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# Oscillatory Behavior of Second Order Neutral Difference Equations with Mixed Neutral Term

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## ABSTRACT

In this paper, we study the oscillatory behavior of solution of second order neutral difference equation with mixed neutral term of the form

$$\Delta(a_n(\Delta z_n)) + q_n x_{\sigma(n)} = 0, \qquad n \in N_0,$$

where $z_n = x_n + b_n x_{n-l} + c_n x_{n+k}$ and $\sum_{s=n_0}^{\infty} \frac{1}{a_s} = \infty$. We obtain some new oscillation criteria for second order neutral difference equation. Examples are presented to illustrate the main results.

**2010 Mathematics Subject Classification:** 39A10.

**Key Words:** Oscillation; second order; neutral difference equation; mixed neutral term.

## INTRODUCTION

This paper is concerned with the oscillatory behavior of solution of second order neutral difference equation with mixed neutral term of the form

$$\Delta(a_n(\Delta z_n)) + q_n x_{\sigma(n)} = 0, \qquad n \in N_0, \tag{1}$$

where $z_n = x_n + b_n x_{n-l} + c_n x_{n+k}$.

Subject to the following conditions:

($H_1$) $\{b_n\}$,$\{c_n\}$ and $\{q_n\}$ are non-negative real sequences with $0 \le b_n < b < \infty$, $0 \le c_n \le c < \infty$ and $q_n > 0$;

($H_2$) $\{a_n\}$ is real sequence with $\sum_{s=n_0}^{\infty} \frac{1}{a_s} = \infty$;

($H_3$) $l, k$ are nonnegative constant, $\sigma(n)$ is a sequence of positive integers with $\Delta\sigma(n) > 0$ and $\Delta$ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

Let $\theta = \max\{l,k\}$. By a solution of equation (1), we mean a real sequence $\{x_n\}$ which is defined for $n \ge n_0 - \theta$ satisfies equation (1) for all $n$. A non trivial solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. The oscillatory behavior of nonlinear neutral delay difference equation of second order have been investigated by several authors, see for example [6, 9, 11, 12] and the references quoted therein. Following this trend, in this paper, we obtain some new oscillation criteria for equation (1) which extend some known results. Some examples are provided to illustrate the main results.

## 2 Main Results

We begin with the following theorem.

**Theorem 2.1** If

$$\sum_{s=n_0}^{\infty} Q_s = \infty \tag{2}$$

where $Q_n = \min\{q_n, q_{n-l}, q_{n+k}\}$, then every solution of equation (1) is oscillatory.

**Proof.** Let $\{x_n\}$ be a nonoscillatory solution of equation (1) for all $n \ge n_0$. Then there exists $n_1 \ge n_0$ such that $\{x_n\} \ne 0$ for all $n \ge n_1$. With out loss of generality, we may assume that $x_n > 0$, $x_{n-l} > 0$, $x_{n+k} > 0$ and $x_{\sigma(n)} > 0$ for all $n \ge n_1$.

From equation (1), we have

$$\Delta(a_n(\Delta z_n)) = -q_n x_{\sigma(n)} < 0 \quad for\ all \quad n \geq n_1. \tag{3}$$

Therefore $a_n(\Delta z_n)$ is nonincreasing and hence $(\Delta z_n)$ is nonincreasing. We shall show that $\Delta z_n > 0$ for all $n \geq n_1$.

If not, there exists $n_2 \geq n_1$ such that $\Delta z_{n_2} < 0$. Then

$$a_n(\Delta z_n) \leq a_{n_2}(\Delta z_{n_2}), \quad n \geq n_2. \tag{4}$$

Summing the above inequality from $n_2$ to $n-1$, we obtain

$$z_n - z_{n_2} \leq a_{n_2}(\Delta z_{n_2}) \sum_{s=n_2}^{n-1} \frac{1}{a_s}$$

$$z_n \leq z_{n_2} + a_{n_2}(\Delta z_{n_2}) \sum_{s=n_2}^{n-1} \frac{1}{a_s}. \tag{5}$$

Letting $n \to \infty$ in (5), we obtain $z_n \to -\infty$, which is a contradiction. We conclude that $\Delta z_n > 0.$ From equation (1) and definition of $z_n$, we have

$$\Delta(a_n(\Delta z_n)) + q_n x_{\sigma(n)} + b\Delta(a_{n-l}(\Delta z_{n-l})) + bq_{n-l} x_{\sigma(n-l)}$$
$$+c\Delta(a_{n+k}(\Delta z_{n+k})) + cq_{n+k} x_{\sigma(n+k)} = 0. \tag{6}$$

Thus,

$$\Delta(a_n(\Delta z_n)) + b\Delta(a_{n-l}(\Delta z_{n-l})) + c\Delta(a_{n+k}(\Delta z_{n+k})) + Q_n z_{\sigma(n)} \leq 0, \quad n \geq n_1. \tag{7}$$

Summing the above inequality from $n_1$ to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} Q_s z_{\sigma(s)} + a_n(\Delta z_n) - a_{n_1}(\Delta z_{n_1}) + b(a_{n-l}(\Delta z_{n-l}))$$
$$-b(a_{n_1-l}(\Delta z_{n_1-l})) + c(a_{n+k}(\Delta z_{n+k})) - c(a_{n_1+k}(\Delta z_{n_1+k})) \leq 0$$
$$\sum_{s=n_1}^{n-1} Q_s z_{\sigma(s)} \leq a_{n_1}(\Delta z_{n_1}) - a_n(\Delta z_n) + b(a_{n_1-l}(\Delta z_{n_1-l}) - a_{n-l}(\Delta z_{n-l}))$$
$$+c(a_{n_1+k}(\Delta z_{n_1+k})) - a_{n+k}(\Delta z_{n+k})). \tag{8}$$

But $\Delta z_n > 0$ for all $n \geq n_1$. There exists a constant $c > 0$ such that $z_n \geq c > 0$ for all $n \geq n_1$. From inequality (8), we have

$$\sum_{s=n_1}^{\infty} Q_s < \infty, \tag{9}$$

which is contradiction to condition (2). This completes the proof.

**Theorem 2.2** Assume that $\sigma(n) = n - m$ such that $m > l$ and $m$ is positive constant. If

$$\liminf_{n\to\infty} \sum_{s=n-m+l}^{n-1} R_{(s-m)} Q_s > (1+b+c)\left(\frac{m}{m+1}\right)^{m+1}, \tag{10}$$

where $R_n = \sum_{s=n_0}^{n} \frac{1}{a_s}$, then every solution of equation (1) is oscillatory.

**Proof.** Let $\{x_n\}$ be a non-oscillatory solution of equation (1) for all $n \geq n_0$. As in the proof of Theorem 2.1, we obtain inequality (7). Let

$$w_n = a_n(\Delta z_n) + b(a_{n-l}(\Delta z_{n-l})) + c(a_{n+k}(\Delta z_{n+k})), \tag{11}$$

then from inequality (7), we obtain

$$\Delta w_n + Q_n z_{n-m} \leq 0, \quad n \geq n_1.$$

Since $a_n(\Delta z_n)$ is non-increasing, we have

$$a_s \Delta z_s \geq a_n(\Delta z_n), \quad n \geq s.$$

Dividing the last inequality by $a_s$, we have

$$\Delta z_s \geq \frac{a_n(\Delta z_n)}{a_s}. \tag{12}$$

Summing from $n_1$ to $n-1$, we obtain

$$z_n \geq a_n(\Delta z_n)\sum_{s=n_1}^{n-1}\frac{1}{a_s}$$

$$z_n \geq R_n a_n(\Delta z_n), \quad n \geq n_1. \tag{13}$$

From equation (11), we obtain

$$w_n \geq a_{n-l}(\Delta z_{n-l})(1+b+c). \tag{14}$$

Combining (12), (13) and (14), we obtain

$$\Delta w_n + \frac{R_{[n-m]}}{(1+b+c)} w_{n+l-m} \leq 0. \tag{15}$$

By Theorem 6.20.5 in [1] and the condition (15) implies that equation (10) has no positive solution. This contradiction completes the proof.

Next to define the operator T by $T{:}\sum_{s=\sigma(n)}^{n-1}\phi(s)g(s)$ and $T[\Delta g(s)] = -T[g(s+1)\chi(s)]$, $\sigma(n) < n$, where $\{\phi_n\}$, $\{g_n\}$ and $\{\chi_n\}$ are positive real sequences.

**Theorem 2.3** Assume that $\sigma(n) \leq n-l$. There exists a positive real sequence $\{k_n\}$ such that

$$\limsup_{n\to\infty} T\left[k_s\ Q_s - (1+b+c)\frac{\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)^2}{4k_s}k_{s+1}^2 a_{\sigma(s)}\right] > 0, \tag{16}$$

where $Q_n$ is defined as in Theorem 2.1. Then every solution of equation (1) is oscillatory.

**Proof.** Let $\{x_n\}$ be a nonoscillatory solution of equation (1). Then there exists $n_1 \geq n_0$ such that $x_n \neq 0$ for all $n \geq n_0$. Without loss of generality, we assume that $x_n > 0$, $x_{n-l} > 0$, $x_{n+k} > 0$ and $x_{\sigma(n)} > 0$ for all $n \geq n_1$.

Define

$$w_n = k_n \frac{a_n(\Delta z_n)}{z_{\sigma(n)}}, \quad n \geq n_1. \tag{17}$$

Thus $w_n > 0$ for $n \geq n_0$, we have

$$\Delta w_n = \frac{k_n}{z\sigma(n)}\Delta(a_n(\Delta z_n)) + \frac{a_{n+1}(\Delta z_{n+1})}{z\sigma(n+1)}\Delta k_n - \frac{k_n a_{n+1}(\Delta z_{n+1})\Delta z\sigma(n)}{z\sigma(n)z\sigma(n+1)}. \tag{18}$$

From (3) and fact that $\Delta z_n > 0$, we have

$$\frac{\Delta z\sigma(n)}{\Delta z_{n+1}} \geq \frac{a_{n+1}}{a\sigma(n)} \quad for\ all \quad n \geq n_1 \geq n_0. \tag{19}$$

Using (17) and (19) in (18), we obtain

$$\Delta w_n \leq \frac{k_n\Delta(a_n(\Delta z_n))}{z\sigma(n)} + \frac{\Delta k_n}{k_{n+1}}w_{n+1} - \frac{k_n}{k_{n+1}^2}\frac{w_{n+1}^2}{a\sigma(n)}. \tag{20}$$

Next we define

$$u_n = k_n \frac{(a_{n-l})(\Delta z_{n-l})}{z_{\sigma(n)}}, \quad n \geq n_1. \tag{21}$$

Then $u_n > 0$ for all $n \geq n_0$, we have

$$\Delta u_n = k_n \frac{\Delta(a_{n-l}(\Delta z_{n-l}))}{z_{\sigma(n)}} + \Delta k_n \frac{(a_{n+1-l}\Delta(z_{n+1-l}))}{z\sigma(n+1)} - \frac{k_n(a_{n+1-l}(\Delta z_{n+1-l})\Delta z\sigma(n))}{z\sigma(n)z\sigma(n+1)}. \tag{22}$$

From (3) and fact that $\Delta z_n > 0$, noting that $\sigma(n) \le n-l$, we obtain

$$\frac{\Delta z\sigma(n)}{\Delta z_{n+1-l}} \ge \frac{a_{n+1-l}}{a\sigma(n)}. \tag{23}$$

Using (21) and (23) in (22), we obtain

$$\Delta u_n \le k_n \frac{\Delta(a_{n-l}(\Delta z_{n-l}))}{z\sigma(n)} + \Delta k_n \frac{u_{n+1}}{k_{n+1}} - k_n \frac{u_{n+1}^2}{k_{n+1}^2 a\sigma(n)}. \tag{24}$$

Define

$$v_n = \frac{k_n(a_{n+k}(\Delta z_{n+k}))}{z\sigma(n)}, \quad n \ge n_1. \tag{25}$$

Then $v_n > 0$ for all $n \ge n_0$, we have

$$\Delta v_n = \frac{k_n\Delta(a_{n+k}(\Delta z_{n+k}))}{z\sigma(n)} + \frac{\Delta k_n(a_{n+1+k}(\Delta z_{n+1+k}))}{z\sigma(n+1)} - \frac{k_n(a_{n+1+k}(\Delta z_{n+1+k}))\Delta z\sigma(n)}{z\sigma(n)z\sigma(n+1)}. \tag{26}$$

From (3) and fact that $\Delta z_n > 0$, noting that $\sigma(n) \le n-l \le n+k$, we have

$$\frac{\Delta z\sigma(n)}{\Delta z_{n+1+k}} \ge \frac{a_{n+1+k}}{a\sigma(n)}. \tag{27}$$

Using (25) and (27) in (26), we obtain

$$\Delta v_n \le \frac{k_n\Delta(a_{n+k}(\Delta z_{n+k}))}{z\sigma(n)} + \frac{\Delta k_n v_{n+1}}{k_{n+1}} - \frac{k_n v_{n+1}^2}{k_{n+1}^2 a\sigma(n)}. \tag{28}$$

Combining (20), (24) and (28), we obtain

$$\begin{aligned}\Delta w_n + b\Delta u_n + c\Delta v_n &\le \frac{k_n\Delta(a_n\Delta z_n)}{z\sigma(n)} + b\,k_n\frac{\Delta(a_{n-l}(\Delta z_{n-l}))}{z\sigma(n)} \\ &+ ck_n\frac{\Delta(a_{n+k}(\Delta z_{n+k}))}{z\sigma(n)} + \frac{\Delta k_n}{k_{n+1}}w_{n+1} - \frac{k_n w_{n+1}^2}{k_{n+1}^2 a\sigma(n)} \\ &+ b\frac{\Delta k_n}{k_{n+1}}u_{n+1} - b\frac{k_n u_{n+1}^2}{k_{n+1}^2 a\sigma(n)} + c\frac{\Delta k_n}{k_{n+1}}v_{n+1} - c\frac{k_n v_{n+1}^2}{k_{n+1}^2 a\sigma(n)}.\end{aligned} \tag{29}$$

From (7) and (29), we obtain

$$\begin{aligned}\Delta w_n + b\Delta u_n + c\Delta v_n &\le -k_n Q_n + \frac{\Delta k_n}{k_{n+1}}w_{n+1} - \frac{k_n w_{n+1}^2}{k_{n+1}^2 a\sigma(n)} \\ &+ b\frac{\Delta k_n}{k_{n+1}}u_{n+1} - b\frac{k_n u_{n+1}^2}{k_{n+1}^2 a\sigma(n)} + c\frac{\Delta k_n}{k_{n+1}}v_{n+1} - c\frac{k_n v_{n+1}^2}{k_{n+1}^2 a\sigma(n)}.\end{aligned} \tag{30}$$

Apply the operator $T$ on (30), we obtain

$$\begin{aligned}T[\Delta w_s + b\Delta u_s + c\Delta v_s] &\le T[-k_s Q_s + \frac{\Delta k_s}{k_{s+1}}w_{s+1} - \frac{k_s w_{s+1}^2}{k_{s+1}^2 a\sigma(s)} \\ &+ b\frac{\Delta k_s}{k_{s+1}}u_{s+1} - b\frac{k_s u_{s+1}^2}{k_{s+1}^2 a\sigma(s)} + c\frac{\Delta k_s}{k_{s+1}}v_{s+1} - c\frac{k_s v_{s+1}^2}{k_{s+1}^2 a\sigma(s)}]\end{aligned} \tag{31}$$

$$\begin{aligned}T[k_s Q_s] &\le T[\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)w_{s+1} + b\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)u_{s+1} + c\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)v_{s+1} \\ &- \frac{k_s w_{s+1}^2}{k_{s+1}^2 a\sigma(s)} - \frac{k_s u_{s+1}^2}{k_{s+1}^2 a\sigma(s)} - \frac{k_s v_{s+1}^2}{k_{s+1}^2 a\sigma(s)}]\end{aligned}$$

$$\begin{aligned}T[k_s Q_s] &\le T[\frac{\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)^2}{4k_s}k_{s+1}^2 a\sigma(s) + b\frac{\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)^2}{4k_s}k_{s+1}^2 a\sigma(s) \\ &+ c\frac{\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)^2}{4k_s}k_{s+1}^2 a\sigma(s)]\end{aligned}$$

or

$$T\left[k_s Q_s - (1+b+c)\frac{\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)^2}{4k_s} k_{s+1}^2 a\sigma(s)\right] \leq 0,$$

which is contradiction to inequality (16). This completes the proof.

**Corollary 2.1** Assume that $\sigma(n) = n - l$ and there exists a sequence $\{k_n\}$ such that

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[(n-s)^{\alpha}(s-n_0)^{\beta} k_s Q_s - (1+b+c)\frac{\left(\chi_s + \frac{\Delta k_s}{k_{s+1}}\right)^2}{4k_s} k_{s+1}^2 a\sigma(s)\right] > 0,$$

where $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ and $\chi_s = \frac{\beta_{(n)} - (b+c)s + \alpha_{n_0}}{(n-s)(s-n_0)}$. Then every solution of equation (1) is oscillatory.

## 3 Examples

In this section, we provide three examples.

**Example 3.1** Consider the second order neutral difference equation of the form

$$\Delta\big(2\Delta(x_n + 2x_{n-2} + x_{n+3})\big) + 16x_{n-3} = 0, \qquad n \geq 4, \tag{32}$$

where $a_n = 2$, $b_n = 2$, $c_n = 1$, $l = 2$, $k = 3$, $\sigma(n) = n - 3$ and $q_n = 16$. Since all the conditions of Theorem 2.2 are satisfied. Hence every solution of equation (32) is oscillatory. In fact one such solution is $x_n = (-1)^n$.

**Example 3.2** Consider the second order neutral difference equation of the form

$$\Delta\left(\frac{1}{10n+13}\Delta(x_n + x_{n-2} + 3x_{n+2})\right) + \frac{2}{n-3}x_{n-3} = 0, \qquad n \geq 4, \tag{33}$$

where $a_n = \frac{1}{10n+13}$, $b_n = 1$, $c_n = 3$, $l = 2$, $k = 2$, $\sigma(n) = n - 3$ and $q_n = \frac{2}{n-3}$, since all the conditions of Theorem 2.3 are satisfied. Hence every solution of equation (33) is oscillatory. In fact one such solution is $x_n = n(-1)^n$.

**Example 3.3** Consider the second order neutral difference equation of the form

$$\Delta^2(x_n + 2x_{n-2} + 5x_{n+3}) + 8x_{n-2} = 0, \quad n \geq 3, \tag{34}$$

where $a_n = 1$, $b_n = 2$, $c_n = 5$, $l = 2$, $k = 3$, $\sigma(n) = n - 2$ and $q_n = 8$. Since all the conditions of Corollary 2.1 are satisfied. Hence every solution of equation (34) is oscillatory. In fact one such solution is $x_n = (-1)^{n+1}$.

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