Stability of Cubic Functional Equation in Random Normed Space

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Abstract

In this paper, we present the Hyers-Ulam stability of Cubic functional equation

$$f\left(\sum_{i=1}^{n} ix_i\right) = \sum_{\substack{i=1 \\ i \neq j \neq k}}^{n} f\left(ix_i + jx_j + kx_k\right) + (3-n)\sum_{\substack{i=1 \\ i \neq j}}^{n} f\left(ix_i + jx_j\right) + \left(\frac{n^2 - 5n + 6}{2}\right)\sum_{i=0}^{n-1}(i+1)^3 f\left(x_{i+1}\right)$$

where n is greater than or equal to 4, in Random Normed Space.

Keywords: Cubic Functional Equation, Fixed Point, Hyers-Ulam Stability, Random Normed Space.

1. Introduction

The theory of random normed spaces (briefly, RN-Spaces) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. The notion of an RN-Space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of possible values of this norm.

Random Theory is a setting in which uncertainty arising from problems in various fields of science, can be modelled. It is a practical tool for handling situations where classical theories fail to explain. Random Theory has many applications in several fields, for example, population dynamics, computer programming, nonlinear dynamical system, nonlinear operators, statistical convergence and so forth. Jun and Kim [5] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and they established the general solution and the generalized Hyers-Ulam stability for the functional equation. The function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a cubic functional equation. The solution and stability of the succeeding cubic functional equation,

$$f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)=2k(k^2-1)f(y) \tag{1.2}$$

$$f\left(\sum_{i=1}^{n}x_i\right)+\sum_{j=1}^{n}\left(-x_j+\sum_{\substack{i=1\\i\neq j}}^{n}x_i\right)=(n-5)\sum_{1\leq i<j<k\leq n}f\left(x_i+x_j+x_k\right)+\left(-n^2+8n-11\right)\sum_{\substack{i=1\\i\neq j}}^{n}f\left(x_i+x_j\right)$$

$$-\sum_{j=1}^{n}f\left(2x_j\right)+\frac{1}{2}\left(n^3-10n^2+23n+2\right)\sum_{i=1}^{n}f\left(x_i\right) \tag{1.3}$$

were dealt by Seong Sik Kim et al., [17], S, Murthy et al., [9]. Some of the non-cubic functional equations discussed in various spaces of papers are used to develop this paper which are [1,3,7,8,10,11,12, 13,14,16,18,19]. In this paper, the authors investigate the general solution and generalized Hyers-Ulam stability of a new type of n-dimensional cubic functional equation

$$f\left(\sum_{i=1}^{n}ix_i\right)=\sum_{\substack{i=1\\i\neq j\neq k}}^{n}f\left(ix_i+jx_j+kx_k\right)+(3-n)\sum_{\substack{i=1\\i\neq j}}^{n}f\left(ix_i+jx_j\right)+\left(\frac{n^2-5n+6}{2}\right)\sum_{i=0}^{n-1}(i+1)^3 f\left(x_{i+1}\right)$$

$$\tag{1.4}$$

where n is greater than or equal to 4, in Random Normed Space by using direct and fixed-point method.

2. Preliminaries

In this part, we evoke some notations and basic definitions used in this article.

Definition 2.1[9] A mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm, if T satisfies the following condition:

a) T is commutative and associative;

b) T is continuous

c) $T(a,1)=a$ for all $a\in[0,1]$

d) $T(a,b) \leq T(c,d)$ when $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_p(a,b) = ab$, $T_m(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (The Lukasiewicz t-norm). Recall [9] that if T is a t-norm and x_n is a given sequence of numbers in [0,1], then $T_{i=1}^n x_{n+i}$ is defined recurrently by $T_{i=1}^1 x_i = x_i$ and $T_{i=1}^n x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right)$ for $n \geq 2$, $T_{i=1}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$. It is known that, for the Lukasiewicz t-norm, the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty$$

Definition 2.2[9] A random normed space (briefly, RN-Space) is a triple (X, μ, T), where X is a vector space. T is a continuous t-norm and μ is a mapping from X into D^+ satisfies the following conditions:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$.

(RN2) $\mu_{\alpha x}(t) = \mu_x\left(\dfrac{t}{|\alpha|}\right)$ for all $x \in X$, and $\alpha \in \Re$ with $\alpha \neq 0$.

(RN3) $\mu_{x+y}(t+s) \geq T\left(\mu_x(t), \mu_y(s)\right)$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3[9] Let (X, μ, T) be a RN-space.

1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n > N$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.

3) A RN-Space (X, μ, T) is said to be complete, if every Cauchy sequence in X is convergent to a point in X.

For more details we can go through [2, 4, 6, 16, 20,21].

3. General Solution of the n-Dimensional Cubic Functional Equation (1.4):

In this part, the authors discuss the general solution of the functional equation (1.4) by considering X and Y are real vector space.

Theorem 3.1 If a mapping $f: X \to Y$ satisfies the functional equation (1.1), then the function $f: X \to Y$ satisfies the functional equation (1.4).

Proof. Assume that $f: X \to Y$ satisfies the functional equation (1.4), for all $x_1, x_2, x_3, ..., x_n \in X$. Substituting $(x_1, x_2, x_3, ..., x_n)$ by $(0,0,0,...,0)$ in (1.4), we receive

$$f(0)=0$$

for all $x \in X$. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x, 0, 0, ..., 0)$ in (1.4), we get

$$f(-x) = -f(x)$$

for all $x \in X$. Hence f is an odd function. Again replacing $(x_1, x_2, x_3, ..., x_n)$ by $\left(x, \dfrac{x}{2}, 0, ..., 0\right)$ in (1.4), we have

$$f(2x) = 2^3 f(x) \tag{3.1}$$

for all $x \in X$. Now, letting x by $2x$ in (3.1), we get

$$f(4x) = 4^3 f(x) \tag{3.2}$$

for all $x \in X$. In general, for any positive integer a, we obtain

$$f(ax) = a^3 f(x) \tag{3.3}$$

Setting $(x_1, x_2, x_3, x_4, ..., x_n)$ by $\left(x, \dfrac{-x}{2}, \dfrac{x}{3}, \dfrac{y}{4}, 0, ..., 0\right)$ in (1.4) and using (3.1), we receive

$$3f(x+y) = -6f(x) + 3f(y) + f(2x+y) + f(x-y) \tag{3.5}$$

for all $x, y \in X$. Replacing y by $-y$ in (3.5), we obtain

$$3f(x-y) = -6f(x) - 3f(y) + f(2x-y) + f(x+y) \tag{3.6}$$

for all $x, y \in X$. Adding (3.5) and (3.6), We achieve our required result (1.1).

All over this paper we use the following notation for a given mapping $f : X \to Y$ as

$$Df(x_1, x_2, x_3, ..., x_n) = f\left(\sum_{i=1}^{n} i x_i\right) - \sum_{\substack{i=1 \\ i \neq j \neq k}}^{n} f\left(i x_i + j x_j + k x_k\right) - (3-n)\sum_{\substack{i=1 \\ i \neq j}}^{n} f\left(i x_i + j x_j\right) - \left(\frac{n^2 - 5n + 6}{2}\right)$$

$$\sum_{i=0}^{n-1} (i+1)^3 f(x_{i+1})$$

for all $x_1, x_2, x_3, ..., x_n \in X$.

4. Random Stability Results: Direct Method

In this part, the generalized Ulam-Hyers Stability of the cubic functional equation (1.4) in RN-Space is provided. All through this part, let us consider X be a linear space (Y, μ, T) is a complete RN-Space.

Theorem 4.1 Let $j = \pm 1$, $f : X \to Y$ be a mapping for which there exists a function $\eta : X^n \to D^+$ with the condition

$$\lim_{k \to \infty} T_{i=0}^{\infty} = \left(\eta_{2^{(k+i)} x_1, 2^{(k+i)} x_2, 2^{(k+i)} x_3, ..., 2^{(k+i)} x_n}\left(2^{3(k+i+1)j} t\right)\right) = 1 \tag{4.1}$$

$$= \lim_{k \to \infty} \eta_{2^{kj} x_1, 2^{kj} x_2, 2^{kj} x_3, ..., 2^{kj} x_n}\left(2^{3kj} t\right) \tag{4.2}$$

such that the functional inequality with $f(0)=0$ such that

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t) \geq \eta_{(x_1,x_2,\ldots,x_n)}(t) \tag{4.3}$$

for all $x_1,x_2,x_3,\ldots,x_n \in X$ and all $t>0$. Then there exists a unique cubic mapping $C:X \to Y$ satisfies the functional equation (1.4) and

$$\mu_{C(x)-f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{2^{(i+1)j}x,2^{(i+1)j}x,0,\ldots,0}\left(2^{3(i+1)j}t\right)\right) \tag{4.4}$$

for all $x \in X$ and all $t>0$. The mapping $C(x)$ is defined by

$$\mu_{C(x)}(t) = \lim_{k\to\infty}\mu_{\frac{f\left(2^{kj}x\right)}{2^{3kj}}}(t) \tag{4.5}$$

for all $x \in X$ and all $t>0$.

Proof. Assume $j=1$. Setting $\left(x_1,x_2,x_3,\ldots,x_n\right)$ by $\left(x,x,0,\ldots,0\right)$ in (4.1), we acquire

$$\mu_{\left(n^2-5n+6\right)f(2x)-8\left(n^2-5n+6\right)f(x)}(t) \geq \eta_{x,x,0,\ldots,0}(t) \tag{4.6}$$

for all $x \in X$ and all $t>0$. It follows from (4.5) and (RN2), we arrive

$$\mu_{\frac{f(2x)}{2^3}-f(x)}(t) \geq \eta_{x,x,0,\ldots,0}\left(8\left(n^2-5n+6\right)t\right) \tag{4.7}$$

for all $x \in X$ and all $t>0$. Replacing x by $2^k x$ in (4.6), we catch

$$\mu_{\frac{f\left(2^{k+1}x\right)}{2^{3(k+1)}}-\frac{f\left(2^k x\right)}{2^{3k}}}(t) \geq \eta_{2^k x,2^k x,0,\ldots,0}\left(8^k\left(n^2-5n+6\right)8t\right) \tag{4.8}$$

$$\geq \eta_{x,x,0,\ldots,0}\left(\frac{8^k\left(n^2-5n+6\right)8}{\alpha^k}t\right)$$

for all $x \in X$ and all $t>0$. It follows from $\dfrac{f\left(2^n x\right)}{8^n}-f(x)=\displaystyle\sum_{k=0}^{n-1}\dfrac{f\left(2^{k+1}x\right)}{8^{k+1}}-\dfrac{f\left(2^k x\right)}{8^k}$ and (4.8) that

$$\mu_{\frac{f\left(2^n x\right)}{8^n}-f(x)}\left(t\sum_{k=0}^{n-1}\frac{\alpha^k}{8^k\left(n^2-5n+6\right)8}\right) \geq T_{k=0}^{n-1}\left(\eta_{x,x,0,\ldots,0}(t)\right)=\eta_{x,x,0,\ldots,0}(t) \tag{4.9}$$

$$\mu_{\frac{f(2^n x)}{8^n} - f(x)}(t) \geq \eta_{x,x,0,\dots,0}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{8^k (n^2 - 5n + 6)8}}\right) \tag{4.10}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^m x$ in (4.10), we arrive that

$$\mu_{\frac{f(2^{n+m} x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}}(t) \geq \eta_{x,x,0,\dots,0}\left(\frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{8^k (n^2 - 5n + 6)8}}\right) \tag{4.11}$$

As $\quad \eta_{x,x,0,\dots,0}\left(\dfrac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{8^k (n^2 - 5n + 6)8}}\right) \to 1$ as $m,n \to \infty$, then $\left\{\dfrac{f(2^n x)}{8^n}\right\}$ is a Cauchy sequence in

(Y, μ, T). Since (Y, μ, T) is a complete RN-Space, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (4.11), we have

$$\mu_{\frac{f(2^n x)}{8^n} - f(x)}(t) \geq \eta_{x,x,0,\dots,0}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{8^k (n^2 - 5n + 6)8}}\right) \tag{4.12}$$

and so, for every $\delta > 0$, we collect

$$\mu_{C(x) - f(x)}(t + \delta) \geq T\left(\mu_{C(x) - \frac{f(2^n x)}{8^n}}(\delta), \mu_{\frac{f(2^n x)}{8^n} - f(x)}(t)\right)$$

$$\geq T\left(\mu_{C(x) - \frac{f(2^n x)}{8^n}}(\delta), \eta_{x,x,0,\dots,0}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{8^k (n^2 - 5n + 6)8t}}\right)\right) \tag{4.13}$$

Taking limit as $n \to \infty$ and using (4.13), we arrive

$$\mu_{C(x) - f(x)}(t + \delta) \geq \eta_{x,x,0,\dots,0}\left((n^2 - 5n + 6)(8 - \alpha)t\right) \tag{4.14}$$

Since δ was arbitrary, by taking $\delta \to 0$ in (4.14), we have

$$\mu_{C(x)-f(x)}(t) \geq \eta_{x,x,0,\dots,0}\left(\left(n^2 - 5n + 6\right)(8-\alpha)t\right) \tag{4.15}$$

Replacing $(x_1, x_2, \dots, x_n)$ by $(2^n x_1, 2^n x_2, \dots, 2^n x_n)$ in (4.3) respectively, we acquire

$$\mu_{Df\left(2^n x_1, 2^n x_2, \dots, 2^n x_n\right)}(t) \geq \eta_{2^n x_1, 2^n x_2, \dots, 2^n x_n}\left(8^n t\right) \tag{4.16}$$

for all $x_1, x_2, \dots, x_n \in X$ and for all $t > 0$. Since

$$\lim_{k \to \infty} T_{i=0}^{\infty} = \left(\eta_{2^{(k+i)} x_1, 2^{(k+i)} x_2, 2^{(k+i)} x_3, \dots, 2^{(k+i)} x_n}\left(2^{3(k+i+1)j} t\right)\right) = 1.$$

We conclude that C fulfils (1.1). To prove the uniqueness of the cubic mapping C, assume that there exists a cubic mapping D from X to Y, which satisfies (4.15). Fix $x \in X$. Clearly, $C(2^n x) = 8^n C(x)$ and $D(2^n x) = 8^n D(x)$ for all $x \in X$. It follows from (4.15) that

$$\mu_{C(x)-D(x)}(t) = \lim_{n \to \infty} \mu_{\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}}(t)$$

$$\mu_{\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}}(t) \geq \min\left\{\mu_{\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}}\left(\frac{t}{2}\right), \mu_{\frac{D(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}}\left(\frac{t}{2}\right)\right\}$$

$$\geq \eta_{2^n x, 2^n x, 0, \dots, 0}\left(8^n\left(n^2 - 5n + 6\right)(8-\alpha)t\right)$$

$$\geq \eta_{x,x,0,\dots,0}\left(\frac{8^n\left(n^2 - 5n + 6\right)(8-\alpha)t}{\alpha^n}\right) \tag{4.17}$$

Since $\lim_{n \to \infty}\left(\dfrac{8^n\left(n^2 - 5n + 6\right)(8-\alpha)t}{\alpha^n}\right) = \infty$, we get $\lim_{n \to \infty} \eta_{x,x,0,\dots,0}\left(\dfrac{8^n\left(n^2 - 5n + 6\right)(8-\alpha)t}{\alpha^n}\right) = 1$. Therefore, it

follows that $\mu_{C(x)-D(x)}(t) = 1$ for all $t > 0$ and so $C(x) = D(x)$. This completes the proof.

The following corollary is an immediate consequence of Theorem 4.1, concerning the stability of (1.4).

Corollary 4.2. Let ε and s be non-negative real numbers. Let a Cubic Function $f : X \to Y$ satisfies the inequality

$$\mu_{Df(x_1,x_2,x_3,\dots,x_n)}(t) \geq \begin{cases} \eta_\varepsilon(t) \\ \eta_\varepsilon \sum_{i=1}^n \|x_i\|^s(t), & s \neq 3 \\ \eta_\varepsilon\left(\prod_{i=1}^n \|x_i\|^s + \sum_{i=1}^n \|x_i\|^{ns}\right)(t), & s \neq \dfrac{3}{n} \end{cases} \tag{4.18}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$. The there exists a unique cubic function $C : X \to Y$ such that

$$\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\varepsilon}{(n^2-5n+6)|2^3-1|}}(t) \\ \eta_{\frac{2\varepsilon\|x\|^s}{(n^2-5n+6)|2^3-2^s|}}(t) \\ \eta_{\frac{2\varepsilon\|x\|^{ns}}{(n^2-5n+6)|2^3-2^{ns}|}}(t) \end{cases} \tag{4.19}$$

for all $x \in X$ and all $t > 0$.

5. Random Stability Results: Fixed Point Method.

In this part, the author presents the generalized Ulam-Hyers Stability of the functional equation (1.4), in Random Normed Space using fixed point method.

Theorem 5.1 Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^n \to D^+$ with the condition

$$\lim_{k\to\infty} \eta_{\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \dots, \delta_i^k x_n}\left(\delta_i^{3k} t\right) = 1 \tag{5.1}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$ and where $\delta_i = \begin{cases} 2, & i = 0; \\ \dfrac{1}{2} & i = 1; \end{cases}$ satisfying the functional inequality

$$\mu_{Df(x_1,x_2,x_3,\dots,x_n)}(t) \geq \eta_{x_1,x_2,x_3,\dots,x_n}(t) \tag{5.2}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x,t) = \eta_{\frac{x}{2},\frac{x}{2},0,\dots,0}\left((n^2 - 5n + 6)t\right),$$

has the property, that

$$\beta(x,t) \leq L\frac{1}{\delta_i^3}\beta(\delta_{ix}, t) \tag{5.3}$$

for all $x \in X$ and $t > 0$. Then there exists a unique cubic function $C : X \to Y$ satisfying the functional equation (1.4) and

$$\mu_{C(x)-f(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t) \qquad (5.4)$$

for all $x \in X$ and $t > 0$.

Proof. Let d be a general metric on Ω, such that $d(g,h) = \inf\left\{k \in (0,\infty) / \mu_{(g(x)-h(x))}(kt) \geq \beta(x,t), x \in X, t > 0\right\}$. It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3} g(\delta_i x)$, for all $x \in X$. Now for $g, h \in \Omega$, we have $d(g,h) \leq K$.

$$\Rightarrow \mu_{(g(x)-h(x))}(Kt) \geq \beta(x,t)$$

$$\Rightarrow \mu_{(Tg(x)-Th(x))}\left(\frac{Kt}{\delta_i^3}\right) \geq \beta(x,t)$$

$$\Rightarrow d(Tg(x), Th(x)) \leq KL$$

$$\Rightarrow d(Tg, Th) \leq Ld(g,h) \qquad (5.5)$$

for all $g, h \in \Omega$. Therefore, T is strictly contractive mapping on Ω with Lipschitz constant L.

It follows from (4.6) that

$$\mu_{(n^2-5n+6)f(2x)-8(n^2-5n+6)f(x)}(t) \geq \eta_{x,x,0,\dots,0}(t) \qquad (5.6)$$

for all $x \in X$. It follows from (5.6) that

$$\mu_{\frac{f(2x)}{8}-f(x)}(t) \geq \eta_{x,x,0,\dots,0}\left((n^2-5n+6)8t\right) \qquad (5.7)$$

for all $x \in X$. Using (5.3) for the case $i = 0$, it reduces to

$$\mu_{\frac{f(2x)}{8}-f(x)}(t) \geq L\beta(x,t)$$

for all $x \in X$. Hence, we obtain

$$d\left(\mu_{Tf(x)-f(x)}\right) \geq L = L^{1-i} < \infty \qquad (5.8)$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (5.7), we get

$$\mu_{\frac{f(x)}{8}-f\left(\frac{x}{2}\right)}(t) \geq \eta_{\frac{x}{2},\frac{x}{2},0,\ldots,0}\left(\left(n^2 - 5n + 6\right)8t\right) \qquad (5.9)$$

for all $x \in X$. Using (5.3) for the case $i = 1$, it reduces to

$$\mu_{8f\left(\frac{x}{2}\right)-f(x)}(t) \geq \beta(x,t) \Rightarrow \mu_{Tf(x)-f(x)}(t) \geq \beta(x,t)$$

for all $x \in X$. Hence, we get

$$d\left(\mu_{Tf(x)-f(x)}\right) \geq L = L^{1-i} < \infty \qquad (5.10)$$

for all $x \in X$. From (5.8) and (5.10), we can conclude

$$d\left(\mu_{Tf(x)-f(x)}\right) \geq L = L^{1-i} < \infty \qquad (5.11)$$

for all $x \in X$. In order to prove $C: X \to Y$ satisfies the functional equation (1.4), the remaining proof is similar by using Theorem 4.1. Since C is unique fixed point of T in the set $\Delta = \left\{f \in \Omega / d\left(f,C\right) < \infty\right\}$. Finally, C is an unique function such that

$$\mu_{f(x)-C(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t)$$

for all $x \in X$ and $t > 0$. This completes the proof of the Theorem.

From the Theorem 5.1, we obtain the following Corollary concerning the stability for the functional equation (1.4).

Corollary 5.2. Suppose that a function $f: X \to Y$ satisfies the inequality

$$\mu_{Df(x_1,x_2,x_3,\ldots,x_n)}(t) \geq \begin{cases} \eta_\varepsilon(t) \\ \eta_\varepsilon \sum_{i=1}^{n}\|x_i\|^s(t), \quad s \neq 3 \\ \eta_\varepsilon\left(\prod_{i=1}^{n}\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right)(t), \quad s \neq \dfrac{3}{n} \end{cases} \qquad (5.12)$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$, where ε, s are constants with $\varepsilon > 0$, then there exists a unique cubic mapping $C: X \to Y$ such that

$$
\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\varepsilon}{\left(n^2-5n+6\right)\left|2^3-1\right|}}(t) \\ \eta_{\frac{2\varepsilon\|x\|^s}{\left(n^2-5n+6\right)\left|2^3-2^s\right|}}(t) \\ \eta_{\frac{2\varepsilon\|x\|^{ns}}{\left(n^2-5n+6\right)\left|2^3-2^{ns}\right|}}(t) \end{cases}
\tag{5.13}
$$

for all $x \in X$ and all $t > 0$.

Proof. Setting

$$
\mu_{Df(x_1,x_2,x_3,\ldots,x_n)}(t) \geq \begin{cases} \eta_\varepsilon(t) \\ \eta_{\varepsilon\sum_{i=1}^n\|x_i\|^s}(t), \\ \eta_{\varepsilon\left(\prod_{i=1}^n\|x_i\|^s + \sum_{i=1}^n\|x_i\|^{ns}\right)}(t), \end{cases}
$$

for all $x_1, x_2, x_3,\ldots, x_n \in X$ and all $t > 0$. Then

$$
\eta_{\left(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3,\ldots,\delta_i^k x_n\right)}\left(\delta_i^{3k}t\right) = \begin{cases} \eta_{\varepsilon\delta_i^{3k}}(t) \\ \eta_{\varepsilon\sum_{i=1}^n\|x_i\|^s \delta_i^{(3-s)k}}(t) \\ \eta_{\varepsilon\left(\prod_{i=1}^n\|x_i\|^s \delta_i^{(3-s)k} + \sum_{i=1}^n\|x_i\|^{ns}\delta_i^{(3-ns)k}\right)}(t) \end{cases}
$$

$$
= \begin{cases} \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \end{cases}
$$

But we have $\beta(x,t) = \eta_{\frac{x}{2},\frac{x}{2},0,\ldots,0}\left(\left(n^2-5n+6\right)t\right)$ has the property $L\frac{1}{\delta_i^3}\beta(\delta_i x,t)$ for all $x \in X$ and $t > 0$.

Now

$$
\beta(x,t) = \begin{cases} \eta_{\frac{\varepsilon}{\left(n^2-5n+6\right)}}(t) \\ \eta_{\frac{2\varepsilon\|x\|^s}{2^s\left(n^2-5n+6\right)}}(t) \\ \eta_{\frac{2\varepsilon\|x\|^{ns}}{2^{ns}\left(n^2-5n+6\right)}}(t) \end{cases}
$$

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$$L\frac{1}{\delta_i^3}\beta(\delta_{ix},t) = \begin{cases} \eta_{\delta_i^{-3}\beta(x)}(t) \\ \eta_{\delta_i^{s-3}\beta(x)}(t) \\ \eta_{\delta_i^{ns-3}\beta(x)}(t) \end{cases}$$

By using Theorem 5.1, we prove the following six cases:

$L = 2^{-3}$ if $i = 0$ and $L = 2^3$ if $i = 1$

$L = 2^{s-3}$ for $s < 3$ if $i = 0$ and $L = 2^{3-s}$ for $s > 3$ if $i = 1$

$L = 2^{ns-3}$ for $s < \dfrac{3}{n}$ if $i = 0$ and $L = 2^{3-ns}$ for $s > \dfrac{3}{n}$ if $i = 1$

Case .1: $L = 2^{-3}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \ge L\frac{1}{\delta_i^3}\beta(\delta_{ix},t)(t) \ge \eta_{\left(\frac{\varepsilon}{(n^2-5n+6)(2^3-1)}\right)}(t)$$

Case.2: $L = 2^3$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \ge L\frac{1}{\delta_i^3}\beta(\delta_{ix},t)(t) \ge \eta_{\left(\frac{\varepsilon}{(n^2-5n+6)(1-2^3)}\right)}(t)$$

Case.3: $L = 2^{s-3}$ for $s < 3$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \ge L\frac{1}{\delta_i^3}\beta(\delta_{ix},t)(t) \ge \eta_{\left(\frac{2\varepsilon\|x\|^s}{(n^2-5n+6)(2^3-2^s)}\right)}(t)$$

Case.4: $L = 2^{3-s}$ for $s > 3$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \ge L\frac{1}{\delta_i^3}\beta(\delta_{ix},t)(t) \ge \eta_{\left(\frac{2\varepsilon\|x\|^s}{(n^2-5n+6)(2^s-2^3)}\right)}(t)$$

Case.5: $L = 2^{ns-3}$ for $s < \dfrac{3}{n}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \ge L\frac{1}{\delta_i^3}\beta(\delta_{ix},t)(t) \ge \eta_{\left(\frac{2\varepsilon\|x\|^{ns}}{(n^2-5n+6)(2^3-2^{ns})}\right)}(t)$$

Case.6: $L = 2^{3-ns}$ for $s > \dfrac{3}{n}$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \ge L\frac{1}{\delta_i^3}\beta(\delta_{ix},t)(t) \ge \eta_{\left(\frac{2\varepsilon\|x\|^{ns}}{(n^2-5n+6)(2^{ns}-2^3)}\right)}(t)$$

Hence the proof is complete.

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