Generalized Fuzzy Soft Connected Sets in Generalized Fuzzy Soft Topological Spaces

F.H. Khedr[1], M. AZAB. Abd-Allah[2], and M.S Malfi[3]

[1,2,3] Mathematics Department, Faculty of Science, Assiut University, Assiut 71516, Egypt

[1]khedrfathi@gmail.com, [2]mazab57@yahoo.com, [3]mulfy_s76@yahoo.com

ABSTRACT

In this paper we introduce some types of generalized fuzzy soft separated sets and study some of their properties. Next, the notion of connectedness in fuzzy soft topological spaces due to Karata et al, Mahanta et al, and Kandil et al., extended to generalized fuzzy soft topological spaces. The relationship between these types of connectedness in generalized fuzzy soft topological spaces is investigated with the help of number of counter examples.

Keywords: Generalized fuzzy soft sets; generalized fuzzy soft topological space; generalized fuzzy soft separated sets; generalized fuzzy soft Q-separated sets; generalized fuzzy soft weakly separated sets; generalized fuzzy soft strongly separated sets; generalized fuzzy soft connected sets.

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1. INTRODUCTION

The concept of soft sets was first introduced by Molodtsov [16] as a general mathematical tool for dealing with uncertain objects. Cagman et al. [2], Shabir et al. [20] introduced soft topological space independently. Maji et al. [13] introduced the concept of fuzzy soft set and some of its properties. Tanay and Kandemir [21] introduced the definition of a fuzzy soft topology over a subset of the initial universe set. Later, Roy and Samanta [18] gave the definition of fuzzy soft topology over the initial universe set. Karal and Ahmed [8] defined the notion of a mapping on classes of fuzzy soft sets.

Majumdar and Samanta [14] introduced the notion of generalized fuzzy soft set as a generalization of fuzzy soft sets and studied some of its basic properties. Chakraborty and Mukherjee. [3] gave the topological structure of generalized fuzzy soft sets. Khedr et al. [9] introduced the concept of a generalized fuzzy soft point, a generalized fuzzy soft base (subbase), a generalized fuzzy soft subspace. Khedr et al. [10] introduced the concept of a generalized fuzzy soft mapping on families of generalized fuzzy soft sets.

The notion of connectedness in fuzzy topological spaces has been studied by Ming and Ming [15], Zheng Chong You [23], Fatteh and Bassan [5], Saha [19], and Ajmal and Kohli [1]. In fuzzy soft setting, connectedness has been introduced by Mahanta et al. [12], Karata et al. [7] and Kandil et al. [6].

Khedr et al. [11] introduced the generalized fuzzy soft connectedness and generalized fuzzy soft C_i-connectedness $(i = 1,2,3,4)$ in generalized fuzzy soft topological space and studied some of its basic properties.

In this paper, we extend the notion of connectedness of fuzzy soft topological spaces to generalized fuzzy soft topological spaces. In Section 3, we introduce different notions of generalized fuzzy soft separated sets and study the relationship between them. Section 4 is devoted to introduce the different notions of connectedness in generalized fuzzy soft topological spaces and study the implications that exist between them. Also, we study some characterizations of connectedness in generalized fuzzy soft setting.

2. Preliminaries

In this section, we will give some basic definitions and theorems about generalized fuzzy soft sets, generalized fuzzy soft topology and generalized fuzzy soft continuous mappings which will be needed in the sequel.

Definition 2.1. [22] Let X be a non-empty set. A fuzzy set A in X is defined by a membership function $\mu_A: X \to [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership" of x in A for $x \in X$. The set of all fuzzy sets in a set X is denoted by I^X, where I is the closed unit interval [0,1].

Definition 2.2. [22] If $A, B \in I^X$, then, we have:

(i) $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x),\ \forall x \in X;$

(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x),\ \forall x \in X;$

(iii) $C = A \vee B \Leftrightarrow \mu_C(x) = \max(\mu_A(x), \mu_B(x)),\ \forall x \in X;$

(iv) $D = A \wedge B \Leftrightarrow \mu_D(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X;$

(v) $E = A^C \Leftrightarrow \mu_E(x) = 1 - \mu_A(x), \forall x \in X.$

Definition 2.3. [16] Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A \subseteq E$. A pair (f, A) is called a soft set over X if f is a mapping from A into $P(X)$, i.e., $f : A \longrightarrow P(X)$. In other words, a soft set is a parameterized family of subsets of the set X. For $e \in A$, $f(e)$ may be considered as the set of e −approximate elements of the soft set (f, A).

Definition 2.4. [18] Let X be an initial universe set and E be a set of parameters. Let $A \subseteq E$. A fuzzy soft set f_A over X is a mapping from E to I^X, i.e., $f_A: E \longrightarrow I^X$, where $f_A(e) \neq \bar{0}$ if $e \in A \subseteq E$, and $f_A(e) = \bar{0}$ if $e \notin A$, where $\bar{0}$ denotes the empty fuzzy set in X.

Definition 2.5. [14] Let X be a universal set of elements and E be a universal set of parameters for X. Let $F : E \longrightarrow I^X$ and μ be a fuzzy subset of E, i.e., $\mu : E \longrightarrow I$. Let F_μ be the mapping $F_\mu : E \longrightarrow I^X \times I$ defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I$. Then F_μ is called a generalised fuzzy soft set ($GFSS$ in short) over (X, E). The family of all generalized fuzzy soft sets over (X, E) is denoted by $GFSS(X, E)$.

Definition 2.6. [14] Let F_μ and G_δ be two $GFSSs$ over (X, E). F_μ is said to be a GFS subset of G_δ or G_δ is said to be a GFS super set of F_μ denoted by $F_\mu \subseteq G_\delta$ if

(i) μ is a fuzzy subset of δ;

(ii) $F(e)$ is also a fuzzy subset of $G(e), \forall e \in E$.

Definition 2.7. [14] Let F_μ be a $GFSS$ over (X, E). The generalized fuzzy soft complement of F_μ, denoted by F_μ^c, is defined by $F_\mu^c = G_\delta$, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e)$, $\forall e \in E$.

Obviously $(F_\mu^c)^c = F_\mu$.

Definition 2.8. [3] Let F_μ and G_δ be two $GFSSs$ over (X, E). The generalized fuzzy soft union (GFS union, in short) of F_μ and G_δ, denoted by $F_\mu \sqcup G_\delta$, is The $GFSS$ H_ν, defined as $H_\nu : E \longrightarrow I^X \times I$ such that

$H_\nu(e) = (H(e), \nu(e))$, where $H(e) = F(e) \vee G(e)$ and $\nu(e) = \mu(e) \vee \delta(e)$, $\forall e \in E$.

Let $\{(F_\mu)_\lambda, \lambda \in \nabla\}$, where ∇ is an index set, be a family of $GFSSs$. The GFS union of these family, denoted by $\sqcup_{\lambda \in \Lambda}(F_\mu)_\lambda$, is The $GFSS$ H_ν, defined as $H_\nu : E \longrightarrow I^X \times I$ such that $H_\nu(e) = (H(e), \nu(e))$, where $H(e) = \vee_{\lambda \in \nabla}(F(e))_\lambda$, and $\nu(e) = \vee_{\lambda \in \nabla}(\mu(e))_\lambda$, $\forall e \in E$.

Definition 2.9. [3] Let F_μ and G_δ be two $GFSSs$ over (X, E). The generalized fuzzy soft Intersection (GFS Intersection, in short) of F_μ and G_δ, denoted by $F_\mu \sqcap G_\delta$, is the $GFSS$ M_σ, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that

$M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = F(e) \wedge G(e)$ and $\sigma(e) = \mu(e) \wedge \delta(e)$, $\forall e \in E$.

Let $\{(F_\mu)_\lambda, \lambda \in \nabla\}$, where ∇ is an index set, be a family of $GFSSs$. The GFS Intersection of these family, denoted by $\sqcap_{\lambda \in \nabla}(F_\mu)_\lambda$, is the $GFSS$ M_σ, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that $M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = \wedge_{\lambda \in \nabla}(F(e))_\lambda$, and $\sigma(e) = \wedge_{\lambda \in \nabla}(\mu(e))_\lambda$, $\forall e \in E$.

Theorem 2.1. [3] Let $\{(F_\mu)_\lambda, \lambda \in \nabla\} \subseteq GFSS(X, E)$. Then the following statements hold,

$[\sqcup_{\lambda \in \nabla} (F_\mu)_\lambda, \lambda \in \nabla]^c = \sqcap_{\lambda \in \nabla} (F_\mu)_\lambda^c$,

$[\sqcap_{\lambda \in \nabla} (F_\mu)_\lambda, \lambda \in \nabla]^c = \sqcup_{\lambda \in \nabla} (F_\mu)_\lambda^c$.

Definition 2.10. [14] A $GFSS$ is said to be a generalized null fuzzy soft set, denoted by $\tilde{0}_\theta$, if $\tilde{0}_\theta : E \longrightarrow I^X \times I$ such that $\tilde{0}_\theta(e) = (\tilde{0}(e), \theta(e))$ where $\tilde{0}(e) = \bar{0}$ $\forall e \in E$ and $\theta(e) = 0$ $\forall e \in E$ (Where $\bar{0}(x) = 0, \forall x \in X$).

Definition 2.11. [14] A $GFSS$ is said to be a generalized absolute fuzzy soft set, denoted by $\tilde{1}_\Delta$, if $\tilde{1}_\Delta : E \longrightarrow I^X \times I$, where $\tilde{1}_\Delta(e)= (\tilde{1}(e), \Delta(e))$ is defined by $\tilde{1}(e) = \bar{1}, \forall e \in E$ and $\Delta(e) = 1, \forall e \in E$ (Where $\bar{1}(x) = 1, \forall x \in X$).

Definition 2.12. [3] Let T be a collection of generalized fuzzy soft sets over (X, E). Then T is said to be a generalized fuzzy soft topology (GFS topology in short) over (X, E) if the following conditions are satisfied:

(i) $\tilde{0}_\theta$ and $\tilde{1}_\Delta$ are in T;

(ii) Arbitrary GFS unions of members of T belong to T;

(iii) Finite GFS intersections of members of T belong to T.

The triple (X, T, E) is called a generalized fuzzy soft topological space ($GFST$-space in short) over (X, E).

The members of T are called generalized fuzzy soft open sets [GFS open in short] in (X, T, E).

Definition 2.13 [3] Let (X, T, E) be a $GFST$ $-$space. A $GFSS$ F_μ over (X, E) is said to be a generalized fuzzy soft closed set in X [*GFS* closed in short], if its complement F_μ^c is GFS open. The collection of all GFS closed sets will be denoted by T^c.

Definition 2.14. [3] Let (X, T, E) be a $GFST$ $-$space and $F_\mu \in GFSS(X, E)$. The generalized fuzzy soft closure of F_μ, denoted by $cl(F_\mu)$, is the intersection of all GFS closed supper sets of F_μ. i.e., $cl(F_\mu) = \sqcap \{H_\nu : H_\nu \in T^c, F_\mu \sqsubseteq H_\nu\}$. Clearly, $cl(F_\mu)$ is the smallest GFS closed set over (X, E) which contains F_μ.

Definition 2.15. [9] The generalized fuzzy soft set $F_\mu \in GFS(X, E)$ is called a generalized fuzzy soft point (*GFS* point in short) if there exist $e \in E$ and $x \in X$ such that

(i) $F(e)(x) = \alpha$ $(0 < \alpha \leq 1)$ and $F(e)(y) = 0$ for all $y \in X - \{x\}$,

(ii) $\mu(e) = \lambda$ $(0 < \lambda \leq 1)$ and $\mu(e') = 0$ for all $e' \in E - \{e\}$. We denote this generalized fuzzy soft point $F_\mu = (x_\alpha, e_\lambda)$.

(x, e) and (α, λ) are called respectively, the support and the value of (x_α, e_λ).

Definition 2.16. [9] Let F_μ be a $GFSS$ over (X, E). We say that $(x_\alpha, e_\lambda) \tilde{\in} F_\mu$ read as (x_α, e_λ) belongs to the *GFSS* F_μ if for the element $e \in E$, $\alpha \leq F(e)(x)$ and $\lambda \leq \mu(e)$.

Definition 2.17. [17] For any two *GFSSs* F_μ and G_δ over (X, E). F_μ is said to be a generalized fuzzy soft quasi-coincident with G_δ, denoted by $F_\mu q G_\delta$, if there exist $e \in E$ and $x \in X$ such that $F(e)(x) + G(e)(x) > 1$ and $\mu(e) + \delta(e) > 1$.

If F_μ is not generalized fuzzy soft quasi-coincident with G_δ, then we write $F_\mu \bar{q} G_\delta$, i.e., for every $e \in E$ and $x \in X$, $F(e)(x) + G(e)(x) \leq 1$ or for every $e \in E$ and $x \in X$, $\mu(e) + \delta(e) \leq 1$.

Definition 2.18. [17] Let (x_α, e_λ) be a GFS point and F_μ be a *GFSS* over (X, E). (x_α, e_λ) is said to be generalized fuzzy soft quasi-coincident with F_μ, denoted by $(x_\alpha, e_\lambda) q F_\mu$, if and only if there exists an element $e \in E$ such that $\alpha + F(e)(x) > 1$ and $\lambda + \mu(e) > 1$.

Theorem 2.2. [17] Let F_μ and G_δ are *GFSSs* over (X, E). Then the following are hold:

(1)$F_\mu \sqsubseteq G_\delta \Leftrightarrow F_\mu \bar{q} (G_\delta)^c$;

(2) $F_\mu q G_\delta \Longrightarrow F_\mu \sqcap G_\delta \neq \tilde{0}_\theta$;

(3) $(x_\alpha, e_\lambda)\bar{q} F_\mu \Leftrightarrow (x_\alpha, e_\lambda) \tilde{\in} (F_\mu)^c$;

(4) $F_\mu \bar{q} (F_\mu)^c$.

Definition 2.19. [10] Let $GFSS(X,E)$ and $GFSS(Y,K)$ be the families of all generalized fuzzy soft sets over (X,E) and (Y,K) , respectively. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be two functions. Then a mapping f_{up} : $GFSS(X,E) \longrightarrow GFSS(Y,K)$ is defined as follows: for a generalized fuzzy soft set $F_\mu \in GFSS(X,E)$, $\forall\, k \in p(E) \subseteq K$ and $y \in Y$,

$$f_{up}(F_\mu)(k)(y) = \begin{cases} \left(\vee_{x\in u^{-1}(y)} \vee_{e\in p^{-1}(k)} F(e)(x)\,, \vee_{e\in p^{-1}(k)} \mu(e)\right) & if \quad u^{-1}(y) \neq \varphi, p^{-1}(k) \neq \varphi, \\ (0,0), & otherwise. \end{cases}$$

f_{up} is called a generalized fuzzy soft mapping [GFS mapping in short] and $f_{up}(F_\mu)$ is called a GFS image of a $GFSS$ F_μ.

Definition 2.20. [10] Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Let $f_{up} : GFSS(X,E) \longrightarrow GFSS(Y,K)$ be a GFS mapping and $G_\delta \in GFSS(Y,K)$. Then, $f_{up}^{-1}(G_\delta) \in GFSS(X,E)$, defined as follows:

$$f_{up}^{-1}(G_\delta)(e)(x) = \Big(G\big(p(e)\big)\big(u(x)\big), \delta\big(p(e)\big)\Big), \text{ for } e \in E,\ x \in X.$$

$f_{up}^{-1}(G_\delta)$ is called a *GFS* inverse image of G_δ.

If u and p are injective then the generalized fuzzy soft mapping f_{up} is said to be injective. If u and p are surjective then the generalized fuzzy soft mapping f_{up} is said to be surjective. The generalized fuzzy soft mapping f_{up} is called constant, if u and p are constant.

Definition 2.21. [10] Let (X,T_1,E) and (Y,T_2,K) be two $GFST$-spaces, and $f_{up} : (X,T_1,E) \longrightarrow (Y,T_2,K)$ be a GFS mapping. Then f_{up} is called

(1) generalized fuzzy soft continuous [*GFS*-continuous in short] if $f_{up}^{-1}(G_\delta) \in T_1$ for all $G_\delta \in T_2$.

(2) generalized fuzzy soft open [*GFS* open in short] if $f_{up}(F_\mu) \in T_2$ for each $F_\mu \in T_1$.

Definition 2.22. [11] Let (X,T,E) be a $GFST$-space and $F_\mu \in GFS(X,E)$. Then, F_μ is called

i. $GFSC_1$-connected if and only if it does not exist two non null GFS open sets H_ν and K_γ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$, $F_\mu \sqcap H_\nu \neq \tilde{0}_\theta$ and $F_\mu \sqcap K_\gamma \neq \tilde{0}_\theta$.

ii. $GFSC_2$-connected if and only if it does not exist two non null GFS open sets H_ν and K_γ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $F_\mu \sqcap H_\nu \sqcap K_\gamma = \tilde{0}_\theta$, $F_\mu \sqcap H_\nu \neq \tilde{0}_\theta$ and $F_\mu \sqcap K_\gamma \neq \tilde{0}_\theta$.

iii. $GFSC_3$-connected if and only if it does not exist two non null GFS open sets H_ν and K_γ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$, $H_\nu \not\sqsubseteq F_\mu^c$ and $K_\gamma \not\sqsubseteq F_\mu^c$.

iv. $GFSC_4$-connected if and only if it does not exist two non null GFS open sets H_ν and K_γ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $F_\mu \sqcap H_\nu \sqcap K_\gamma = \tilde{0}_\theta$, $H_\nu \not\sqsubseteq F_\mu^c$ and $K_\gamma \not\sqsubseteq F_\mu^c$.

Otherwise, F_μ is called not $GFSC_i$-connected set for $i = 1,2,3,4$.

In the above definition, if we take $\tilde{1}_\Delta$ instead of F_μ , then the $GFST$-space (X,T,E) is called $GFSC_i$-connected space $(i = 1,2,3,4)$.

Remark 2.1. [11] The relationship between $GFSC_i$-connectedness $(i = 1,2,3,4)$ can be described by the following diagram:

$GFSC_1 \Longrightarrow GFSC_2$

$\Downarrow \qquad\qquad \Downarrow$

$GFSC_3 \Longrightarrow GFSC_4$

Remark 2.2. [11] The reverse implications is not true in general (see Examples 4.2, 4.3, 4.4, 4.5, 4.6 in [11]).

3 GENERALIZED FUZZY SOFT SEPARATED SETS IN GENERALIZED FUZZY SOFT TOPOLOGICAL SPACES

In this section, we will introduce different notions of generalized fuzzy soft separated sets and study the relation between these notions. Also, we will investigate the characterizations of the generalized fuzzy soft separated sets.

Definition 3.1. Two non-null $GFSS$ sets F_μ and G_δ in $GFST$-space (X,T,E) are said to be generalized fuzzy soft Q −separated [$GFS\ Q$ −separated, in short] if $cl(F_\mu) \sqcap G_\delta = F_\mu \sqcap cl(G_\delta) = \tilde{0}_\theta$.

Theorem 3.1. Let (X,T,E) be a $GFST$-space, F_μ and G_δ be two GFS closed sets in (X,E). Then F_μ and G_δ are $GFS\ Q$ −separated sets if and only if $F_\mu \sqcap G_\delta = \tilde{0}_\theta$.

Proof. Suppose that F_μ and G_δ are $GFS\ Q$ −separated sets. Then $cl(F_\mu) \sqcap G_\delta = F_\mu \sqcap cl(G_\delta) = \tilde{0}_\theta$. Since F_μ and G_δ are GFS closed sets then, $F_\mu \sqcap G_\delta = \tilde{0}_\theta$.

Conversely, let $F_\mu \sqcap G_\delta = \tilde{0}_\theta$. Since F_μ and G_δ are GFS closed sets, then $cl(F_\mu) \sqcap G_\delta = F_\mu \sqcap G_\delta = \tilde{0}_\theta$ and $F_\mu \sqcap cl(G_\delta) = F_\mu \sqcap G_\delta = \tilde{0}_\theta$. It follows that, F_μ and G_δ are $GFS\ Q$ −separated sets.

Theorem 3.2. Let H_ν, K_γ be $GFS\ Q$ −separated sets of $GFST$-space (X,T,E) and $F_\mu \sqsubseteq H_\nu, G_\delta \sqsubseteq K_\gamma$.Then, F_μ, G_δ are $GFSQ$ −separated sets.

Proof. Let $F_\mu \sqsubseteq H_\nu$. Then, $cl(F_\mu) \sqsubseteq cl(H_\nu)$. It follows that, $cl(F_\mu) \sqcap G_\delta \sqsubseteq cl(F_\mu) \sqcap K_\gamma \sqsubseteq cl(H_\nu) \sqcap K_\gamma = \tilde{0}_\theta$. Also, since $G_\delta \sqsubseteq K_\gamma$. Then, $cl(G_\delta) \sqsubseteq cl(K_\gamma)$. Hence, $F_\mu \sqcap cl(G_\delta) \sqsubseteq H_\nu \sqcap cl(K_\gamma) = \tilde{0}_\theta$. Thus F_μ, G_δ are $GFSQ$ −separated sets.

Definition 3.2. Two non- null $GFSSs$ F_μ and G_δ in $GFST$-space (X,T,E) are said to be generalized fuzzy soft weakly separated [in short, GFS weakly separated] if $cl(F_\mu)\overline{q}G_\delta$ and $F_\mu\overline{q}cl(G_\delta)$.

Theorem 3.3. Let (X,T,E) be a $GFST$-space and $F_\mu, G_\delta \in GFS(X,E)$. Then, F_μ and G_δ are GFS weakly separated sets if and only if there exist GFS open sets H_ν and K_γ such that $F_\mu \sqsubseteq H_\nu, G_\delta \sqsubseteq K_\gamma$, and $F_\mu\overline{q}K_\gamma$ and $G_\delta\overline{q}H_\nu$.

Proof. Let F_μ and G_δ are GFS weakly separated sets in (X,T,E). Then $cl(F_\mu)\overline{q}G_\delta$ and $F_\mu\overline{q}cl(G_\delta)$. Therefore, $G_\delta \sqsubseteq [cl(F_\mu)]^c$ and $F_\mu \sqsubseteq [cl(G_\delta)]^c$. Taking $H_\nu = [cl(G_\delta)]^c$ and $K_\gamma = [cl(F_\mu)]^c$. Then, $H_\nu, K_\gamma \in T$, $F_\mu\overline{q}K_\gamma$ and $G_\delta\overline{q}H_\nu$. The converse is obvious.

Remark 3.1. From Definitions 3.1, 3.2 if F_μ and G_δ are $GFS\ Q$ −separated sets, then F_μ and G_δ are GFS weakly separated sets.

Remark 3.2. Two GFS weakly separated sets may not be $GFS\,Q$ −separated as shown by the following example.

Example 3.1. Let $X=\{x_1,x_2\}$, $E=\{e_1,e_2\}$ and $T=\{\tilde{0}_\theta,\tilde{1}_\Delta,\{(e_1=\{\frac{x_1}{0.3},\frac{x_2}{0.2}\},0.4),(e_2=\{\frac{x_1}{0.5},\frac{x_2}{0.3}\},0.6)\}\}$ be a GFS topology over (X,E). If $F_\mu=\{(e_1=\{\frac{x_1}{0.1}\},0.2)\}$ and $G_\delta=\{(e_2=\{\frac{x_1}{0.1},\frac{x_2}{0.1}\},0.3)\}$. Then F_μ and G_δ are GFS weakly separated sets, but F_μ and G_δ are not $GFS\ Q$ −separated.

Definition 3.3. Two non- null $GFSSs$ F_μ and G_δ in $GFST$-space (X,T,E) are said to be generalized fuzzy soft separated [in short, GFS separated] if there exist GFS open sets H_ν and K_γ such that $F_\mu\sqsubseteq H_\nu, G_\delta\sqsubseteq K_\gamma$ and $F_\mu\sqcap K_\gamma=G_\delta\sqcap H_\nu=\tilde{0}_\theta$.

Remark 3.3. Two GFS separated sets are GFS weakly separated sets.

Proof. From Definitions 3.3 and Theorem 3.3 it follows that.

Remark 3.4. Two GFS weakly separated sets may not be GFS separated. In fact, F_μ and G_δ defined in Example 3.1, are GFS weakly separated, but not GFS separated.

Remark 3.5. The notions of GFS separated sets and $GFS\ Q$ −separated are independent to each others as shown by the following example.

Example 3.2. Let $X=\{x_1,x_2\}$, $E=\{e_1,e_2\}$ and

$T=\{\tilde{0}_\theta,\tilde{1}_\Delta,H_\nu=\{(e_1=\{\frac{x_1}{0.5}\},0.3)\},K_\gamma=\{(e_2=\{\frac{x_2}{0.5}\},0.3)\},H_\nu\sqcup K_\gamma\}$ be a GFS topology over (X,E).

If $F_\mu=\{(e_1=\{\frac{x_1}{0.2}\},0.1)\}$ and $G_\delta=\{(e_2=\{\frac{x_2}{0.2}\},0.1)\}$. Then there exist GFS open sets H_ν and K_γ such that $F_\mu\sqsubseteq H_\nu, G_\delta\sqsubseteq K_\gamma$ and $F_\mu\sqcap K_\gamma=G_\delta\sqcap H_\nu=\tilde{0}_\theta$. So, F_μ and G_δ are GFS separated sets.

But F_μ and G_δ are not $GFS\ Q$ −separated. Since, $cl(F_\mu)=\{(e_1=\{\frac{x_1}{0.5},\frac{x_2}{1}\},0.7),(e_2=\{\frac{x_1}{1},\frac{x_2}{0.5}\},0.7)\}$ and $cl(F_\mu)\sqcap G_\delta\neq\tilde{0}_\theta$.

Example 3.3. Let $X=\{x_1,x_2\}$, $E=\{e_1,e_2\}$ and

$T=\{\tilde{0}_\theta,\tilde{1}_\Delta,\{(e_1=\{\frac{x_1}{0.3},\frac{x_2}{0.2}\},0.4),(e_2=\{\frac{x_1}{1},\frac{x_2}{1}\},1)\},\{(e_1=\{\frac{x_1}{1},\frac{x_2}{1}\},1),(e_2=\{\frac{x_1}{0.1},\frac{x_2}{0.4}\},0.3)\},$

$\{(e_1=\{\frac{x_1}{0.3},\frac{x_2}{0.2}\},0.4),(e_2=\{\frac{x_1}{0.1},\frac{x_2}{0.4}\},0.3)\}\}$ be a GFS topology over (X,E). Let $F_\mu=\{(e_1=\{\frac{x_1}{0.2}\},0.3)\}$ and $G_\delta=\{(e_2=\{\frac{x_2}{0.3}\},0.2)\}$. Then F_μ and G_δ are $GFS\ Q$ −separated sets, but not GFS separated.

Definition 3.4. Let $F_\mu\in GFS(X,E)$. The generalized fuzzy soft support (in short, GFS support) of F_μ defined by $S(F_\mu)$ is the set, $S(F_\mu)=\{x\in X,e\in E: F(e)(x)>0 \text{ and } \mu(e)>0\}$.

Definition 3.5. Two non- null $GFSSs$ F_μ and G_δ are said to be GFS quasi-coincident with respect to F_μ if $F(e)(x)+G(e)(x)>1$ and $\mu(e)+\delta(e)>1$ for every $x,e\in S(F_\mu)$.

Definition 3.6. Two non- null $GFSSs$ F_μ and G_δ in a $GFST$ −space (X,T,E) are said to be generalized fuzzy soft strongly separated [in short, GFS strongly separated] if there exist GFS open sets H_ν and K_γ such that

$i.$ $F_\mu\sqsubseteq H_\nu, G_\delta\sqsubseteq K_\gamma$ and $F_\mu\sqcap K_\gamma=G_\delta\sqcap H_\nu=\tilde{0}_\theta$,

$ii.$ F_μ and H_ν are GFS quasi-coincident with respect to F_μ,

$iii.$ G_δ and K_γ are GFS quasi-coincident with respect to G_δ.

Remark 3.6. From Definitions 3.3 and Remark 3.3 if F_μ and G_δ are GFS strongly separated, then F_μ and G_δ are GFS separated and GFS weakly separated.

Remark 3.7. Two GFS separated sets may not be GFS strongly separated as shown by the following example.

Example 3.4. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{0.3}, \frac{x_2}{0.2}\right\}, 0.3\right)\right\}, \left\{\left(e_2 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.2}\right\}, 0.4\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{0.3}, \frac{x_2}{0.2}\right\}, 0.3\right), \left(e_2 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.2}\right\}, 0.4\right)\right\}\right\}$ be a GFS topology over (X, E). If $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{0.1}\right\}, 0.2\right)\right\}$ and $G_\delta = \left\{\left(e_2 = \left\{\frac{x_2}{0.2}\right\}, 0.3\right)\right\}$. Then F_μ and G_δ are GFS separated sets, but not GFS strongly separated.

Remark 3.8. The notions of $GFS\ Q$ −separated and GFS strongly separated are independent to each others as shown by the following example:

Example 3.5. In Example 3.3, F_μ and G_δ are $GFS\ Q$ −separated sets, but not GFS strongly separated.

Example 3.6. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{0.7}, \frac{x_2}{0.2}\right\}, 0.8\right)\right\}, \left\{\left(e_2 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.7}\right\}, 0.6\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{0.7}, \frac{x_2}{0.2}\right\}, 0.8\right), \left(e_2 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.7}\right\}, 0.6\right)\right\}\right\}$ be a GFS topology over (X, E). Let $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{0.5}\right\}, 0.6\right)\right\}$ and $G_\delta = \left\{\left(e_2 = \left\{\frac{x_2}{0.4}\right\}, 0.5\right)\right\}$. Then F_μ and G_δ are GFS strongly separated, but not $GFS\ Q$ −separated.

Remark 3.9. In $GFST$ −space (X, T, E) the relationship between different notions of generalized fuzzy soft separated sets can be discribed by the following diagram.

$$GFS \text{ strongly separated}$$

$$\Downarrow$$

$$GFS \text{ separated}$$

$$\Downarrow$$

$$GFS\ Q - \text{separated} \Longrightarrow GFS \text{ weakly separated}$$

Theorem 3.4. Let F_μ and G_δ are $GFS\ Q$ −separated (respectively, separated, strongly separated, weakly separated) sets in (X, E) and $H_\nu \sqsubseteq F_\mu, K_\gamma \sqsubseteq G_\delta$. Then, H_ν and K_γ are $GFS\ Q$ −separated (respectively, separated, strongly separated, weakly separated) sets in (X, E).

Proof. As a sample, we will prove the case $GFS\ Q$ −separated. Let F_μ and G_δ are $GFS\ Q$ −separated in (X, E). Then, $cl(F_\mu) \sqcap G_\delta = F_\mu \sqcap cl(G_\delta) = \tilde{0}_\theta$. Since $H_\nu \sqsubseteq F_\mu, K_\gamma \sqsubseteq G_\delta$, then

$cl(H_\nu) \sqcap K_\gamma = H_\nu \sqcap cl(K_\gamma) = \tilde{0}_\theta$, therefore, H_ν and G_δ are $GFS\ Q$ −separated set in (X, E).

Theorem 3.5. Let (X, T, E) be a $GFST$ −space and $F_\mu, G_\delta \in GFS(X, E)$. Then, F_μ and G_δ are $GFS\ Q$ −separated in (X, E) if and only if there exist GFS closed sets H_ν and K_γ such that $F_\mu \sqsubseteq H_\nu, G_\delta \sqsubseteq K_\gamma$ and $F_\mu \sqcap K_\gamma = G_\delta \sqcap H_\nu = \tilde{0}_\theta$.

Proof. Let F_μ and G_δ are $GFS\ Q$ −separated in (X, E). Then, $cl(F_\mu) \sqcap G_\delta = F_\mu \sqcap cl(G_\delta) = \tilde{0}_\theta$. Taking $H_\nu = cl(F_\mu)$ and $K_\gamma = cl(G_\delta)$. Therefore, H_ν and K_γ are GFS closed sets in (X, E) such that $F_\mu \sqsubseteq H_\nu, G_\delta \sqsubseteq K_\gamma$ and $F_\mu \sqcap K_\gamma = G_\delta \sqcap H_\nu = \tilde{0}_\theta$. The converse is obvious.

Definition 3.7. Let (X,T,E) be a $GFST$ −space over (X,E) and G_δ be GFS subset of (X,E). Then $T_{G_\delta} = \{G_\delta \sqcap F_\mu : F_\mu \in T\}$ is called a GFS relative topology and $(G_\delta, T_{G_\delta}, E)$ is called a GFS subspace of (X,T,E). If $G_\delta \in T$ (resp, $G_\delta \in T^c$) then $(G_\delta, T_{G_\delta}, E)$ is called generalized fuzzy soft open (resp. closed) subspace of (X,T,E).

Theorem 3.6. Let (X,T,E) be a $GFST$ −space and $G_\delta \sqsubseteq F_\mu \tilde{\in} GFSS(X,E)$.Then, $cl_{F_\mu}(G_\delta) = cl(G_\delta) \sqcap F_\mu$. Where $cl_{F_\mu}(G_\delta)$ denotes the GFS closure in the GFS subspace (F_μ, T_{F_μ}, E).

Proof. We know $cl(G_\delta)$ is GFS closed set in $(X,T,E) \Longrightarrow cl(G_\delta) \sqcap F_\mu$ is GFS closed set in (F_μ, T_{F_μ}, E).

Now, $G_\delta \sqsubseteq cl(G_\delta) \sqcap F_\mu$ and GFS closure of G_δ in (F_μ, T_{F_μ}, E) is the smallest GFS closed set containing G_δ, so, GFS closure of G_δ in (F_μ, T_{F_μ}, E) is contained in $cl(G_\delta) \sqcap F_\mu$ i.e., $cl_{F_\mu}(G_\delta) \sqsubseteq cl(G_\delta) \sqcap F_\mu$.

Conversely,

let $cl_{F_\mu}(G_\delta)$ be a GFS closure of G_δ in (F_μ, T_{F_μ}, E). Since, $cl_{F_\mu}(G_\delta)$ is GFS closed set in $(F_\mu, T_{F_\mu}, E) \Longrightarrow cl_{F_\mu}(G_\delta) = K_\gamma \sqcap F_\mu$ where K_γ is GFS closed set in (X,T,E). Then, K_γ is GFS closed set containing $G_\delta \Longrightarrow cl(G_\delta) \sqsubseteq K_\gamma \Longrightarrow cl(G_\delta) \sqcap F_\mu \sqsubseteq K_\gamma \sqcap F_\mu \sqsubseteq cl_{F_\mu}(G_\delta)$.

Theorem 3.7. Let (X,T,E) be a $GFST$ −space and $G_\delta \sqsubseteq F_\mu \in GFS(X,E)$. If H_ν and K_γ are GFS separated (respectively, Q −separated, strongly separated, weakly separated) in (F_μ, T_{F_μ}, E), then H_ν and K_γ are GFS separated (respectively, Q −separated, strongly separated, weakly separated) in $(G_\delta, T_{G_\delta}, E)$.

Proof. As a sample, we will prove the case GFS weakly separated. Let H_ν and K_γ be GFS weakly separated sets in (F_μ, T_{F_μ}, E). Then, $cl_{F_\mu}(H_\nu)\overline{q}K_\gamma$ and $H_\nu \overline{q} cl_{F_\mu}(K_\gamma)$. Since, $G_\delta \sqsubseteq F_\mu$.Then, $cl_{G_\delta}(H_\nu) = cl_{F_\mu}(H_\nu) \sqcap G_\delta \sqsubseteq cl_{F_\mu}(H_\nu)$ and $cl_{G_\delta}(K_\gamma) = cl_{F_\mu}(K_\gamma) \sqcap G_\delta \sqsubseteq cl_{F_\mu}(K_\gamma)$. Therefore, $cl_{G_\delta}(H_\nu)\overline{q}K_\gamma$ and $H_\nu \overline{q} cl_{G_\delta}(K_\gamma)$. Thus, H_ν and K_γ be GFS weakly separated in $(G_\delta, T_{G_\delta}, E)$.

Remark 3.10. The converse of Theorem 3.6 is not true in general as shown by the following example:

Example 3.7. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $T^0 = \{\tilde{0}_\theta, \tilde{1}_\Delta\}$ be the GFS indiscrete topology over (X,E).

If $H_\nu = \{(e_1 = \{\frac{x_1}{0.1}, \frac{x_2}{0.2}\}, 0.1)\} \sqsubseteq F_\mu$, $K_\gamma = \{(e_2 = \{\frac{x_1}{0.1}, \frac{x_2}{0.3}\}, 0.2)\} \sqsubseteq F_\mu$, where

$F_\mu = \{(e_1 = \{\frac{x_1}{0.1}, \frac{x_2}{0.2}\}, 0.1), (e_2 = \{\frac{x_1}{0.1}, \frac{x_2}{0.3}\}, 0.2)\}$. Then, H_ν and K_γ are GFS weakly separated sets in (F_μ, T_{F_μ}, E) but H_ν and K_γ are not GFS weakly separated sets in (X,T,E).

4 GENERALIZED FUZZY SOFT CONNECTED SETS IN GENERALIZED FUZZY SOFT TOPOLOGICAL SPACES

In this section, we introduce different notions of connectedness of $GFSSs$ and study the relation between these notions. Also, we will investegate the characterizations of the generalized fuzzy soft connected sets.

Definition 4.1. A $GFSS$ F_μ in a $GFST$-space (X,T,E) is called GFS Q −connected set if there does not two non-null GFS Q −separated sets H_ν and K_γ such that $F_\mu = H_\nu \sqcup K_\gamma$, Otherwise, F_μ is called not GFS Q −connected set.

Definition 4.2. A $GFSS$ F_μ in a $GFST$-space (X,T,E) is called GFS weakly−connected set if there does not two non-null GFS weakly separated sets H_ν and K_γ such that $F_\mu = H_\nu \sqcup K_\gamma$, Otherwise, F_μ is called not GFS weakly−connected set.

Definition 4.3. A $GFSS$ F_μ in a $GFST$-space (X,T,E) is called GFS s −connected (respectively, GFS strongly−connected) set if there does not two non-null GFS separated (respectively, not strongly separated) sets H_ν and K_γ such that $F_\mu = H_\nu \sqcup K_\gamma$, Otherwise, F_μ is called not $GFS\, s$ −connected (respectively, GFS strongly−connected) set.

Definition A $GFSS\ F_\mu$ in a $GFST$-space (X,T,E) is called generalized fuzzy soft clopen set (GFS clopen set, in shoft) if $F_\mu, F_\mu^c \in T$.

Definition 4.4. A $GFSS\ F_\mu$ in a $GFST$-space (X,T,E) is called GFS clopen−connected set in (X,E) if there does not exist any non-null proper GFS clopen set in (F_μ, T_{F_μ}, E).

In the above definitions, if we take $\tilde{1}_\Delta$ instead of F_μ, then the $GFST$-space (X,T,E) is called $GFS\ Q -$ connected (respectively, GFS weakly−connected, $GFS\ s$ −connected, GFS strongly−connected, GFS clopen−connected) space.

Theorem 4.1. The GFS −weakly connected set in (X,E) is a $GFS\ Q$ −connected.

Proof. Let F_μ be a GFS −weakly connected set in (X,E). Suppose F_μ is not a $GFS\ Q$ −connected. Then, there exist two non-null $GFS\ Q$ −separated sets H_ν and K_γ such that $F_\mu = H_\nu \sqcup K_\gamma$. By Remark 3.1, H_ν and K_γ are non-null GFS weakly separated sets in (X,E) such that $F_\mu = H_\nu \sqcup K_\gamma$. Therefore, F_μ is not a GFS −weakly connected set in (X,E), a contradiction. Hence, F_μ is a $GFS\ Q$ −connected.

Remark 4.1. A $GFS\ Q$ −connected set may not be GFS weakly−connected as shown by the following example.

Example 4.1. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{0.3}, \frac{x_2}{0.2}\right\}, 0.3\right), \left(e_2 = \left\{\frac{x_1}{0.5}, \frac{x_2}{0.3}\right\}, 0.4\right)\right\}\right\}$ be a GFS topology over (X,E). Let $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{0.1}, \frac{x_2}{0.1}\right\}, 0.3\right)\right\}$. Then there exist $H_\nu = \left\{\left(e_1 = \left\{\frac{x_1}{0.1}\right\}, 0.2\right)\right\}$ and $K_\gamma = \left\{\left(e_1 = \left\{\frac{x_2}{0.1}\right\}, 0.3\right)\right\}$ such that $cl(H_\nu)\bar{q}K_\gamma$ and $H_\nu \bar{q}\, cl(K_\gamma)$, $F_\mu = H_\nu \sqcup K_\gamma$. So, F_μ is not a GFS weakly−connected. If we take $M_\psi = \left\{\left(e_1 = \left\{\frac{x_1}{0.1}, \frac{x_2}{\beta}\right\}, \lambda\right)\right\}$, $N_\eta = \left\{\left(e_1 = \left\{\frac{x_1}{\alpha}, \frac{x_2}{0.1}\right\}, 0.3\right)\right\}$ where $\alpha, \beta \le 0.1$ and $\lambda \le 0.3$. Then $cl(M_\psi) \sqcap N_\eta \ne \tilde{0}_\theta$ and $M_\psi \sqcap cl(N_\eta) \ne \tilde{0}_\theta$. Therefore, M_ψ and N_η are not $GFS\ Q$ separated sets. Hence, F_μ is a $GFS\ Q$ −connected.

Theorem 4.2. A $GFSC_1$ −connected set in (X,E) is GFS weakly−connected.

Proof. Let F_μ be a $GFSC_1$ −connected set in (X,E). Suppose F_μ is not GFS weakly−connected. Then, there exist two non-null GFS weakly separated sets H_ν and K_γ such that $F_\mu = H_\nu \sqcup K_\gamma$. By Theorem 3.3, there exist GFS open sets M_ψ and N_η such that $H_\nu \sqsubseteq M_\psi, K_\gamma \sqsubseteq N_\eta$, $H_\nu \bar{q} N_\eta$ and $M_\psi \bar{q} K_\gamma$. Then, $F_\mu \sqsubseteq M_\psi \sqcup N_\eta$. Also, $F_\mu \sqcap M_\psi \ne \tilde{0}_\theta$. For, if $F_\mu \sqcap M_\psi = \tilde{0}_\theta$, then $F_\mu \sqcap H_\nu = \tilde{0}_\theta$ so that $H_\nu = \tilde{0}_\theta$ (since $F_\mu = H_\nu \sqcup K_\gamma$ implies that $H_\nu \sqsubseteq F_\mu$), which contradiction that H_ν is a non-null. Similarly, $F_\mu \sqcap N_\eta \ne \tilde{0}_\theta$.

Also, $M_\psi \sqcap N_\eta \sqsubseteq (F_\mu)^c$. For, if $M_\psi \sqcap N_\eta \not\sqsubseteq F_\mu^c$, then there exist $x \in X, e \in E$ such that

$M(e)(x) > 1 - F(e)(x)$, $\psi(e) > 1 - \mu(e)$ and $N(e)(x) > 1 - F(e)(x)$, $\eta(e) > 1 - \mu(e)$.

This means $M(e)(x) + F(e)(x) > 1$, $\psi(e) + \mu(e) > 1$ and $N(e)(x) + F(e)(x) > 1$, $\eta(e) + \mu(e) > 1$. Since, $F_\mu = H_\nu \sqcup K_\gamma$, then $M(e)(x) + H(e)(x) > 1$, $\psi(e) + \nu(e) > 1$ or $M(e)(x) + K(e)(x) > 1$, $\psi(e) + \gamma(e) > 1$ and

$N(e)(x) + H(e)(x) > 1$, $\eta(e) + \nu(e) > 1$ or $N(e)(x) + K(e)(x) > 1$, $\eta(e) + \gamma(e) > 1$. Hence, ($M_\psi q H_\nu$ or $M_\psi q K_\gamma$) and ($N_\eta q H_\nu$ or $N_\eta q K_\gamma$). This a contradiction. So, F_μ is a GFS weakly−connected .

Remark 4.2. The GFS weakly−connected set may not be a $GFSC_1$ −connected as shown by the following example.

Example 4.2. Let $X = \{x_1, x_2\}$, $E = \{e_1\}$ and $T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{0.7}, \frac{x_2}{0.8}\right\}, 0.6\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.3}\right\}, 0.1\right)\right\}\right\}$ be a GFS topology over (X,E) and $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{0.4}, \frac{x_2}{0.4}\right\}, 0.5\right)\right\}$. Then, there exist two GFS open sets $H_\nu = \{(e_1 =$

$\left\{\frac{x_1}{0.7},\frac{x_2}{0.8}\right\}, 0.6)\}$ and $K_\gamma = \{(e_1 = \{\frac{x_1}{0.2},\frac{x_2}{0.3}\}, 0.1)\}$ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$, $F_\mu \sqcap H_\nu \neq \tilde{0}_\theta$ and $F_\mu \sqcap K_\gamma \neq \tilde{0}_\theta$. So, F_μ is not a $GFSC_1$ −connected. If we take $M_\psi = \left\{\left(e_1 = \left\{\frac{x_1}{0.4},\frac{x_2}{\beta}\right\},\lambda\right)\right\}$, $N_\eta = \{(e_1 = \{\frac{x_1}{\alpha},\frac{x_2}{0.4}\}, 0.5)\}$ where $\alpha,\beta \leq 0.4$ and $\lambda \leq 0.5$. Then $cl(M_\psi)qN_\eta$ and $M_\psi qcl(N_\eta)$. Therefore, M_ψ and N_η are not GFS weakly separated sets. Hence, F_μ is a GFS weakly−connected.

Theorem 4.3. A GFS weakly−connected set in (X,E) is $GFSC_2$ −connected.

Proof. Let F_μ be a GFS weakly−connected set in (X,E). Suppose F_μ is not $GFSC_2$ −connected. Then, there exist H_ν and $K_\gamma \in T$ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $F_\mu \sqcap H_\nu \sqcap K_\gamma = \tilde{0}_\theta$, $F_\mu \sqcap H_\nu \neq \tilde{0}_\theta$ and $F_\mu \sqcap K_\gamma \neq \tilde{0}_\theta$. Then, $F_\mu = M_\psi \sqcup N_\eta$ where $M_\psi = F_\mu \sqcap H_\nu \sqsubseteq H_\nu$ and $N_\eta = F_\mu \sqcap K_\gamma \sqsubseteq K_\gamma$. Since $F_\mu \sqcap H_\nu \sqcap K_\gamma = \tilde{0}_\theta$ and $M_\psi \sqsubseteq H_\nu$, then $F_\mu \sqcap M_\psi \sqcap K_\gamma = \tilde{0}_\theta$. Also, since $M_\psi \sqsubseteq F_\mu$, then $M_\psi \sqcap K_\gamma = \tilde{0}_\theta$. Therefore, $M_\psi \overline{q} K_\gamma$, Similarly, $N_\eta \overline{q} H_\nu$. Hence, F_μ is not a GFS weakly−connected . This complete the proof.

Theorem 4.4. A GFS weakly−connected set in (X,E) is $GFSC_3$ −connected.

Proof. Let F_μ be a The GFS weakly−connected set in (X,E). Suppose F_μ is not $GFSC_3$ −connected. Then, there exist H_ν and $K_\gamma \in T$ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$, $H_\nu \not\sqsubseteq F_\mu^c$ and $K_\gamma \not\sqsubseteq F_\mu^c$. Then, $F_\mu = M_\psi \sqcup N_\eta$ where $M_\psi = F_\mu \sqcap H_\nu \sqsubseteq H_\nu$ and $N_\eta = F_\mu \sqcap K_\gamma \sqsubseteq K_\gamma$. Let J_σ and $L_\rho \in GFS(X,E)$ defined by:

$$J_\sigma = \begin{cases} M_\psi, & H_\nu \sqsupseteq K_\gamma, \\ \tilde{0}_\theta, & \text{otherwise} \end{cases}$$

$$L_\rho = \begin{cases} N_\eta, & K_\gamma \sqsupset H_\nu, \\ \tilde{0}_\theta, & \text{otherwise} \end{cases}$$

Then $F_\mu = J_\sigma \sqcup L_\rho$.

Now, $J(e)(x) \neq 0$, $\sigma(e) \neq 0$. For, $J(e)(x) = 0$, $\sigma(e) = 0$. Since, $H_\nu \not\sqsubseteq F_\mu^c$, then there exist $x \in X, e \in E$ such that $H(e)(x) + F(e)(x) > 1$, $\nu(e) + \mu(e) > 1$. Then, $H(e)(x) > K(e)(x)$, $\nu(e) > \gamma(e)$. For, $H(e)(x) \leq K(e)(x)$, $\nu(e) \leq \gamma(e)$ implies $K(e)(x) + F(e)(x) > 1$, $\gamma(e) + \mu(e) > 1$ and hence $(H_\nu \sqcap K_\gamma)(e)(x) > 1 - F_\mu(e)(x)$ i.e., $H(e)(x) > 1 - F(e)(x)$, $\nu(e) > 1 - \mu(e)$ and $K(e)(x) > 1 - F(e)(x)$, $\gamma(e) > 1 - \mu(e)$ this is a contradiction with $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$. So, $J(e)(x) \neq 0$, $\sigma(e) \neq 0$. Similarly, $L(e)(x) \neq 0$, $\rho(e) \neq 0$. Also, $J_\sigma \sqsubseteq M_\psi \sqsubseteq H_\nu$ and $L_\rho \sqsubseteq N_\eta \sqsubseteq K_\gamma$. Now, $J_\sigma \overline{q} K_\gamma$. For, if $J_\sigma q K_\gamma$, then there exist $x \in X, e \in E$ such that $J(e)(x) + K(e)(x) > 1$, $\sigma(e) + \gamma(e) > 1$ and hence $J(e)(x) > 0$, $\sigma(e) > 0$. This means $H(e)(x) \geq K(e)(x)$, $\nu(e) \leq \gamma(e)$ and so $F(e)(x) = M(e)(x)$, $\mu(e) = \psi(e)$ implying $F(e)(x) + H(e)(x) > 1$, $\mu(e) + \nu(e) > 1$ and thus $(H_\nu \sqcap K_\gamma)(e)(x) > 1 - F_\mu(e)(x)$ which is a contradiction with $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$. Similarly, $L_\rho \overline{q} H_\nu$. Thus, J_σ and L_ρ are GFS weakly separated and $F_\mu = J_\sigma \sqcup L_\rho$. So, F_μ is not a GFS weakly−connected. This a contradiction. Then F_μ is a $GFSC_3$ −connected.

Remark 4.3. The $GFSC_3$ −connected set (respectively, $GFSC_2$ −connected) may not be a GFS weakly−connected as shown by the following example.

Example 4.3. Let $X = \{x_1, x_2\}$, $E = \{e_1\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{2/3},\frac{x_2}{1/3}\right\}, {}^1/_3\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{1/3},\frac{x_2}{2/3}\right\}, {}^2/_3\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{1/3},\frac{x_2}{1/3}\right\}, {}^1/_3\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{2/3},\frac{x_2}{2/3}\right\}, {}^2/_3\right)\right\}\right\}$ be a GFS topology over (X,E) and $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{1/3},\frac{x_2}{1/3}\right\}, {}^1/_3\right)\right\}$. Then, F_μ is $GFSC_3$ −connected (respectively, $GFSC_2$ −connected). But F_μ is not a GFS weakly−connected as there exist GFS weakly separated sets $H_\nu = \left\{\left(e_1 = \left\{\frac{x_1}{1/3}\right\}, {}^1/_3\right)\right\}$, $K_\gamma = \left\{\left(e_1 = \left\{\frac{x_2}{1/3}\right\}, {}^1/_3\right)\right\}$ such that $F_\mu = H_\nu \sqcup K_\gamma$.

Theorem 4.5. The $GFSC_3$ −connected set in (X,E) is a $GFS\ Q$ −connected.

Proof. Let F_μ be a $GFSC_3$ −connected set in (X, E). Suppose F_μ is not $GFS\ Q$ −connected. Then, there exist two non-null $GFS\ Q$ −separated sets H_ν and K_γ such that $F_\mu = H_\nu \sqcup K_\gamma$, $cl(H_\nu) \sqcap K_\gamma = H_\nu \sqcap cl(K_\gamma) = \tilde{0}_\theta$. This implies that $K_\gamma \sqsubseteq [cl(H_\nu)]^c$ and $H_\nu \sqsubseteq [cl(K_\gamma)]^c$. Let $M_\psi = [cl(H_\nu)]^c$ and $N_\eta = [cl(K_\gamma)]^c$. Then, M_ψ and N_η are non- null GFS open sets such that $F_\mu \sqsubseteq M_\psi \sqcup N_\eta$. Now, $M_\psi \sqcap N_\eta = [cl(H_\nu)]^c \sqcap [cl(K_\gamma)]^c = [cl(H_\nu) \sqcup cl(K_\gamma)]^c = [cl(H_\nu \sqcup K_\gamma)]^c \sqsubseteq F_\mu^c$. Aso, $M_\psi \not\sqsubseteq F_\mu^c$. For, if $M_\psi \sqsubseteq F_\mu^c$, then $F_\mu \sqsubseteq M_\psi^c = cl(H_\nu)$ which would imply $K_\gamma = \tilde{0}_\theta$ (since $cl(H_\nu) \sqcap K_\gamma = \tilde{0}_\theta$). This is a contradiction. Similarly, $N_\eta \not\sqsubseteq F_\mu^c$. Therefore, F_μ is not $GFSC_3$ −connected. So, F_μ is $GFS\ Q$ −connected.

Remark 4.4. A $GFS\ Q$ −connected set may not be $GFSC_3$ −connected as shown by the following example.

Example 4.4. Let $X = \{x_1, x_2\}$, $E = \{e_1\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{0.6}, \frac{x_2}{0.2}\right\}, 0.3\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.7}\right\}, 0.4\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{0.6}, \frac{x_2}{0.7}\right\}, 0.4\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.2}\right\}, 0.3\right)\right\}\right\}$, be a GFS topology over (X, E) and $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{0.6}, \frac{x_2}{0.7}\right\}, 0.4\right)\right\}$.

Then, there exist non- null GFS open sets $H_\nu = \left\{\left(e_1 = \left\{\frac{x_1}{0.6}, \frac{x_2}{0.2}\right\}, 0.3\right)\right\}$ and $K_\gamma = \left\{\left(e_1 = \left\{\frac{x_1}{0.2}, \frac{x_2}{0.7}\right\}, 0.4\right)\right\}$ such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$, $H_\nu \not\sqsubseteq F_\mu^c$ and $K_\gamma \not\sqsubseteq F_\mu^c$. So, F_μ is not $GFSC_3$ −connected. However, F_μ is $GFS\ Q$ −connected.

Theorem 4.6. A $GFSS\ F_\mu$ in (X, E) is $GFSC_2$ −connected if and only if F_μ is $GFS\ s$ −connected.

Proof. Let F_μ be a $GFSC_2$ −connected set in (X, E). Suppose F_μ is not a $GFS\ s$ −connected. Then there exist non-null GFS separated sets H_ν and K_γ in (X, E) such that $F_\mu = H_\nu \sqcup K_\gamma$. Then, there exist two non- null GFS open sets M_ψ and N_η such that $H_\nu \sqsubseteq M_\psi$, $K_\gamma \sqsubseteq N_\eta$, and $H_\nu \sqcap N_\eta = K_\gamma \sqcap M_\psi = \tilde{0}_\theta$. Then, $F_\mu \sqsubseteq M_\psi \sqcup N_\eta$.

Now, $F_\mu \sqcap M_\psi \sqcap N_\eta = (H_\nu \sqcup K_\gamma) \sqcap M_\psi \sqcap N_\eta = (H_\nu \sqcap M_\psi \sqcap N_\eta) \sqcup (K_\gamma \sqcap M_\psi \sqcap N_\eta) = \tilde{0}_\theta$ and $F_\mu \sqcap M_\psi = (H_\nu \sqcup K_\gamma) \sqcap M_\psi = (H_\nu \sqcap M_\psi) \sqcup (K_\gamma \sqcap M_\psi) = H_\nu \neq \tilde{0}_\theta$. Similarly, $F_\mu \sqcap N_\eta \neq \tilde{0}_\theta$. So, F_μ is not $GFSC_2$ −connected which is a contradiction.

Conversely, let F_μ be $GFS\ s$ −connected. Suppose that F_μ is not $GFSC_2$ −connected. Then there exist two non-null GFS open sets M_ψ and N_η such that $F_\mu \sqsubseteq M_\psi \sqcup N_\eta$, $F_\mu \sqcap M_\psi \sqcap N_\eta = \tilde{0}_\theta$, $F_\mu \sqcap M_\psi \neq \tilde{0}_\theta$, $F_\mu \sqcap N_\eta \neq \tilde{0}_\theta$. Hence, $F_\mu = H_\nu \sqcup K_\gamma$ where $H_\nu = F_\mu \sqcap M_\psi \sqsubseteq M_\psi$ and $K_\gamma = F_\mu \sqcap N_\eta \sqsubseteq N_\eta$. Also, $K_\gamma \sqcap M_\psi = (F_\mu \sqcap N_\eta) \sqcap M_\psi = \tilde{0}_\theta$, Similarly, $H_\nu \sqcap N_\eta = \tilde{0}_\theta$. So, F_μ is not $GFS\ s$ −connected and this complete the proof.

Theorem 4.7. The $GFSC_4$ −connected set in (X, E) is a GFS strongly−connected.

Proof. Let F_μ be a $GFSC_4$ −connected set in (X, E). Suppose F_μ is not a GFS strongly−connected. Then there exist two non-null GFS strongly separated sets H_ν and K_γ in (X, E) such that $F_\mu = H_\nu \sqcup K_\gamma$. So, there exist two non- null GFS open sets M_ψ and N_η such that

$H_\nu \sqsubseteq M_\psi$, $K_\gamma \sqsubseteq N_\eta$, and $H_\nu \sqcap N_\eta = K_\gamma \sqcap M_\psi = \tilde{0}_\theta$,

H_ν and M_ψ GFS quasi-coincident with respect to H_ν, and K_γ and N_η GFS quasi-coincident with respect to K_γ.

Then, for every $x, e \in S(H_\nu)$ we have $H(e)(x) + M(e)(x) > 1$ and $\nu(e) + \psi(e) > 1$ and for every $x, e \in S(K_\gamma)$ we have $K(e)(x) + N(e)(x) > 1$ and $\gamma(e) + \eta(e) > 1$. Then, $F_\mu \sqsubseteq M_\psi \sqcup N_\eta$. Also, $F_\mu \sqcap M_\psi \sqcap N_\eta = \tilde{0}_\theta$.

Again, $F(e)(x) + M(e)(x) > H(e)(x) + M(e)(x)$ and $\mu(e) + \psi(e) > \nu(e) + \psi(e) >$ for every $x, e \in S(H_\nu)$. Therefore, $M_\psi \not\sqsubseteq F_\mu^c$, Similarly, $N_\eta \not\sqsubseteq F_\mu^c$. Thus, F_μ is not a $GFSC_4$ −connected. This is a contradiction. So, F_μ is a GFS strongly−connected.

Remark 4.5. A GFS strongly−connected set may not be $GFSC_4$ −connected as shown by the following example.

Example 4.5. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \{(e_1 = \{\frac{x_1}{0.7}\}, 0.9)\}, \{(e_2 = \{\frac{x_2}{0.7}, \frac{x_3}{0.8}\}, 0.6)\}, \{(e_1 = \{\frac{x_1}{0.7}\}, 0.9), (e_2 = \{\frac{x_2}{0.7}, \frac{x_3}{0.8}\}, 0.6)\}\right\}$, be a GFS topology over (X, E).

Let $F_\mu = \{(e_1 = \{\frac{x_1}{0.7}\}, 0.9), (e_2 = \{\frac{x_2}{0.7}, \frac{x_3}{0.8}\}, 0.6)\}$ and $H_\nu = \{(e_1 = \{\frac{x_1}{0.7}\}, 0.9)\}, K_\gamma = \{(e_2 = \{\frac{x_2}{0.7}, \frac{x_3}{0.8}\}, 0.6)\} \in T$.

Then, $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $F_\mu \sqcap H_\nu \sqcap K_\gamma = \tilde{0}_\theta$, $H_\nu \not\sqsubseteq F_\mu^c$ and $K_\gamma \not\sqsubseteq F_\mu^c$. So, F_μ is not a $GFSC_4$ −connected. However, F_μ is GFS strongly−connected.

Remark 4.6. A $GFS\ Q$ −connected set and GFS strongly−connected are independent concepts as shown by the following examples.

Example 4.6. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \{(e_1 = \{\frac{x_1}{0.8}\}, 0.9)\}, \{(e_2 = \{\frac{x_2}{0.9}, \frac{x_3}{0.9}\}, 0.7)\}, \{(e_1 = \{\frac{x_1}{0.8}\}, 0.9), (e_2 = \{\frac{x_2}{0.9}, \frac{x_3}{0.9}\}, 0.7)\}\right\}$ be a GFS topology over (X, E). Let $F_\mu = \{(e_1 = \{\frac{x_1}{0.6}\}, 0.7), (e_2 = \{\frac{x_2}{0.7}, \frac{x_3}{0.8}\}, 0.6)\}$.

Then, there exist two non-null GFS strongly separated $H_\nu = \{(e_1 = \{\frac{x_1}{0.6}\}, 0.7)\}$ and $K_\gamma = \{(e_2 = \{\frac{x_2}{0.7}, \frac{x_3}{0.8}\}, 0.6)\}$ such that $F_\mu = H_\nu \sqcup K_\gamma$. So, F_μ is not GFS strongly−connected. However, F_μ is $GFS\ Q$ −connected as $cl(H_\nu) \sqcap K_\gamma \neq \tilde{0}_\theta$ and also $H_\nu \sqcap cl(K_\gamma) \neq \tilde{0}_\theta$.

Example 4.7. Let $X = \{x_1, x_2\}$,$E = \{e_1, e_2\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \{(e_1 = \{\frac{x_1}{0.4}\}, 0.4), (e_2 = \{\frac{x_1}{1}, \frac{x_2}{1}\}, 1)\}, \{(e_1 = \{\frac{x_1}{1}, \frac{x_2}{1}\}, 1), (e_2 = \{\frac{x_2}{0.4}\}, 0.4)\}, \{(e_1 = \{\frac{x_1}{0.4}\}, 0.4), (e_2 = \{\frac{x_2}{0.4}\}, 0.4)\}\right\}$ be a GFS topology over (X, E). Let $F_\mu = \{(e_1 = \{\frac{x_1}{0.4}\}, 0.4), (e_2 = \{\frac{x_2}{0.4}\}, 0.4)\}$. Then, there exist non- null $GFS\ Q$ −separated sets $H_\nu = \left\{\left((e_1 = \{\frac{x_1}{0.4}\}, 0.4)\right)\right\}$ and $K_\gamma = \{(e_2 = \{\frac{x_2}{0.4}\}, 0.4)\}$ such that $F_\mu = H_\nu \sqcup K_\gamma$. So, F_μ is not $GFS\ Q$ −connected. However, F_μ is GFS strongly−connected as H_ν and K_γ are not GFS strongly separated.

Remark 4.7. A $GFSC_2$ −connected set may not be $GFS\ Q$ −connected as shown by the following example.

Example 4.8. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{1/3}, \frac{x_2}{1}\right\}, 1/3\right), \left(e_2 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right), \left(e_2 = \left\{\frac{x_1}{1}, \frac{x_2}{1/3}\right\}, 1/3\right)\right\},\right.$

$\left.\left\{\left(e_1 = \left\{\frac{x_1}{1/3}, \frac{x_2}{1}\right\}, 1/3\right), \left(e_2 = \left\{\frac{x_1}{1}, \frac{x_2}{1/3}\right\}, 1/3\right)\right\}\right\}$ be a GFS topology over (X, E).

Let $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{2/3}\right\}, 2/3\right), \left(e_2 = \left\{\frac{x_2}{2/3}\right\}, 2/3\right)\right\}$. Then, F_μ can be expressed as union of two non-null $GFS\ Q$ −separated sets $H_\nu = \left\{\left(e_1 = \left\{\frac{x_1}{2/3}\right\}, 1/3\right)\right\}$ and $K_\gamma = \left\{\left(e_2 = \left\{\frac{x_2}{2/3}\right\}, 2/3\right)\right\}$. So, F_μ is not a $GFS\ Q$ −connected. However, F_μ is a $GFSC_2$ −connected as if we take

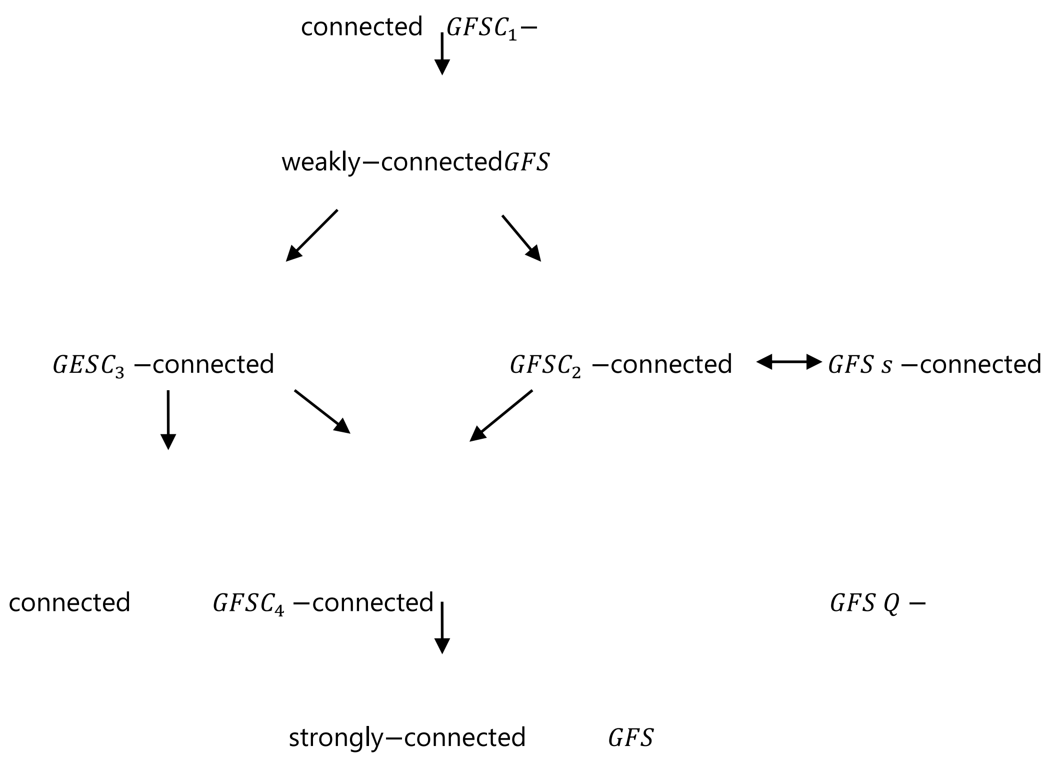
$M_\psi = \left\{\left(e_1 = \left\{\frac{x_1}{1/3}, \frac{x_2}{1}\right\}, 1/3\right), \left(e_2 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right)\right\}$ and $N_\eta = \left\{\left(e_1 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right), \left(e_2 = \left\{\frac{x_1}{1}, \frac{x_2}{1/3}\right\}, 1/3\right)\right\} \in T$, then $F_\mu \sqsubseteq M_\psi \sqcup N_\eta$, but $F_\mu \sqcap M_\psi \sqcap N_\eta \neq \tilde{0}_\theta$.

Remark 4.8. A GFS clopen−connected set may not be a $GFS\,s$ −connected (respectively, GFS strongly−connected, $GFS\ Q$ −connected, GFS weakly−connected, $GFSC_i$ −connected for $i = 1,2,3,4$). In fact, F_μ defined in Example 4.6 is a GFS clopen−connected, but it is not a GFS strongly−connected set and in Example 4.8 is a GFS clopen−connected, but it is not a $GFS\ Q$ −connected set. Therefore, it is not a $GFS\ s$ −connected, not a GFS weakly−connected set and not a $GFSC_i$ −connected set for $i = 1,2,3,4$.

Remark 4.9. A $GFS\,s$ −connected (respectively, GFS strongly−connected, $GFS\ Q$ −connected, GFS weakly−connected, $GFSC_i$ −connected for $i = 1,2,3,4$) set may not be GFS clopen−connected as shown by the following example.

Example 4.9. Let $X = \{x_1, x_2\}$, $E = \{e_1\}$ and $T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{0.3}\right\}, 0.3\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{0.5}, \frac{x_2}{0.6}\right\}, 0.5\right)\right\}\right\}$ be a GFS topology over (X, E). Let $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{0.7}\right\}, 0.7\right)\right\}$. Then, F_μ is a $GFS\ s$ −connected, GFS strongly−connected, $GFS\ Q$ −connected, GFS weakly−connected, $GFSC_i$ −connected for $i = 1,2,3,4$). But since $\left\{\left(e_1 = \left\{\frac{x_1}{0.5}\right\}, 0.5\right)\right\}$ is a non-null proper clopen $GFSS$ in F_μ. So, F_μ is not a GFS clopen−connected.

Remark 4.10. In a $GFST$-space (X, T, E). The classes of $GFS\,s$ −connected, GFS strongly−connected, $GFS\ Q$ −connected, GFS weakly−connected, $GFSC_i$ −connected for $i = 1,2,3,4$, can be discribed by the following diagram.



Theorem 4.8. Let (X, T_1, E) and (Y, T_2, K) be a $GFST$-spaces and $f_{up}: (X, T_1, E) \longrightarrow (Y, T_1, K)$ be a GFS-continuous bijective mapping. If F_μ is a $GFSC_i$ −connected (respectively, $GFS\ s$ −connected, GFS strongly−connected, GFS weakly−connected, GFS clopen−connected) set in (X, E) for $i = 1,2$, then $f_{up}(F_\mu)$ is a $GFSC_i$ −connected (respectively, $GFS\ s$ −connected, GFS strongly−connected, GFS weakly−connected, GFS clopen−connected) set in (Y, K) for $i = 1,2$.

Proof. The case of $GFSC_i$ −connected set **(**$i = 1,2$) previously proved (see Theorem 4.7 in [11]). Now, we will prove the case of GFS clopen−connected. Let F_μ be a GFS −clopen connected set in (X, E). Suppose $f_{up}(F_\mu)$ is not a GFS clopen−connected set in (Y, K). Then, $f_{up}(F_\mu)$ has non-null proper clopen GFS subset of J_σ. So, there exist $S_\varepsilon \in T_2$ and $L_\rho \in T_2^c$ such that $J_\sigma = f_{up}(F_\mu) \sqcap S_\varepsilon = f_{up}(F_\mu) \sqcap L_\rho$. Since, f_{up} is injective mapping, then $f_{up}^{-1}(J_\sigma) = F_\mu \sqcap f_{up}^{-1}(S_\varepsilon) = F_\mu \sqcap f_{up}^{-1}(L_\rho)$. Also, since $S_\varepsilon \in T_2$ and $L_\rho \in T_2^c$ and f_{up} is a GFS- continuous mapping, then $f_{up}^{-1}(S_\varepsilon) \in T_1$ and $f_{up}^{-1}(L_\rho) \in T_1^c$. Hence, $f_{up}^{-1}(J_\sigma)$ is non-null proper clopen GFS subset of F_μ which is a contradiction. Therefore, $f_{up}(F_\mu)$ is a GFS −clopen connected set in (Y, K).

The cases of $GFSC_3$ −connected and $GFSC_4$ −connected sets we need to the GFS-continuous surjective mapping previously proved (see Theorem 4.8 in [11]).

Theorem 4.9. Let (X, T_1, E) and (Y, T_2, K) be a $GFST$-spaces and $f_{up}: (X, T_1, E) \longrightarrow (Y, T_1, K)$ be a GFS injective mapping. If F_μ is a $GFS\ Q$ −connected set in (X, E), then $f_{up}(F_\mu)$ is a $GFS\ Q$ −connected set in (Y, K).

Proof. Let F_μ be a $GFS\ Q$ −connected set in (X, E). Suppose $f_{up}(F_\mu)$ is not a $GFS\ Q$ −connected set in (Y, K). Then, there exist two non- null $GFS\ Q$ separated sets J_σ and L_ρ in (X, E) such that

$$f_{up}(F_\mu) = J_\sigma \sqcup L_\rho,\ cl(J_\sigma) \sqcap L_\rho = J_\sigma \sqcap cl(L_\rho) = \tilde{0}_{\theta_Y}.$$

Since, f_{up} is injective mapping, then $f_{up}^{-1}(f_{up}(F_\mu)) = f_{up}^{-1}(J_\sigma) \sqcup f_{up}^{-1}(L_\rho)$,

$$cl(f_{up}^{-1}(J_\sigma)) \sqcap f_{up}^{-1}(L_\rho) \sqsubseteq f_{up}^{-1}(cl(J_\sigma)) \sqcap f_{up}^{-1}(L_\rho) = f_{up}^{-1}\big(cl(J_\sigma) \sqcap L_\rho\big) = f_{up}^{-1}(\tilde{0}_{\theta_Y}) = \tilde{0}_{\theta_X},$$

$$f_{up}^{-1}(J_\sigma) \sqcap cl(f_{up}^{-1}(L_\rho)) \sqsubseteq f_{up}^{-1}(J_\sigma \sqcap f_{up}^{-1}(cl(L_\rho)) = f_{up}^{-1}\big(L_\rho \sqcap cl(L_\rho)\big) = f_{up}^{-1}(\tilde{0}_{\theta_Y}) = \tilde{0}_{\theta_X}.$$

This mains that, $f_{up}^{-1}(J_\sigma)$, $f_{up}^{-1}(L_\rho)$ are $GFS\ Q$ separated sets of F_μ in (X, E), which is contradicts of the $GFS\ Q$ −connectedness of F_μ in (X, E). Therefore, $f_{up}(F_\mu)$ is a $GFS\ Q$ −connected set in (Y, K).

Theorem 4.9. Let (X, T_1, E) and (Y, T_2, K) be a $GFST$-spaces and $f_{up}: (X, T_1, E) \longrightarrow (Y, T_1, K)$ be a GFS- bijective open mapping. If G_δ is a $GFSC_i$ −connected(respectively, $GFS\ s$ −connected, GFS strongly−connected, $GFS\ Q$ −connected, GFS weakly−connected, GFS clopen−connected) set in (Y, E) for $i = 1,2,3,4$, then $f_{up}^{-1}(G_\delta)$ is a $GFSC_i$ −connected (respectively, $GFS\ s$ −connected, GFS strongly−connected, $GFS\ Q$ −connected,GFS weakly−connected, GFS −clopen connected) set in (Y, E) for $i = 1,2,3,4$.

Proof. The case of $GFSC_i$ −connected set **(**$i = 1,2,3,4$) previously proved (see Theorem 4.13 in [11]). Now, we will prove the case of $GFS\ s$ −connected. Let G_δ is a $GFS\ s$ −connected set in (Y, K). Suppose $f_{up}^{-1}(G_\delta)$ is not a $GFS\ s$ −connected set in (X, E). Then, there exist two non- null GFS separated sets H_ν and K_γ in (X, E) such that $f_{up}^{-1}(G_\delta) = H_\nu \sqcup K_\gamma$. Therefore, there exist two non- null GFS open sets M_ψ and N_η in (X, E) such that $H_\nu \sqsubseteq M_\psi$ and $K_\gamma \sqsubseteq N_\eta$ and $H_\nu \sqcap N_\eta = K_\gamma \sqcap M_\psi = \tilde{0}_\theta$. Since, f_{up} is a GFS surjective mapping, then $f_{up}(f_{up}^{-1}(G_\delta)) = G_\delta$ and so $G_\delta = f_{up}\big(H_\nu \sqcup K_\gamma\big) = f_{up}(H_\nu) \sqcup f_{up}(K_\gamma)$. Since, f_{up} is a GFS open mapping, then $f_{up}(M_\psi)$ and $f_{up}(N_\eta)$ are non- null GFS open sets in (Y, K) such that $f_{up}(H_\nu) \sqsubseteq f_{up}(M_\psi)$, $f_{up}(K_\gamma) \sqsubseteq f_{up}(N_\eta)$. Since, f_{up} is a GFS injective mapping, then $f_{up}(H_\nu) \sqcap f_{up}(N_\eta) = f_{up}(H_\nu \sqcap N_\eta) = \tilde{0}_{\theta_Y}$ and $f_{up}\big(K_\gamma\big) \sqcap f_{up}(M_\psi) = \tilde{0}_{\theta_Y}$. It follows that G_δ is not a $GFS\ s$ −connected set, a contradiction.

Theorem 4.10. If F_μ and G_δ are intersecting $GFSC_1$ −(respectively, $GFSC_2$ −connected, $GFS\ s$ −connected, GFS weakly−connected, $GFS\ Q$ −connected, GFS strongly−connected) sets in (X, E). Then, $F_\mu \sqcup G_\delta$ is a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, $GFS\ s$ −connected, GFS weakly−connected, $GFS\ Q$ −connected, GFS strongly−connected) set in (X, E).

Proof. The cases of $GFSC_1$ −connected and $GFSC_2$ −connected sets is previously proved (see Theorem 4.9 in [11]). Now, we will prove the case of $GFS\ Q$ −connected sets. Let F_μ and G_δ are intersecting $GFS\ Q$ −connected sets in (X,E). Suppose$F_\mu \sqcup G_\delta$ is not a $GFS\ Q$ −connected set. Then, there exist two non- null $GFS\ Q$ −separated sets H_ν and K_γ in (X,E) such that $F_\mu \sqcup G_\delta = H_\nu \sqcup K_\gamma$. Therefore, $F_\mu \sqcap H_\nu$, $F_\mu \sqcap K_\gamma$, $G_\delta \sqcap H_\nu$ and $G_\delta \sqcap K_\gamma$ are non- null $GFS\ Q$ −separated sets in (X,E) as subsets of H_ν and K_γ. Since, $F_\mu = (F_\mu \sqcap H_\nu) \sqcup (F_\mu \sqcap K_\gamma)$ and $G_\delta = (F_\mu \sqcap H_\nu) \sqcup (F_\mu \sqcap K_\gamma)$, then F_μ and G_δ are not $GFS\ Q$ −connected which is a contradiction.

Theorem 4.11. Let $\{(F_\mu)_i : i \in J\}$ be a family of a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, $GFS\ s$ −connected, GFS weakly−connected, $GFS\ Q$ −connected, GFS strongly−connected) sets in (X,E) such that for $i,j \in J$, the $GFSSs$ $(F_\mu)_i$ and $(F_\mu)_j$ are intersecting. Then, $F_\mu = \sqcup_{i\in J}(F_\mu)_i$ is a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, $GFS\ s$ −connected, GFS weakly−connected, $GFS\ Q$ −connected, GFS strongly−connected) set in (X,E).

Proof. The case of $GFSC_1$-connected set previously proved (see Theorem 4.11 in [11]). Now, we will prove the case of $GFSC_2$ −connected set. Let $\{(F_\mu)_i : i \in J\}$ be family of $GFSC_2$-connected sets in (X,E). Suppose that F_μ is not a $GFSC_2$-connected set in (X,E). Then, there exist two GFS open sets H_ν and K_γ in (X,E) such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $F_\mu \sqcap H_\nu \sqcap K_\gamma = \tilde{0}_\theta$, $F_\mu \sqcap H_\nu \neq \tilde{0}_\theta$ and $F_\mu \sqcap K_\gamma \neq \tilde{0}_\theta$.

Now, let $(F_\mu)_{i_0}$ be any $GFSS$ of the given family. Then, $(F_\mu)_{i_0} \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq (F_\mu)_{i_0}^c$. But, $(F_\mu)_{i_0}$ is a $GFSC_2$-connected set. Hence, $(F_\mu)_{i_0} \sqcap H_\nu = \tilde{0}_\theta$ or $(F_\mu)_{i_0} \sqcap K_\gamma = \tilde{0}_\theta$. Now if $(F_\mu)_{i_0} \sqcap H_\nu = \tilde{0}_\theta$, we can prove that $(F_\mu)_i \sqcap H_\nu = \tilde{0}_\theta$ for each $i \in J - \{i_0\}$ and so $F_\mu \sqcap H_\nu = \tilde{0}_\theta$. This complete the proof.

Corollary 4.1. If $\{(F_\mu)_i : i \in J\}$ is a family of a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, $GFS\ s$ −connected, GFS weakly−connected, $GFS\ Q$ −connected, GFS strongly−connected) sets in X and $\sqcap_{i\in J}(F_\mu)_i \neq \tilde{0}_\theta$, then $F_\mu = \sqcup_{i\in J}(F_\mu)_i$ is a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, $GFS\ s$ −connected, GFS weakly−connected, $GFS\ Q$ −connected, GFS strongly−connected) set in (X,E).

The following examples show that Theorem 4.10 fails for $GFSC_3$ −connected (respectively, $GFSC_4$ −connected) spaces.

Example 4.11. Let $X = \{x_1, x_2\}$, $E = \{e_1\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{4/5}, \frac{x_2}{2/5}\right\}, 4/5\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{2/5}, \frac{x_2}{4/5}\right\}, 2/5\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{2/5}, \frac{x_2}{2/5}\right\}, 2/5\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{4/5}, \frac{x_2}{4/5}\right\}, 4/5\right)\right\}\right\}$ be a GFS topology over (X,E). Let $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{1/5}, \frac{x_2}{2/5}\right\}, 1/5\right)\right\}$ and $G_\delta = \left\{\left(e_1 = \left\{\frac{x_1}{2/5}, \frac{x_2}{1/5}\right\}, 2/5\right)\right\}$. Hence, $F_\mu \sqcap G_\delta \neq \tilde{0}_\theta$ and F_μ and G_δ are $GFSC_3$ −connected sets in (X,E), but $F_\mu \sqcup G_\delta$ is not $GFSC_3$ −connected set in (X,E).

Example 4.12. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and

$T = \left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{3/5}, \frac{x_2}{2/5}\right\}, 2/5\right)\right\}, \left\{\left(e_2 = \left\{\frac{x_1}{2/5}, \frac{x_2}{3/5}\right\}, 3/5\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{3/5}, \frac{x_2}{2/5}\right\}, 2/5\right), \left(e_2 = \left\{\frac{x_1}{2/5}, \frac{x_2}{3/5}\right\}, 3/5\right)\right\}\right\}$ be a GFS topology over (X,E). Let $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{3/5}\right\}, 2/5\right), \left(e_2 = \left\{\frac{x_1}{2/5}\right\}, 2/5\right)\right\}$ and $G_\delta = \left\{\left(e_1 = \left\{\frac{x_1}{1/5}, \frac{x_2}{2/5}\right\}, 1/5\right), \left(e_2 = \left\{\frac{x_2}{3/5}\right\}, 2/5\right)\right\}$. Hence, $F_\mu \sqcap G_\delta \neq \tilde{0}_\theta$ and F_μ and G_δ are $GFSC_4$ −connected sets in (X,E), but $F_\mu \sqcup G_\delta$ is not $GFSC_4$ −connected set in (X,E).

Theorem 4.12. If F_μ and G_δ are GFS quasi-coincident $GFSC_3$ −connected (respectively, $GFSC_4$ −connected) sets in (X,E) , then $F_\mu \sqcup G_\delta$ is a $GFSC_3$ −connected (respectively, $GFSC_4$ −connected) set in (X,E).

Proof. As a sample, we will prove the case $GFSC_3$ −connected. Let F_μ and G_δ be GFS quasi-coincident $GFSC_3$ −connected sets in (X,E). Suppose there exist two non-null GFS open sets H_ν and K_γ in (X,E) such that

$F_\mu \sqcup G_\delta \sqsubseteq H_\nu \sqcup K_\gamma$ and $H_\nu \sqcap K_\gamma \sqsubseteq (F_\mu \sqcup G_\delta)^c$. (1) [we prove that $H_\nu \sqsubseteq (F_\mu \sqcup G_\delta)^c$ or $K_\gamma \sqsubseteq (F_\mu \sqcup G_\delta)^c$]

Therefore, $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$, $G_\delta \sqsubseteq H_\nu \sqcup K_\gamma$ and $H_\nu \sqcap K_\gamma \sqsubseteq G_\delta^c$. Since, F_μ and G_δ are $GFSC_3$ −connected, then ($H_\nu \sqsubseteq F_\mu^c$ or $K_\gamma \sqsubseteq F_\mu^c$) and ($H_\nu \sqsubseteq G_\delta^c$ or $K_\gamma \sqsubseteq G_\delta^c$).

Moreover, since F_μ and G_δ are GFS quasi-coincident, there exist $x \in X, e \in E$ such that

$F(e)(x) > 1 - G(e)(x)$ and $\mu(e) > 1 - \delta(e)$. (2)

Now, consider the following cases:

case 1. Suppose $H_\nu \sqsubseteq F_\mu^c$. Then, by (2) we have, $1 - H(e)(x) \geq F(e)(x) > 1 - G(e)(x)$ and $1 - \nu(e) \geq \mu(e) > 1 - \delta(e) \Longrightarrow H(e)(x) < G(e)(x)$ and $\nu(e) < \delta(e)$. (3)

We claim that, $K_\gamma \not\sqsubseteq G_\delta^c$. For if not, then

$K(e)(x) \leq 1 - G(e)(x) < F(e)(x)$ and $\gamma(e) \leq 1 - \delta(e) < \mu(e)$. (4)

Now by (3) and (4), we have $H(e)(x) \vee K(e)(x) < F(e)(x) \vee G(e)(x)$ and $\nu(e) \vee \gamma(e) < \mu(e) \vee \delta(e)$ which implies $F_\mu \sqcup G_\delta \not\sqsubseteq H_\nu \sqcup K_\gamma$, this contradicts (1). Hence, $H_\nu \sqsubseteq G_\delta^c$. Therefore, $H_\nu \sqsubseteq F_\mu^c \sqcap G_\delta^c = (F_\mu \sqcup G_\delta)^c$.

case 2. Suppose $K_\gamma \sqsubseteq F_\mu^c$. Here, we can show as in Case 1 that $H_\nu \not\sqsubseteq G_\delta^c$. Therefore, $K_\gamma \sqsubseteq G_\delta^c$. Hence, $K_\gamma \sqsubseteq G_\delta^c$. Therefore, $K_\gamma \sqsubseteq F_\mu^c \sqcap G_\delta^c = (F_\mu \sqcup G_\delta)^c$. This complete the proof.

Theorem 4.13. Let $\{(F_\mu)_i : i \in J\}$ be a family of $GFSC_3$ −connected (respectively, $GFSC_4$ −connected,) sets in (X, E) such that for $i, j \in J$, the $GFSSs$ $(F_\mu)_i$ and $(F_\mu)_j$ are GFS quasi-coincident. Then, $F_\mu = \sqcup_{i\in J}(F_\mu)_i$ is a $GFSC_3$ −connected (respectively, $GFSC_4$ −connected) set in (X, E).

Proof. Let $\{(F_\mu)_i : i \in J\}$ be family of $GFSC_3$-connected sets in (X, E). Suppose there exist two GFS open sets H_ν and K_γ in (X, E) such that $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$ and $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$. Let $(F_\mu)_{i_0}$ be any $GFSS$ of the given family. Then, $(F_\mu)_{i_0} \sqsubseteq H_\nu \sqcup K_\gamma$, $H_\nu \sqcap K_\gamma \sqsubseteq (F_\mu)_{i_0}^c$. Since, $(F_\mu)_{i_0}$ is a $GFSC_3$-connected set, we have $H_\nu \sqsubseteq (F_\mu)_{i_0}^c$ or $K_\gamma \sqsubseteq (F_\mu)_{i_0}^c$. Now, the result follows in view of the facts that $(F_\mu)_{i_0} \sqsubseteq H_\nu^c$, then $(F_\mu)_i \sqsubseteq H_\nu^c$ for each $i \in J - \{i_0\}$, since $(F_\mu)_{i_0}$ and $(F_\mu)_i$ are GFS quasi-coincident $GFSC_3$ −connected sets, and $H_\nu \sqsubseteq [\sqcap_{i\in J}(F_\mu)_i]^c = F_\mu^c$. Hence, F_μ is a $GFSC_3$-connected. Similarly, if $\{(F_\mu)_i : i \in J\}$ is family of $GFSC_4$-connected sets in (X, E) such that for $i, j \in J$, the $GFSSs$ $(F_\mu)_i$ and $(F_\mu)_j$ are GFS quasi-coincident, then, $F_\mu = \sqcup_{i\in J}(F_\mu)_i$ is a $GFSC_4$ −connected set in (X, E). This complete the proof.

Corollary 4.2. Let $\{(F_\mu)_i : i \in J\}$ be a family of a $GFSC_3$ −connected (respectively, $GFSC_4$ −connected,) sets in (X, E) and (x_α, e_λ) be a GFS point such that $\alpha > \frac{1}{2}$, $\lambda > \frac{1}{2}$ and $(x_\alpha, e_\lambda) \in \sqcap_{i\in J}(F_\mu)_i$. Then $\sqcup_{i\in J}(F_\mu)_i$ is a $GFSC_3$ −connected (respectively, $GFSC_4$ −connected) set in (X, E).

Proof. Since $(x_\alpha, e_\lambda) \in \sqcap_{i\in J}(F_\mu)_i$, then $(x_\alpha, e_\lambda) \in (F_\mu)_i$ for each $i \in J$. Therefore, $(F_\mu)_i$ and $(F_\mu)_j$ are GFS quasi-coincident for each $i, j \in J$. By Theorem 4.13, $\sqcup_{i\in J}(F_\mu)_i$ is a $GFSC_3$ −connected (respectively, $GFSC_4$ −connected) set in (X, E).

Theorem 4.14. If F_μ is a $GFSC_3$ −connected (respectively, $GFSC_4$ −connected, GFS strongly−connected, $GFS\ Q$ −connected) set in (X, E) and $F_\mu \sqsubseteq G_\delta \sqsubseteq cl(F_\mu)$, then G_δ is also a $GFSC_3$ −connected (respectively, $GFSC_4$ −connected, GFS strongly−connected, $GFS\ Q$ −connected) set in (X, E). In particular $cl(F_\mu)$ is $GFSC_3$ −connected (respectively, $GFSC_4$ −connected, GFS strongly−connected, $GFS\ Q$ −connected) set in (X, E).

Proof. As a sample, we will prove the case $GFSC_3$ −connected. Let H_ν and K_γ be GFS open sets in (X,E) such that $G_\delta \sqsubseteq H_\nu \sqcup K_\gamma$ and $H_\nu \sqcap K_\gamma \sqsubseteq G_\delta^c$. Then, $F_\mu \sqsubseteq H_\nu \sqcup K_\gamma$ and $H_\nu \sqcap K_\gamma \sqsubseteq F_\mu^c$. Since F_μ is a $GFSC_3$ −connected set, we have $F_\mu \sqsubseteq H_\nu^c$ or $F_\mu \sqsubseteq K_\gamma^c$. But, if $F_\mu \sqsubseteq H_\nu^c$, then $cl(F_\mu) \sqsubseteq H_\nu^c$ and on the other hand, if $F_\mu \sqsubseteq K_\gamma^c$, then $cl(F_\mu) \sqsubseteq K_\gamma^c$. Therefore, $G_\delta \sqsubseteq cl(F_\mu) \sqsubseteq H_\nu^c$ or $G_\delta \sqsubseteq cl(F_\mu) \sqsubseteq K_\gamma^c$. Hence, G_δ is a $GFSC_3$ −connected set in (X,E).

However, the above theorem fails in case of $GFSC_1$ −connectedness (respectively, $GFSC_2$ −connectedness, GFS clopen−connectedness, GFS weakly−connectedness, $GFS\ s$ −connectedness) which is a departure from general topology. The following example will illustrate that the closure of a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, GFS clopen−connected, GFS weakly−connected, $GFS\ s$ −connected) set need not be a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, GFS clopen−connected, GFS weakly−connected, $GFS\ s$ −connected).

Example 4.13. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and

$T = \left\{\left\{\tilde{0}_\theta, \tilde{1}_\Delta, \left\{\left(e_1 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right)\right\}, \left\{\left(e_2 = \left\{\frac{x_1}{2/3}, \frac{x_2}{2/3}\right\}, 2/3\right)\right\}, \left\{\left(e_1 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right), \left(e_2 = \left\{\frac{x_1}{2/3}, \frac{x_2}{2/3}\right\}, 2/3\right)\right\}\right\}\right\}$ be a GFS topology over (X,E).

Here, $F_\mu = \left\{\left(e_1 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right)\right\}$ is a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, GFS clopen−connected, GFS weakly−connected, $GFS\ s$ −connected) set, but $cl(F_\mu) = \left\{\left(e_1 = \left\{\frac{x_1}{1}, \frac{x_2}{1}\right\}, 1\right), \left(e_2 = \left\{\frac{x_1}{1/3}, \frac{x_2}{1/3}\right\}, 1/3\right)\right\}$ is not a $GFSC_1$ −connected (respectively, $GFSC_2$ −connected, GFS clopen−connected, GFS weakly−connected, $GFS\ s$ −connected).

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