$ be a non-negative and continuous function. If $f,g:[\gamma,\Gamma]\to\mathbb{R}$ are continuous and both f and g are h-synchronous (h-asynchronous) on $[\gamma,\Gamma]$, then*

$$\begin{aligned}\left\langle \phi\left(h^2(B)\right)y,y\right\rangle\cdot\langle\varphi(f(A)g(A))x,x\rangle \\ +\left\langle\varphi\left(h^2(A)\right)x,x\right\rangle\cdot\langle\phi(f(B)g(B))y,y\rangle \\ \geq\langle\varphi(h(A)f(A))x,x\rangle\cdot\langle\phi(h(B)g(B))y,y\rangle \\ +\langle\varphi(h(A)g(A))x,x\rangle\cdot\langle\phi(h(B)f(B))y,y\rangle\end{aligned}\tag{2.1}$$

for each $x,y\in H$ with $\|x\|=\|y\|=1$.

$$\begin{aligned}\left\langle \phi\left(h^2(A)\right)y,y\right\rangle\cdot\langle\varphi(f(A)g(A))x,x\rangle \\ +\left\langle\varphi\left(h^2(A)\right)x,x\right\rangle\cdot\langle\phi(f(A)g(A))y,y\rangle \\ \geq(\leq)\langle\varphi(h(A)f(A))x,x\rangle\cdot\langle\phi(h(A)g(A))y,y\rangle \\ +\langle\varphi(h(A)g(A))x,x\rangle\cdot\langle\phi(h(A)f(A))y,y\rangle\end{aligned}\tag{2.2}$$

for each $x\in H$ with $\|x\ =1$.

Proof. Since f and g are h-synchronous then

$$(h(s)f(t)-h(t)f(s))(h(s)g(t)-h(t)g(s))\geq 0,$$

and this is allow us to write

$$\begin{aligned}h^2(s)f(t)g(t)+h^2(t)f(s)g(s) \\ \geq h(s)h(t)f(t)g(s)+h(s)h(t)g(t)f(s)\end{aligned}\tag{2.3}$$

for all $t,s\in[a,b]$. We fix $s\in[a,b]$ and apply the functional calculus; property (1.1) for inequality (2.3) for the operator A, then we have for each $x\in H$ with $\|x\|=1$, that

$$\begin{aligned}h^2(s)1_H\cdot f(A)g(A)+h^2(A)\cdot f(s)g(s)1_H \\ \geq h(A)f(A)\cdot h(s)g(s)1_H+h(A)g(A)\cdot h(s)f(s)1_H,\end{aligned}$$

and since φ is normalized positive linear map we get

$$\begin{aligned}h^2(s)1_H\cdot\varphi(f(A)g(A))+\varphi\left(h^2(A)\right)\cdot f(s)g(s)1_H \\ \geq\varphi(h(A)f(A))\cdot h(s)g(s)1_H+\varphi(h(A)g(A))\cdot h(s)f(s)1_H,\end{aligned}$$

and this is equivalent to write

$$\begin{aligned}h^2(s)1_H\cdot\langle\varphi(f(A)g(A))x,x\rangle+\left\langle\varphi\left(h^2(A)\right)x,x\right\rangle\cdot f(s)g(s)1_H \\ \geq\langle\varphi(h(A)f(A))x,x\rangle\cdot h(s)g(s)1_H+\langle\varphi(h(A)g(A))x,x\rangle\cdot h(s)f(s)1_H,\end{aligned}\tag{2.4}$$

Applying property (1.1) again for inequality (2.4) but for the operator B, then we have for each $y \in H$ with $\|y\| = 1$, that

$$\begin{aligned}
&h^2(B) \cdot \langle \varphi(f(A)g(A))x, x\rangle + \left\langle \varphi\left(h^2(A)\right)x, x\right\rangle \cdot f(B)g(B) \\
&\qquad \geq \langle \varphi(h(A)f(A))x, x\rangle \cdot h(B)g(B) + \langle \varphi(h(A)g(A))x, x\rangle \cdot h(B)f(B),
\end{aligned}$$

and since ϕ is normalized positive linear map we get

$$\begin{aligned}
&\left\langle \phi\left(h^2(B)\right)y, y\right\rangle \cdot \langle \varphi(f(A)g(A))x, x\rangle + \left\langle \varphi\left(h^2(A)\right)x, x\right\rangle \cdot \langle \phi(f(B)g(B))y, y\rangle \\
&\geq \langle \varphi(h(A)f(A))x, x\rangle \cdot \langle \phi(h(B)g(B))y, y\rangle + \langle \varphi(h(A)g(A))x, x\rangle \cdot \langle \phi(h(B)f(B))y, y\rangle,
\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, which gives the required results in (2.1). To obtain (2.2) we set $B = A$ in (2.1). The revers case follows trivially, and this completes the proof. □

Corollary 2.2. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}_+$ be a non-negative and continuous function. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both f and g are synchronous (asynchronous) on $[\gamma, \Gamma]$, then*

$$\begin{aligned}
&\langle \varphi(f(A)g(A))x, x\rangle + \langle \phi(f(B)g(B))y, y\rangle \\
&\qquad \geq (\leq) \langle \varphi(f(A))x, x\rangle \langle \phi(g(B))y, y\rangle + \langle \varphi(g(A))x, x\rangle \langle \phi(f(B))y, y\rangle
\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. In special case, the following Čebyšev inequality for positive linear maps of selfadjoint operator is valid

$$\begin{aligned}
&\langle \varphi(f(A)g(A))x, x\rangle + \langle \varphi(f(A)g(A))x, x\rangle \\
&\qquad \geq (\leq) \langle \varphi(f(A))x, x\rangle \langle \varphi(g(A))x, x\rangle + \langle \varphi(g(A))x, x\rangle \langle \varphi(f(A))x, x\rangle
\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Setting $h(t) = 1$ in both (2.1) and (2.2). Also, in (2.2) take $\phi = \varphi$, $B = A$ and $y = x$. □

Remark 2.3. Setting $\phi = \varphi$, $B = A$ and $y = x$ in (2.1), we get

$$\begin{aligned}
&\left\langle \varphi\left(h^2(A)\right)x, x\right\rangle \cdot \langle \varphi(f(A)g(A))x, x\rangle \\
&\qquad + \left\langle \varphi\left(h^2(A)\right)x, x\right\rangle \cdot \langle \varphi(f(A)g(A))x, x\rangle \\
&\qquad \geq (\leq) \langle \varphi(h(A)f(A))x, x\rangle \cdot \langle \varphi(h(A)g(A))x, x\rangle \\
&\qquad\qquad + \langle \varphi(h(A)g(A))x, x\rangle \cdot \langle \varphi(h(A)f(A))x, x\rangle
\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

The following generalization of Cauchy-Schwarz inequality holds.

Corollary 2.4. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}_+$ be a non-negative and continuous* *function. If* $f:[\gamma,\Gamma]\to\mathbb{R}$ *is continuous and* h*-synchronous on* $[\gamma,\Gamma]$*, then*

$$\left\langle \phi\left(h^2\left(B\right)\right)y,y\right\rangle \cdot \left\langle \varphi\left(f^2\left(A\right)\right)x,x\right\rangle + \left\langle \varphi\left(h^2\left(A\right)\right)x,x\right\rangle \cdot \left\langle \phi\left(f^2\left(B\right)\right)y,y\right\rangle \\ \geq 2\left\langle \varphi\left(h\left(A\right)f\left(A\right)\right)x,x\right\rangle \cdot \left\langle \phi\left(h\left(B\right)f\left(B\right)\right)y,y\right\rangle \quad (2.5)$$

for each $x,y\in H$ *with* $\|x\|=\|y\|=1$*. In particular, we have*

$$\left\langle \varphi\left(h^2\left(A\right)\right)x,x\right\rangle \cdot \left\langle \varphi\left(f^2\left(A\right)\right)x,x\right\rangle \geq \left\langle \varphi\left(h\left(A\right)f\left(A\right)\right)x,x\right\rangle^2 \quad (2.6)$$

for each $x\in H$ *with* $\|x\|=1$*.*

Proof. Setting $f=g$ in both (2.1) and (2.2). Also, in (2.2) take $\phi=\varphi$, $B=A$ and $y=x$, so that the desired results hold. □

Corollary 2.5. *Let* A *be a selfadjoint operator with* $\operatorname{sp}(A)\subset[\gamma,\Gamma]$ *for some real numbers* γ,Γ *with* $0<\gamma<\Gamma$*. If* $f,g:[\gamma,\Gamma]\to\mathbb{R}$ *are continuous and* t*-synchronous (*t*-asynchronous) on* $[\gamma,\Gamma]$*, then*

$$\left\langle \phi\left(B^2\right)y,y\right\rangle \cdot \left\langle \varphi\left(f\left(A\right)g\left(A\right)\right)x,x\right\rangle + \left\langle \varphi\left(A^2\right)x,x\right\rangle \cdot \left\langle \phi\left(f\left(B\right)g\left(B\right)\right)y,y\right\rangle \\ \geq (\leq)\left\langle \varphi\left(Af\left(A\right)\right)x,x\right\rangle \cdot \left\langle \phi\left(Bg\left(B\right)\right)y,y\right\rangle \\ + \left\langle \varphi\left(Ag\left(A\right)\right)x,x\right\rangle \cdot \left\langle \phi\left(Bf\left(B\right)\right)y,y\right\rangle \quad (2.7)$$

for each $x,y\in H$ *with* $\|x\|=\|y\|=1$*.*

Proof. Setting $h(t)=t$ in (2.1) we get the desired result. □

Before we state our next remark, we interested to give the following example.

Example 2.6.
(1) If $f(s)=s^p$ and $g(s)=s^q$ $(s>0)$, then f and g are t^r-synchronous for all $p,q>r>0$ and t^r-asynchronous for all $p>r>q>0$.
(2) If $f(s)=s^p$ and $g(s)=\log(s)$ $(s>1)$, then f is t^r-synchronous for all $p<r<0$ and t^r-asynchronous for all $r<p<0$.
(3) If $f(s)=\exp(s)=g(s)$, then f is t^r-synchronous for all for all $r\in\mathbb{R}$.

Remark 2.7. Using Example 2.6 we can observe the following special cases:

(1) If $f\left(s\right)=s^p$ and $g\left(s\right)=s^q$ $(s>0)$, then f and g are t^r-synchronous for all $p,q>r>0$, so that we have

$$\left\langle \phi\left(B^{2r}\right)y,y\right\rangle \left\langle \varphi\left(A^{p+q}\right)x,x\right\rangle + \left\langle \varphi\left(A^{2r}\right)x,x\right\rangle \left\langle \phi\left(B^{p+q}\right)y,y\right\rangle \\ \geq \left\langle \varphi\left(B^{q+r}\right)y,y\right\rangle \left\langle \phi\left(A^{p+r}\right)x,x\right\rangle + \left\langle \varphi\left(A^{q+r}\right)x,x\right\rangle \left\langle \phi\left(B^{p+r}\right)y,y\right\rangle.$$

If $p>r>q>0$, then f and g are t^r-asynchronous and thus the reverse inequality holds.

(2) If $f\left(s\right)=s^p$ and $g\left(s\right)=\log s$ $(s>1)$, then f and g are t^r-synchronous for all $p<r<0$ we have

$$\left\langle \phi\left(B^{2r}\right)y,y\right\rangle \left\langle \varphi\left(A^p\log\left(A\right)\right)x,x\right\rangle + \left\langle \varphi\left(A^{2r}\right)x,x\right\rangle \left\langle \phi\left(B^p\log\left(B\right)\right)y,y\right\rangle \\ \geq \left\langle \varphi\left(B^r\log\left(B\right)\right)y,y\right\rangle \left\langle \phi\left(A^{p+r}\right)x,x\right\rangle + \left\langle \varphi\left(A\log\left(A\right)\right)x,x\right\rangle \left\langle \phi\left(B^{p+r}\right)y,y\right\rangle.$$

If $r < p < 0$, then f and g are t^r-asynchronous and thus the reverse inequality holds.

(3) If $f(s) = \exp(s) = g(s)$, then f and g are t^r-synchronous for all $r \in \mathbb{R}$, so that we have

$$\left\langle \phi\left(B^{2r}\right) y, y\right\rangle \left\langle \varphi\left(\exp\left(2A\right)\right) x, x\right\rangle + \left\langle \varphi\left(A^{2r}\right) x, x\right\rangle \left\langle \phi\left(\exp\left(2B\right)\right) y, y\right\rangle \\ \geq 2\left\langle \varphi\left(A^{r}\exp\left(A\right)\right) x, x\right\rangle \left\langle \phi\left(B^{r}\exp\left(B\right)\right) y, y\right\rangle .$$

Therefore, by choosing an appropriate function h such that the assumptions in Remark 2.7 are fulfilled then one may generate family of inequalities from (2.1).

Corollary 2.8. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $0 < \gamma < \Gamma$. If $f : [\gamma, \Gamma] \to \mathbb{R}$ is continuous and f is t-synchronous on $[\gamma, \Gamma]$, then*

$$\left\langle \varphi\left(A^{2}\right) x, x\right\rangle \cdot \left\langle \varphi\left(f^{2}\left(A\right)\right) x, x\right\rangle \geq \left\langle \varphi\left(Af\left(A\right)\right) x, x\right\rangle^{2} \tag{2.8}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Setting $f = g$, $\phi = \varphi$, $B = A$ and $y = x$ in Corollary 2.5 we get the desired result. □

Corollary 2.9. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f : [\gamma, \Gamma] \to \mathbb{R}$ is continuous and h-synchronous, then*

$$\left\langle \phi\left(h^{2}\left(B\right)\right) y, y\right\rangle \cdot \left\langle \varphi\left(f\left(A\right)\right) x, x\right\rangle + \left\langle \varphi\left(h^{2}\left(A\right)\right) x, x\right\rangle \cdot \left\langle \phi\left(f\left(B\right)\right) y, y\right\rangle \\ \geq \left\langle \varphi\left(h\left(A\right) f\left(A\right)\right) x, x\right\rangle \cdot \left\langle \phi\left(h\left(B\right)\right) y, y\right\rangle \\ + \left\langle \varphi\left(h\left(A\right)\right) x, x\right\rangle \cdot \left\langle \phi\left(h\left(B\right) f\left(B\right)\right) y, y\right\rangle \tag{2.9}$$

for each $x \in H$ with $\|x\| = 1$. In particular, we have

$$\left\langle \phi\left(h^{2}\left(A^{-1}\right)\right) x, x\right\rangle \cdot \left\langle \varphi\left(f\left(A\right)\right) x, x\right\rangle + \left\langle \varphi\left(h^{2}\left(A\right)\right) x, x\right\rangle \cdot \left\langle \phi\left(f\left(A^{-1}\right)\right) x, x\right\rangle \\ \geq \left\langle \varphi\left(h\left(A\right) f\left(A\right)\right) x, x\right\rangle \cdot \left\langle \phi\left(h\left(A^{-1}\right)\right) x, x\right\rangle \\ + \left\langle \varphi\left(h\left(A\right)\right) x, x\right\rangle \cdot \left\langle \phi\left(h\left(A^{-1}\right) f\left(A^{-1}\right)\right) x, x\right\rangle \tag{2.10}$$

Proof. Setting $g = 1$ in (2.1) we get the first inequality (2.9). The second inequality holds by setting $B = A^{-1}$ and $y = x$ in (2.9). □

Theorem 2.10. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both f and g are h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then*

$$\left\langle \phi\left(h^{2}\left(B\right)\right) y, y\right\rangle \cdot f\left(\left\langle \varphi\left(A\right) x, x\right\rangle\right) g\left(\left\langle \varphi\left(A\right) x, x\right\rangle\right) \\ + h^{2}\left(\left\langle \varphi\left(A\right) x, x\right\rangle\right) \cdot \left\langle \phi\left(f\left(B\right) g\left(B\right)\right) y, y\right\rangle \\ \geq (\leq) \left\langle \phi\left(h\left(B\right) g\left(B\right)\right) y, y\right\rangle f\left(\left\langle \varphi\left(A\right) x, x\right\rangle\right) h\left(\left\langle \varphi\left(A\right) x, x\right\rangle\right) \\ + \left\langle \phi\left(f\left(B\right) h\left(B\right)\right) y, y\right\rangle h\left(\left\langle \varphi\left(A\right) x, x\right\rangle\right) g\left(\left\langle \varphi\left(A\right) x, x\right\rangle\right) \tag{2.11}$$

for any $x \in K$ *with* $\|x\| = \|y\| = 1$.

Proof. Since $\gamma 1_H \leq \langle Ax, x\rangle \leq \Gamma 1_H$ then by employing φ, we get $\gamma 1_K \leq \varphi(A) \leq \Gamma 1_K$. So that $\gamma \leq \langle \varphi(A)x, x\rangle \leq \Gamma$ for any $x \in K$ with $\|x\| = 1$. Since f, g are synchronous

$$\begin{aligned}[t] &[(h(\langle \varphi(A)x,x\rangle) f(t) - h(t) f(\langle \varphi(A)x,x\rangle)] \\ &\qquad \times [h(\langle \varphi(A)x,x\rangle) g(t) - h(t) g(\langle \varphi(A)x,x\rangle)] \geq 0 \end{aligned} \tag{2.12}$$

for any $t \in [\gamma, \Gamma]$ for any $x \in K$ with $\|x\| = 1$.

Simplyfying the terms we have

$$\begin{aligned} &h^2(t) f(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle) \\ &\qquad + h^2(\langle \varphi(A)x,x\rangle) \cdot f(t) g(t) \\ &\geq h(t) g(t) f(\langle \varphi(A)x,x\rangle) h(\langle \varphi(A)x,x\rangle) \\ &\qquad + f(t) h(t) h(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle). \end{aligned} \tag{2.13}$$

Fix $x \in K$ with $\|x\| = 1$. By utilizing the continuous functional calculus for the operator B we have by the property (1.1) for inequality (2.13) we have

$$\begin{aligned} &h^2(B) f(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle) \\ &\qquad + h^2(\langle \varphi(A)x,x\rangle) \cdot f(B) g(B) \\ &\geq h(B) g(B) f(\langle \varphi(A)x,x\rangle) h(\langle \varphi(A)x,x\rangle) \\ &\qquad + f(B) h(B) h(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle). \end{aligned} \tag{2.14}$$

Taking the map ϕ in the inequality (2.14), we get

$$\begin{aligned} &\phi\left(h^2(B)\right) f(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle) \\ &\qquad + h^2(\langle \varphi(A)x,x\rangle) \cdot \phi(f(B) g(B)) \\ &\geq \phi(h(B) g(B)) f(\langle \varphi(A)x,x\rangle) h(\langle \varphi(A)x,x\rangle) \\ &\qquad + \phi(f(B) h(B)) h(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle). \end{aligned} \tag{2.15}$$

for any bounded linear operator B with $\operatorname{sp}(B) \subseteq [\gamma, \Gamma]$ and $y \in H$ with $\|y\| = 1$.

So that we can write (2.15) in the form

$$\begin{aligned} &\left\langle \phi\left(h^2(B)\right) y, y\right\rangle f(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle) \\ &\qquad + h^2(\langle \varphi(A)x,x\rangle) \cdot \langle \phi(f(B) g(B)) y, y\rangle \\ &\geq \langle \phi(h(B) g(B)) y, y\rangle f(\langle \varphi(A)x,x\rangle) h(\langle \varphi(A)x,x\rangle) \\ &\qquad + \langle \phi(f(B) h(B)) y, y\rangle h(\langle \varphi(A)x,x\rangle) g(\langle \varphi(A)x,x\rangle). \end{aligned}$$

for each $x, y \in K$ with $\|x\| = \|y\| = 1$, which proves the inequality in (2.11). The reverse sense follows similarly, and the proof is completed. □

Remark 2.11. Taking $\phi = \varphi$ in (2.12) we get

$$\begin{aligned}\left\langle \varphi\left(h^2(B)\right) y, y\right\rangle \cdot f(\langle\varphi(A)x,x\rangle)\, g(\langle\varphi(A)x,x\rangle) \\ + h^2(\langle\varphi(A)x,x\rangle)\cdot\langle\varphi(f(B)g(B))y,y\rangle\cdot \\ \ge (\le) \langle\varphi(h(B)g(B))y,y\rangle f(\langle\varphi(A)x,x\rangle)\, h(\langle\varphi(A)x,x\rangle) \\ + \langle\varphi(f(B)h(B))y,y\rangle h(\langle\varphi(A)x,x\rangle)\, g(\langle\varphi(A)x,x\rangle).\end{aligned}$$

Also, by setting $B = A$ in (2.12) we get

$$\begin{aligned}\left\langle \phi\left(h^2(A)\right) y, y\right\rangle \cdot f(\langle\varphi(A)x,x\rangle)\, g(\langle\varphi(A)x,x\rangle) \\ + h^2(\langle\varphi(A)x,x\rangle)\cdot\langle\phi(f(A)g(A))y,y\rangle \\ \ge (\le) \langle\phi(h(A)g(A))y,y\rangle f(\langle\varphi(A)x,x\rangle)\, h(\langle\varphi(A)x,x\rangle) \\ + \langle\phi(f(A)h(A))y,y\rangle h(\langle\varphi(A)x,x\rangle)\, g(\langle\varphi(A)x,x\rangle).\end{aligned}$$

Corollary 2.12. *Let A be a selfadjoint operator with $\mathrm{sp}(A) \subset [\gamma,\Gamma]$ for some real numbers γ,Γ with $\gamma < \Gamma$. Let $h : [\gamma,\Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f : [\gamma,\Gamma] \to \mathbb{R}$ is continuous and h-synchronous on $[\gamma,\Gamma]$, then*

$$\begin{aligned}\left\langle \phi\left(h^2(B)\right) y, y\right\rangle \cdot f^2(\langle\varphi(A)x,x\rangle) + \left\langle \phi\left(f^2(B)\right) y, y\right\rangle \cdot h^2(\langle\varphi(A)x,x\rangle) \\ \ge (\le) 2\langle\phi(h(B)f(B))y,y\rangle f(\langle\varphi(A)x,x\rangle)\, h(\langle\varphi(A)x,x\rangle)\end{aligned} \tag{2.16}$$

for any $x \in K$ with $\|x\| = \|y\| = 1$. In particular, we have

$$\begin{aligned}\left\langle \varphi\left(h^2(B)\right) y, y\right\rangle \cdot f^2(\langle\varphi(A)x,x\rangle) + \left\langle \varphi\left(f^2(B)\right) y, y\right\rangle \cdot h^2(\langle\varphi(A)x,x\rangle) \\ \ge (\le) 2\langle\varphi(h(B)f(B))y,y\rangle f(\langle\varphi(A)x,x\rangle)\, h(\langle\varphi(A)x,x\rangle),\end{aligned}$$

also, we have

$$\begin{aligned}\left\langle \phi\left(h^2(A)\right) y, y\right\rangle \cdot f^2(\langle\varphi(A)x,x\rangle) + \left\langle \phi\left(f^2(A)\right) y, y\right\rangle \cdot h^2(\langle\varphi(A)x,x\rangle) \\ \ge (\le) 2\langle\phi(h(A)f(A))y,y\rangle f(\langle\varphi(A)x,x\rangle)\, h(\langle\varphi(A)x,x\rangle).\end{aligned}$$

for any $x \in K$ with $\|x\| = \|y\| = 1$.

Proof. Setting $f = g$ in (2.11), respectively, we get the required results. □

Corollary 2.13. *Let A be a selfadjoint operator with $\mathrm{sp}(A) \subset [\gamma,\Gamma]$ for some real numbers γ,Γ with $0 < \gamma < \Gamma$. If $f : [\gamma,\Gamma] \to \mathbb{R}$ are continuous and t-synchronous on $[\gamma,\Gamma]$, then*

$$\begin{aligned}\left\langle \phi\left(B^2\right) y, y\right\rangle \cdot f^2(\langle\varphi(A)x,x\rangle) + \left\langle \phi\left(f^2(B)\right) y, y\right\rangle \cdot \langle\varphi(A)x,x\rangle^2 \\ \ge (\le) 2\langle\phi(Bf(B))y,y\rangle f(\langle\varphi(A)x,x\rangle)\langle\varphi(A)x,x\rangle\end{aligned} \tag{2.17}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Setting $h(t) = t$ in (2.16), respectively, we get the required results. □

Theorem 2.14. *Let A be a selfadjoint operator with $\mathrm{sp}(A) \subset [\gamma,\Gamma]$ for some real numbers γ,Γ with $\gamma < \Gamma$. Let $h : [\gamma,\Gamma] \to \mathbb{R}_+$ be a positive function on $[\gamma,\Gamma]$. If*

$f,g:[\gamma,\Gamma]\to\mathbb{R}_+$ are both positve, convex and h-synchronous on $[\gamma,\Gamma]$, then

$$\begin{aligned} &h^2(\langle Ax,x\rangle)\langle f(B)y,y\rangle\cdot\langle g(B)y,y\rangle+h^2(\langle By,y\rangle)\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle \\ &\geq h(\langle Ax,x\rangle)h(\langle By,y\rangle)[f(\langle By,y\rangle)g(\langle Ax,x\rangle)+f(\langle Ax,x\rangle)g(\langle By,y\rangle)] \end{aligned} \tag{2.18}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Proof. Since f,g are h-synchronous and $\gamma\leq\langle Ax,x\rangle\leq\Gamma$, $\gamma\leq\langle By,y\rangle\leq\Gamma$ for any $x,y\in H$ with $\|x\|=\|y\|=1$, we have

$$\begin{aligned} &(h(\langle Ax,x\rangle)f(\langle By,y\rangle)-h(\langle By,y\rangle)f(\langle Ax,x\rangle)) \\ &\qquad\times(h(\langle Ax,x\rangle)g(\langle By,y\rangle)-h(\langle By,y\rangle)g(\langle Ax,x\rangle))\geq 0 \end{aligned} \tag{2.19}$$

for any $t\in[a,b]$ for any $x\in H$ with $\|x\|=1$.

Employing property (1.1) for inequality (2.19) we have

$$\begin{aligned} &h^2(\langle Ax,x\rangle)f(\langle By,y\rangle)g(\langle By,y\rangle) \\ &\qquad+h^2(\langle By,y\rangle)f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) \\ &\qquad-h(\langle Ax,x\rangle)h(\langle By,y\rangle)f(\langle By,y\rangle)g(\langle Ax,x\rangle) \\ &\qquad-h(\langle By,y\rangle)h(\langle Ax,x\rangle)f(\langle Ax,x\rangle)g(\langle By,y\rangle)\geq 0 \end{aligned} \tag{2.20}$$

for any bounded linear operator B with $\operatorname{sp}(B)\subseteq[\gamma,\Gamma]$ and $y\in H$ with $\|y\|=1$.

Now, since f and g are convex then we have

$$\begin{aligned} &h^2(\langle Ax,x\rangle)\langle f(B)y,y\rangle\cdot\langle g(B)y,y\rangle+h^2(\langle By,y\rangle)\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle \\ &\geq h^2(\langle Ax,x\rangle)f(\langle By,y\rangle)\cdot g(\langle By,y\rangle)+h^2(\langle By,y\rangle)f(\langle Ax,x\rangle)\cdot g(\langle Ax,x\rangle) \\ &\geq h(\langle Ax,x\rangle)h(\langle By,y\rangle)[f(\langle By,y\rangle)g(\langle Ax,x\rangle)+f(\langle Ax,x\rangle)g(\langle By,y\rangle)] \end{aligned} \tag{2.21}$$

for each $x,y\in H$ with $\|x\|=\|y\|=1$. Setting $B=A^{-1}$ and $y=x$ in (2.21) we get the required result in (2.18). The reverse sense follows similarly. □

Theorem 2.15. *Let A be a selfadjoint operator with $\operatorname{sp}(A)\subset[\gamma,\Gamma]$ for some real numbers γ,Γ with $\gamma<\Gamma$. Let $h:[\gamma,\Gamma]\to\mathbb{R}_+$ be a positive function on $[\gamma,\Gamma]$. If $f,g:[\gamma,\Gamma]\to\mathbb{R}_+$ are both positve, convex and h-synchronous on $[\gamma,\Gamma]$, then*

$$\begin{aligned} &h^2(\langle\varphi(A)x,x\rangle)\langle\phi(f(B))y,y\rangle\cdot\langle\phi(g(B))y,y\rangle \\ &\qquad+h^2(\langle\phi(B)y,y\rangle)\langle\varphi(f(A))x,x\rangle\cdot\langle\varphi(g(A))x,x\rangle \\ &\geq h^2(\langle\varphi(A)x,x\rangle)f(\langle\phi(B)y,y\rangle)\cdot g(\langle\phi(B)y,y\rangle) \\ &\qquad+h^2(\langle\phi(B)y,y\rangle)f(\langle\varphi(A)x,x\rangle)\cdot g(\langle\varphi(A)x,x\rangle) \\ &\geq h(\langle\varphi(A)x,x\rangle)h(\langle\phi(B)y,y\rangle)\times[f(\langle\phi(B)y,y\rangle)g(\langle\varphi(A)x,x\rangle) \\ &\qquad\qquad+f(\langle\varphi(A)x,x\rangle)g(\langle\phi(B)y,y\rangle)] \end{aligned} \tag{2.22}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Proof. Since $\gamma\cdot 1_H\leq A,B\leq\Gamma\cdot 1_H$ then $\gamma\cdot 1_K\leq\varphi(A)\leq\Gamma\cdot 1_K$ and $\gamma\cdot 1_K\leq\phi(B)\leq\Gamma\cdot 1_K$. So that for any $x,y\in H$ with $\|x\|=\|y\|=1$, we have $\gamma\leq\langle\varphi(A)x,x\rangle\leq\Gamma$

and $\gamma \le \langle \phi(B) y, y\rangle \le \Gamma$

$$\begin{aligned}&\left(h\left(\langle\varphi\left(A\right)x,x\rangle\right)f\left(\langle\phi\left(B\right)y,y\rangle\right)-h\left(\langle\phi\left(B\right)y,y\rangle\right)f\left(\langle\varphi\left(A\right)x,x\rangle\right)\right)\\&\times\left(h\left(\langle\varphi\left(A\right)x,x\rangle\right)g\left(\langle\phi\left(B\right)y,y\rangle\right)-h\left(\langle\phi\left(B\right)y,y\rangle\right)g\left(\langle\varphi\left(A\right)x,x\rangle\right)\right)\ge 0 \qquad (2.23)\end{aligned}$$

for any $t \in [a,b]$ for any $x \in H$ with $\|x\| = 1$.

Employing property (1.1) for inequality (2.23) we have

$$\begin{aligned}&h^2\left(\langle\varphi\left(A\right)x,x\rangle\right)f\left(\langle\phi\left(B\right)y,y\rangle\right)g\left(\langle\phi\left(B\right)y,y\rangle\right)\\&\quad+h^2\left(\langle\phi\left(B\right)y,y\rangle\right)f\left(\langle\varphi\left(A\right)x,x\rangle\right)g\left(\langle\varphi\left(A\right)x,x\rangle\right)\\&\quad-h\left(\langle\varphi\left(A\right)x,x\rangle\right)h\left(\langle\phi\left(B\right)y,y\rangle\right)f\left(\langle\phi\left(B\right)y,y\rangle\right)g\left(\langle\varphi\left(A\right)x,x\rangle\right)\\&\quad-h\left(\langle\phi\left(B\right)y,y\rangle\right)h\left(\langle\varphi\left(A\right)x,x\rangle\right)f\left(\langle\varphi\left(A\right)x,x\rangle\right)g\left(\langle\phi\left(B\right)y,y\rangle\right)\ge 0. \qquad (2.24)\end{aligned}$$

Now, since f and g are postive convex functions then we have

$$\begin{aligned}&h^2\left(\langle\varphi\left(A\right)x,x\rangle\right)\langle\phi\left(f\left(B\right)\right)y,y\rangle\cdot\langle\phi\left(g\left(B\right)\right)y,y\rangle\\&\quad+h^2\left(\langle\phi\left(B\right)y,y\rangle\right)\langle\varphi\left(f\left(A\right)\right)x,x\rangle\cdot\langle\varphi\left(g\left(A\right)\right)x,x\rangle\\&\ge h^2\left(\langle\varphi\left(A\right)x,x\rangle\right)f\left(\langle\phi\left(B\right)y,y\rangle\right)\cdot g\left(\langle\phi\left(B\right)y,y\rangle\right)\\&\quad+h^2\left(\langle\phi\left(B\right)y,y\rangle\right)f\left(\langle\varphi\left(A\right)x,x\rangle\right)\cdot g\left(\langle\varphi\left(A\right)x,x\rangle\right)\\&\ge h\left(\langle\varphi\left(A\right)x,x\rangle\right)h\left(\langle\phi\left(B\right)y,y\rangle\right)\times\left[f\left(\langle\phi\left(B\right)y,y\rangle\right)g\left(\langle\varphi\left(A\right)x,x\rangle\right)\right.\\&\qquad\left.+f\left(\langle\varphi\left(A\right)x,x\rangle\right)g\left(\langle\phi\left(B\right)y,y\rangle\right)\right]\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, which proves the required result in (2.22). The reverse sense follows similarly. □

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Department of Mathematics, Faculty of Science and Information Technology, Irbid National University, P.O. Box 2600, Irbid, P.C. 21110, Jordan.

Email address: mwomath@gmail.com