Stability of Fibonacci Functional Equation

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Abstract. In this paper, we solve the Fibonacci functional equation, f(x) = f(x-1) + f(x-2) and discuss its generalized Hyers-Ulam-Rassias stability in Banach spaces and stability in Fuzzy normed space.

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Introduction.

A question in the theory of functional equations is the following "When is it true that a function which approximately satisfies a functional equation $\in$ must be close to an exact solution $\in$?" If the problem accepts a solution, we say that the equation $\in$ is stable.

In 1940, S.M. Ulam [8] gave a wide-ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphism:

Let $(G_1, *)$ be a group and $(G_2, \circ, d)$ be a metric group with the metric d. Given $\in >0$, does there exists a $\delta_\in >0$ such that if a mapping h: $G_1 \rightarrow G_2$ satisfies the inequality $d(h(x*y),h(x)\circ h(y))< \delta_\in$ $\forall$ x, y$\in G_1$, then there is a mapping H: $G_1 \rightarrow G_2$ such that for each x, y $\in G_1$ $H(x*y)=H(x) \circ H(y)$ and $d(h(x),H(x))< \in$?

In the next year, D. H. Hyers [3], gave answer to the above question for additive groups under the assumption that groups are Banach spaces. In 1978, T. M. Rassias [7] proved a generalization of Hyers' theorem for additive mapping as a special case in the form of following result.

Suppose that E and F are real normed spaces with F a complete normed space, f: E$\rightarrow$F is a mapping such that for each fixed x$\in$E the mapping t$\rightarrow$f (tx) is continuous on R, and let there exist $\in \geq 0$ and p$\in$[0,1) s.t

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p) \text{ x, y}\in\text{E.}$$

Then there exists a unique linear mapping T: E$\rightarrow$F s.t $\|f(x)-T(x)\| \leq \varepsilon \frac{\|x\|^p}{(1-2^{p-1})}$, x$\in$E.

In this paper we discuss the stability of Fibonacci functional equation

f(x) = f(x-1) + f(x-2). (1)

A function f: R→X will be called a Fibonacci functional equation if it satisfies (1), for all xϵ R, where X is a real vector space. By α and β we denote the positive and negative roots respectively of the quadratic equation x^2-x-1=0. i.e., $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$ for any xϵR. M. M. Parizi and M. E. Gordji [11] proved the stability of Fibonacci functional equation in Modular functional spaces. S. M. Jung [10] also proved the stability of Fibonacci functional equation in real Banach space as following:

Theorem1: Let (X,||.||) be a real Banach space. If a function f: R→X satisfies the inequality, || f(x)-f(x-1)-f(x-2)||≤ϵ (1.1)

for all xϵR and for some ϵ>0, Then there exists a Fibonacci function F: R→X such that $||f(x)-F(x)||\leq\left(1+\frac{2}{\sqrt{5}}\right)\varepsilon$ (1.2)

for all xϵR.

Proof. We get from (1.1),

||f(x)-αf(x-1)-β[f(x-1)-αf(x-2)]||≤ϵ, (1.3)

For each xϵR. If we replace x by x-k in (1.3), then we have,

||f(x-k)-αf(x-k-1)-β[f(x-k-1)-αf(x-k-2)]||≤ϵ

And

$||\beta^k$ [f(x-k)-αf(x-k-1)]-β^{k+1} [f(x-k-1)-αf(x-k-2)]||≤ β^k ϵ (1.4)

Thus, we have,

$\|f(x)-\alpha f(x--1)-\beta^n\ [f(x-n)-\alpha f(x-n-1)]\| \le \sum_{k=0}^{n-1} \left\| \beta^k[f(x-k)-\alpha f(x-k-1)]-\beta^{k+1}[f(x-k-1)-\alpha f(x-k-2)] \right\| \le \sum_{k=0}^{n-1} |\beta|^k \epsilon$ (1.5)

From (1.4), we get $\{\beta^k[f(x-n)-\alpha f(x-n-1)\}$ is a Cauchy sequence. Therefore, we can define a function F_1: R→X by

$F_1=\lim_{n\to\infty} \beta^k [f(x-n)-\alpha f(x-n-1)]$, since X is complete so F_1 is in X. We obtain that

$$F_1(x-1)+F_1(x-2) = \beta^{-1} \lim_{n\to\infty} \beta^{n+1}[f(x-n-1)-\alpha f(x-n)]$$

$$+\beta^{-2} \lim_{n\to\infty} \beta^{n+2}[f(x-(n+2))-\alpha f(x-(n+2)-1)]$$

$$=\beta^{-1}F_1(x)+\beta^{-2}F_1(x)=F_1(x),$$

For all xϵR. Hence F_1 is a Fibonacci function. If n goes to infinity, then (1.5) implies

$\|f(x)-\alpha f(x-1)-F_1(x)\| \le \frac{3+\sqrt{5}}{2}\epsilon$ (1.6)

For every xϵR.

From (1.1)

$\|f(x)-\beta f(x-1)-\alpha[f(x-1)-\beta f(x-2)]\| \le \epsilon,$ (1.7)

For each xϵR. If we replace x by x+k in (1.7), then we have,

$\|f(x+k)-\beta f(x+k-1)-\alpha[f(x+k-1)-\beta f(x+k-2)]\| \le \epsilon$

And

$\|\alpha^{-k}\ [f(x+k)-\beta f(x+k-1)]-\alpha^{-k+1}\ [f(x+k-1)-\beta f(x+k-2)]\| \le \alpha^{-k}\ \epsilon$ (1.8)

Thus, we have,

$\|\alpha^{-n}[f(x+n)-\beta f(x+n-1)-[f(x)-\beta f(x-1)]\| \le \sum_{k=0}^{n} \left\|\alpha^{-k}[f(x+k)-\beta f(x+k-1)]-\alpha^{-k+1}[f(x+k-1)-\beta f(x+k-2)]\right\| \le \sum_{k=0}^{n} \alpha^{-k}\epsilon$ (1.9)

From (1.8), we get $\{\alpha^{-n}[f(x+n)-\beta f(x+n-1)\}$ is a Cauchy sequence. Therefore, we can define a function F_2: R→X by

$F_2=\lim_{n\to\infty} \alpha^{-n}[f(x+n)-\beta f(x+n-1)]$, since X is complete so F_2 is in X. We obtain that

$$F_2(x-1)+F_2(x-2)= \alpha^{-1} \lim_{n\to\infty} \alpha^{-(n-1)}[f(x+n-1)-\beta f(x+n-1)]$$

$$+\alpha^{-2} \lim_{n\to\infty} \alpha^{-n+2}[f(x+n-2))-\beta f(x+(n-2)-1)]$$

$$=\alpha^{-1}F_2(x)+\alpha^{-2}F_2(x)=F_2(x),$$

For all xϵR. Hence F_2 is a Fibonacci function. If n goes to infinity, then (1.9) implies

$\|F_2(x)-f(x)+\beta f(x-1)\| \le \frac{\sqrt{5}+1}{2}\epsilon$ (1.10)

For every xϵR.

From (1.6) and (1.10), we have

$||f(x)-\left[\frac{\beta}{\beta-\alpha}F_1(x)-\frac{\alpha}{\beta-\alpha}F_2\right]||=\frac{1}{|\beta-\alpha|}||(\beta-\alpha)f(x)-[\beta F_1(x)-\alpha F_2(x)]||$

$\leq\frac{1}{\alpha-\beta}||\beta f(x)-\alpha\beta f(x-1)-\beta F_1(x)||+\frac{1}{\alpha-\beta}||\alpha F_2(x)-\alpha f(x)+\alpha\beta f(x-1)||$

$\leq\left(1+\frac{2}{\sqrt{5}}\right)\epsilon$

For all xϵR. Now we set

$F(x)=\frac{\beta}{\beta-\alpha}F_1(x)-\frac{\alpha}{\beta-\alpha}F_2$

Clearly F(x) is the Fib0nacci function.

Now we prove the stability of Fibonacci functional equation in fuzzy normed space.

Theorem 2: Let (X, N) and (Y, N′) be fuzzy normed spaces. If f: R→X satisfies the inequality

$N(f(x)-f(x-1)-f(x-2), t) \geq N'(\phi(x), t)$ (2.1)

for all xϵR, then there exists a Fibonacci function F: R→X such that

$N(f(x)-F(x)) \geq N'(\phi(x), \left(1+\frac{2}{\sqrt{5}}\right)t)$.

Proof. We get from (2.1),

$N(f(x)-\alpha f(x-1)-\beta[f(x-1)-\alpha f(x-2)], t) \geq N'(\phi(x), t)$ (2.2)

For each xϵR. If we replace x by x-k in (2.2), then we have,

$N(f(x-k)-\alpha f(x-k-1)-\beta[f(x-k-1)-\alpha f(x-k-2)], t) \geq N'(\phi(x-k), t)$

And

$N(\beta^k[f(x-k)-\alpha f(x-k-1)]-\beta^{k+1}[f(x-k-1)-\alpha f(x-k-2)], \beta^k t) \geq N'(\phi(x-k), t)$ (2.3)

Thus, we have,

$N(f(x)-\alpha f(x-1)-\beta^n[f(x-n)-\alpha f(x-n-1)],\sum_{k=0}^{n-1}|\beta|^k t)\geq\min\{N(\beta^k f(x-k)-\alpha f(x-k-1)-\beta^{k+1}[f(x-k-1)-\alpha f(x-k-2),\beta^k t),k=o,1,.....n-1\}\geq N'(\phi(x),\sum_{k=0}^{n-1}|\beta|^k t)$ (2.4)

From (2.4), we get $\{\beta^n[f(x-n)-\alpha f(x-n-1)\}$ is a Cauchy sequence. Therefore, we can define a function F_1: R→X by

$F_1=\lim_{n\to\infty}\beta^k[f(x-n)-\alpha f(x-n-1)]$, since X is complete so F_1 is in X. We obtain that

$F_1(x-1)+F_1(x-2)=\beta^{-1}\lim_{n\to\infty}\beta^{n+1}[f(x-n-1)-\alpha f(x-n)]$

$+\beta^{-2}\lim_{n\to\infty}\beta^{n+2}[f(x-(n+2))-\alpha f(x-(n+2)-1)]$

$=\beta^{-1}F_1(x)+\beta^{-2}F_1(x)=F_1(x)$,

For all xϵR. Hence F_1 is a Fibonacci function. If n goes to infinity, then (2.4) implies

$N(f(x)-\alpha f(x-1)-F_1(x),t) \geq N'(\phi(x), \left(\frac{3+\sqrt{5}}{2}\right)t)$ (2.5)

For every $x \epsilon R$.

From (2.1)

$N(f(x)-\beta f(x-1)-\alpha[f(x-1)-\beta f(x-2)],t) \geq N'(\phi(x), t),$ (2.6)

For each $x \epsilon R$. If we replace x by x+k in (2.6), then we have,

$N(f(x+k)-\beta f(x+k-1)-\alpha[f(x+k-1)-\beta f(x+k-2)],t) \geq N'(\phi(x+k), t)$

And

$N(\alpha^{-k} [f(x+k)-\beta f(x+k-1)]-\alpha^{-k+1} [f(x+k-1)-\beta f(x+k-2)], \alpha^{-k} t) \geq N'(\phi(x+k), t)$ (2.7)

Thus, we have,

$N(\alpha^{-n}[f(x+n)-\beta f(x+n-1)-[f(x)-\beta f(x-1)],\sum_{k=0}^{n-1} \alpha^{-k}t)])$

$\geq \min\{N(\alpha^{-k}f(x+k)-\beta f(x+k-1)-\alpha^{-k+1}[f(x+k-1)-\beta f(x+k-2),\alpha^{-k}t) ,k=o,1,\ldots..n-1\}$

$\geq N'(\phi(x), \sum_{k=0}^{n-1} \alpha|^{-k}t)$ (2.8)

From (2.7), we get $\{\alpha^{-n}[f(x+n)-\beta f(x+n-1)\}$ is a Cauchy sequence. Therefore, we can define a function F_2:R→X by

$F_2=\lim_{n\to\infty} \alpha^{-n}[f(x+n)-\beta f(x+n-1)]$, since X is complete so F_2 is in X. We obtain that

$$F_2(x-1)+F_2(x-2)= \alpha^{-1} \lim_{n\to\infty} \alpha^{-(n-1)}[f(x+n-1)-\beta f(x+n-1)]$$

$$+\alpha^{-2} \lim_{n\to\infty} \alpha^{-n+2}[f(x+n-2))-\beta f(x+(n-2)-1)]$$

$$=\alpha^{-1}F_2(x)+ \alpha^{-2}F_2(x)=F_2(x),$$

For all $x \epsilon R$. Hence F_2 is a Fibonacci function. If n goes to infinity, then (2.8) implies

$N(F_2(x)-f(x)+\beta f(x-1),t)\geq N'(\phi(x), \left(\frac{\sqrt{5}+1}{2}\right)t)$ (2.9)

For every $x \epsilon R$.

From (2.5) and (2.9), we have

$$N(f(x)-\left[\frac{\beta}{\beta-\alpha}F_1(x)-\frac{\alpha}{\beta-\alpha}F_2\right],t)=N((\beta-\alpha)f(x)-[\beta F_1(x)-\alpha F_2(x)], |\beta-\alpha|t)$$

$$\geq\min\{N(\beta f(x)-\alpha\beta f(x-1)-\beta F_1(x), \tfrac{\alpha-\beta}{2}t), N(\alpha F_2(x)-\alpha f(x)+\alpha\beta f(x-1), \tfrac{\alpha-\beta}{2}t)$$

$$\geq N'(\phi(x), \left(1+\frac{2}{\sqrt{5}}\right)t)$$

For all $x \epsilon R$. Now we set

$$F(x)=\frac{\beta}{\beta-\alpha}F_1(x)-\frac{\alpha}{\beta-\alpha}F_2$$

Clearly F(x) is the Fibonacci function.

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