On Almost Alpha Kenmotsu (κ, μ)-Spaces

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Abstract

In this paper, the geometry of almost alpha Kenmotsu (κ, μ)-spaces are studied. Finally, we give an illustrative example on almost alpha Kenmotsu (κ, μ)-space of dimension 3.

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Introduction

Manifolds known as Kenmotsu manifolds have been studied by K. Kenmotsu (see [8]). The author set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension. A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.Kenmotsu manifolds can be qualifed through their Levi-Civita connection, given by $(\nabla X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$, for any vector fields X and Y. Kenmotsu described a certain structure similar to the warped product and it was characterized by tensor equations. The author showed that such a manifold M^{2n+1} is locally a warped product $(-\varepsilon, +\varepsilon) \times f\ N^{2n}$ being a Kaehlerian manifold and $f(t) = cet$ where c is a positive constant. Moreover, Kenmotsu showed locally symmetric Kenmotsu manifolds are of constant curvature -1 that means locally symmetry is a strong restriction for Kenmotsu manifolds.

It is well known that there exist contact metric manifolds $(M^{2n+1}, \varphi, \xi, \eta, g)$, for which the curvature tensor R and the direction of the characteristic vector field ξ satisfy $R(X,Y)\xi = 0$, for any vector fields on M^{2n+1}. Using a D-homothetic deformation to a contact metric manifold with $R(X,Y)\xi = 0$ we get a contact metric manifold satisfying the following special condition

$$R(X,Y)\xi = \eta(Y)(\kappa I + \mu h)X - \eta(X)(\kappa I + \mu h)Y, \tag{1.1}$$

where κ, μ are constants and h is the self-adjoint (1,1)-tensor field. This condition is called (κ, μ)-nullity on M^{2n+1}. Contact metric manifolds with (κ, μ)-nullity condition studied for $\kappa, \mu = const.$ (see [1]).

Moreover, Pastore and Dileo are studied the curvature properties of almost Kenmotsu manifolds, with special attention to (κ, μ)-nullity condition for $\kappa, \mu = const.$ and $\nu = 0$ ((see [6]). The authors prove that an almost Kenmotsu manifolds M^{2n+1} is locally a warped product of an almost Kaehler manifold and an open interval. If additionally M^{2n+1} is locally symmetric then it is locally isometric to the hyperbolic space H^{2n+1} of constant sectional curvature $c = -1$. It is recall that model spaces for almost cosymplectic case were given by Olszak (see [4, 5]).

In 2009, Öztürk et al. studied $(M, \varphi, \xi, \eta, g)$ almost α-Kenmotsu manifold in the light of the following relation

$$R(X,Y)\xi = \eta(Y)(\kappa I + \mu h + \nu\varphi h)X - \eta(X)(\kappa I + \mu h + \nu\varphi h)Y, \tag{1.2}$$

where $\kappa, \mu, \nu \in R_\eta M$ such that $df \wedge \eta = 0$ and $h = \left(\frac{1}{2}\right)\left(L_\xi \varphi\right)$ (see [12]). Such manifolds are said to be almost α-Kenmotsu (κ, μ, ν)-spaces and (φ, ξ, η, g) be called almost α-Kenmotsu (κ, μ, ν)-structure.

In this paper, the geometry of almost alpha Kenmotsu (κ, μ)-spaces are studied. Finally, we give an illustrative example on almost alpha Kenmotsu (κ, μ)-space with dimension 3.

Preliminaries

Let M^{2n+1} almost contact manifold be an odd-dimensional manifold. The triple (φ, ξ, η) is defined as follow. It transports a field φ of endomorphisms of the tangent spaces, ξ is a vector field that is called characteristic or Reeb vector field , and η is a 1-form such that $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. The mapping defined by $I: TM^{2n+1} \to TM^{2n+1}$, is called identity mapping. By using the definition of these it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and that the (1,1)-tensor field φ has constant rank $2n$ (see [1]). An almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$ is said to be normal if the Nijenhuis torsion tensor $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes for any vector fields X, Y on M^{2n+1}. If M^{2n+1} admits a Riemannian metric g, such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

for any vector fields X, Y on M^{2n+1}, then this metric g is said to be a compatible metric and the manifold M^{2n+1} together with the structure $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold. Hence, (2.1) means that $\eta(X) = g(X, \xi)$ for any vector field X on M^{2n+1}. On such a manifold, the fundamental 2-form Φ of M^{2n+1} is defined by $\Phi(X,Y) = g(\varphi X, Y)$. An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. An almost contact metric manifold M^{2n+1} is said to be almost alpha Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant. It is obvious that a normal almost cosymplectic manifold is called a cosymplectic manifold and a normal almost Kenmotsu manifold is called Kenmotsu manifold.

Considering the deformed structure for Kenmotsu metric structure (φ, ξ, η, g)

$$\eta^* = (1/\alpha)\eta, \;\; \xi^* = \alpha\xi, \;\; \varphi^* = \varphi,$$

$$g^* = (1/\alpha^2)g, \;\; \alpha \neq 0, \;\; \alpha \in R, \tag{2.2}$$

where α is a non-zero real constant. Thus we obtain an almost alpha Kenmotsu structure $(\varphi^*, \; \xi^*, \eta^*, g^*)$. This deformation called a homothetic deformation on M^{2n+1} (see [10]).

Now, we set $A = -\nabla\xi$ and $h = (1/2)(L_\xi\varphi)$. These definitions requires that $A(\xi) = 0$ and $h(\xi) = 0$. Furthermore, A and h are symmetric operators and holds the following relations

$$\nabla_X\xi = -\alpha\varphi^2X - \varphi hX, \tag{2.3}$$

$$(\varphi \circ h)X + (h \circ \varphi)X = 0, \tag{2.4}$$

$$(\varphi \circ A)X + (A \circ \varphi)X = -2\alpha\varphi, \tag{2.5}$$

$$(\nabla_X\eta)Y \; = \; \alpha[g(X,Y) - \eta(X)\eta(Y)] + g(\varphi Y, hX), \tag{2.6}$$

$$\delta\eta \; = \; -2\alpha n, \;\; tr(h) = 0, \tag{2.7}$$

for any vector fields $X, Y \; on \; M^{2n+1}$. It is clear that h vanishes iff $\nabla\xi = -\alpha\varphi^2$.

Some Curvature Properties

Lemma 3.1 The following relations are held for an almost alpha Kenmotsu manifolds

$$R(X,Y)\xi = (\alpha^2 \; + \xi(\alpha)) + ([\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\varphi hY - \eta(Y)\varphi hX]$$

$$+(\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y, \tag{3.1}$$

$$R(X,\xi)\xi = (\alpha^2 \; + \xi(\alpha))\, \varphi^2X + 2\alpha\varphi hX - h^2X + \varphi(\nabla_\xi h)X, \tag{3.2}$$

$$(\nabla_\xi h)X = -\varphi R(X,\xi)\xi - (\alpha^2 \; + \xi(\alpha))\, \varphi X - 2\alpha hX - \varphi h^2X, \tag{3.3}$$

$$R(X,\xi)\xi - \varphi R(\varphi X,\xi)\xi = 2[(\alpha^2 \; + \xi(\alpha))\, \varphi^2X - h^2X], \tag{3.4}$$

$$S(X,\xi) = -2n[\alpha^2 + \xi(\alpha)]\eta(X) - \big(div(\varphi h)\big)X, \tag{3.5}$$

$$S(\xi,\xi) = -[2n(\alpha^2 \; + \xi(\alpha)) \; + tr(h^2)], \tag{3.6}$$

for any vector fields on X, Y on M^{2n+1} where α be a smooth function such that $d\alpha \wedge \eta = 0$. In these formulas, ∇ is the Levi-Civita connection and R the Riemannian curvature tensor of M^{2n+1}.

Some Results

Now, we are especially interested in almost almost alpha Kenmotsu manifolds whose almost alpha Kenmotsu structure (φ, ξ, η, g) satisfies the condition (1.1) for $\kappa, \mu \in R_{\eta}(M^{2n+1})$. Such manifolds are said to be almost alpha Kenmotsu (κ, μ)-spaces and (φ, ξ, η, g) be called almost alpha Kenmotsu (κ, μ)-structure.

Proposition 4.1 The following relations are held for an almost alpha Kenmotsu (κ, μ)-space

$$l = -\kappa\varphi^2 + \mu h, \tag{4.1}$$

$$l\varphi - \varphi l = 2\mu h\varphi, \tag{4.2}$$

$$h^2 = (\kappa + \alpha^2)\varphi^2, \qquad \kappa \leq -\alpha^2, \qquad (4.3$$

$$(\nabla_{\xi} h) = -\mu[\varphi h + 2\alpha]h, \tag{4.4}$$

$$\nabla_{\xi} h^2 = -4\alpha(\kappa + \alpha^2)\varphi^2, \tag{4.5}$$

$$\xi(\kappa) = -4\alpha(\kappa + \alpha^2), \tag{4.6}$$

$$R(\xi, X)Y = \kappa(g(Y, X)\xi - \eta(Y)X) + \mu(g(hY, X)\xi - \eta(Y)hX) \tag{4.7}$$

$$Q\xi = 2n\kappa\xi, \qquad (4.8$$

$$(\nabla_X \varphi)Y = g(\alpha\varphi X + hX, Y)\xi - \eta(Y)(\alpha\varphi X + hX), \tag{4.9}$$

$$(\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X = -(\kappa + \alpha^2)(\eta(Y)X - \eta(X)Y) - \mu(\eta(Y)hX - \eta(X)hY)$$

$$+\alpha(\eta(Y)\varphi hX - \eta(X)\varphi hY), \tag{4.10}$$

$$(\nabla_X h)Y - (\nabla_Y h)X = (\kappa + \alpha^2)(\eta(Y)\varphi X - \eta(X)\varphi Y + 2g(\varphi X, Y)\xi) \tag{4.11}$$

$$+\mu(\eta(Y)\varphi hX - \eta(X)\varphi hY) + \alpha(\eta(Y)hX - \eta(X)hY),$$

$$Q\varphi - \varphi Q = 2h[\mu\varphi], \tag{4.12}$$

for all vector fields X, Y on M^{2n+1} and and $\xi(\alpha) = 0$.

Proof. The above relations can be proved with the help of the same techniques that used by Öztürk et al. where $\xi(\alpha) = 0$ and $\kappa, \mu \in R_{\eta}(M^{2n+1})$, (see [12]).

Theorem 4.1 For almost alpha Kenmotsu (κ, μ)-space, the following relation holds

$$0 = \xi(\kappa)(\eta(Y)X - \eta(X)Y) + \xi(\mu)(\eta(Y)hX - \eta(X)hY) - X(\kappa)\varphi^2 Y + X(\mu)hY$$

$$-Y(\mu)hX + Y(\kappa)\varphi^2 X + 2(\kappa + \alpha^2)\mu g(\varphi X, Y)\xi + 2\mu g(hX, \varphi hY)\xi. \tag{4.13}$$

here $\xi(\alpha) = 0$.

Proof. By the means of [12], we have the desired result for $\xi(\alpha) = 0$.

Lemma 4.1 Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost alpha Kenmotsu (κ, μ)-space. For every $p \in N$, there exists neighborhood W of p and orthonormal local vector fields X_i, φX_i and ξ for $i = 1, \dots, n$ defined on W, such that

$$hX_i = \lambda X_i, \qquad h\varphi X_i = -\lambda X_i, \qquad h\xi = 0, \tag{4.14}$$

for $i = 1, \dots, n$ where $\lambda = \sqrt{-(\kappa + \alpha^2)}$.

Proof. According to Öztürk et al. (see [12]), the proof can be easily seen for almost alpha Kenmotsu (κ, μ)-space with $\nu = 0$ and $\xi(\alpha) = 0$.

Now, we explain why the smooth functions κ and ν are element of $R_\eta(M^{2n+1})$. With the help of above Lemma 4.1, we state the following theorem.

Theorem 4.2 Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost alpha Kenmotsu manifolds. If the manifold satisfies the conditions given in Lemma 4.1 then there exists almost alpha Kenmotsu (κ, μ)-space where the κ and μ functions are non-constants defined $df \wedge \eta = 0$ in $R_\eta(M^{2n+1})$.

Proof. By means of Lemma 1, using the local orthonormal basis $\{X_i, \varphi X_i, \xi\}$ and (4.13) we have

$$[e_i(\kappa) - \lambda e_i(\mu)]\varphi e_i + [-\lambda\varphi e_i(\mu) - \varphi e_i(\kappa)] = 0,$$

for $X = e_i$, $Y = \varphi e_i$ and for $\xi(\alpha) = 0$. Since $\{e_i, eX_i\}$ is linearly independent, we obtain $e_i(\kappa) - \lambda e_i(\mu) = 0$ and $\lambda\varphi e_i(\mu) - \varphi e_i(\kappa) = 0$. Then replacing X and Y by e_i and e_j, respectively, for $i \neq j$, (4.13) shows that

$$e_i(\kappa) + \lambda e_i(\mu) = 0.$$

Also, substituting $X = \varphi e_i$ and $Y = \varphi e_j$ in (4.13) for $i \neq j$, we have

$$\varphi e_i(\kappa) - \lambda\varphi e_i(\mu) = 0.$$

In view of the last three equations, we deduce

$$e_i(\kappa) = e_i(\mu) = \varphi e_i(\kappa) = \varphi e_i(\mu) = 0.$$

For an arbitrary function κ, we obtain $d\kappa = \xi(\kappa)\eta$ in the last equation system. Thus we have

$$0 = d^2\kappa = d(d\kappa) = d\xi(\kappa) \wedge \eta + \xi(\kappa)d\eta.$$

Since $d\eta = 0$, it follows that $d\xi(\kappa) \wedge \eta = 0$. Similarly, the same method can be used for an arbitrary function μ. Therefore, there exists almost alpha Kenmotsu (κ, μ)-space where the κ and μ functions are non-constants defined $df \wedge \eta = 0$ in $R_\eta(M^{2n+1})$.

Example 4.1 Suppose that three dimensional manifold is defined by

$$M^3 = \{(x, y, z) \in R^3, \qquad z \neq 0\},$$

where (x, y, z) are the cartesian coordinates in R^3. We define three vector fields on M^3 as

$$e = \left(\frac{\partial}{\partial x}\right),$$

$$\varphi e = \left(\frac{\partial}{\partial y}\right),$$

$$\xi = \left[\alpha x - y\left(e^{\{-2\alpha z\}} + z\right)\right]\left(\frac{\partial}{\partial x}\right)$$

$$+\left[x\left(z - e^{\{-2\alpha z\}}\right) + \alpha y\right]\left(\frac{\partial}{\partial y}\right) + \left(\frac{\partial}{\partial z}\right).$$

We easily get

$$[e, \varphi e] = 0,$$

$$[e, \xi] = \alpha e + (z - e{}^\wedge\{-2\alpha z\})\varphi e,$$

$$[\varphi e, \xi] = -(e{}^\wedge\{-2\alpha z\} + z)e + \alpha\varphi e.$$

Moreover, the matrice form of the metric tensor g, the tensor fields ϕ and h are given by

$$g = \begin{pmatrix} 1 & 0 & -d \\ 0 & 1 & -k \\ -d & -k & 1 + d^2 + k^2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & -d & k \\ 1 & 0 & -d \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} e{}^\wedge\{-2z\} & 0 & k - de{}^\wedge\{-2z\} \\ 0 & -e{}^\wedge\{-2z\} & ke{}^\wedge\{-2z\} \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$d \;=\; \alpha x - y(e{}^\wedge\{-2\alpha z\} + z),$$

$$k \;=\; x(z - e{}^\wedge\{-2\alpha z\} + \alpha y.$$

Let η be the 1-form defined by $\eta = k_1dx + k_2dy + k_3dz$ for all vector fields on M^3. Since $\eta(X) = g(X, \xi)$, we can easily obtain that $\eta(e) = 0$, $\eta(\varphi e) = 0$ and $\eta(\xi) = 1$. By using these equations, we get $\eta = dz$ for all vector fields. Since $d\eta = d(dz) = d^2z$, we obtain $d\eta = 0$. Using Koszul's formula, we have seen that $d\Phi = 2\alpha\eta \wedge \Phi$. Hence, it has been showed that M^3 is an almost alpha Kenmotsu manifold. Thus we obtain

$$R(X, Y)\xi = -(e{}^\wedge\{-4\alpha z\} + \alpha^2)[\eta(Y)X - \eta(X)Y] + 2z[\eta(Y)hX - \eta(X)hY],$$

where $\kappa = -\left(e^{\{-4\alpha z\}} + \alpha^2\right) and\ \mu = 2z$. Also, we remark that this example is provided according to Theorem 7.3.1 in [12] for $\xi(\alpha) = 0$.

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