ISSN 2347-1921
Volume 13 Number 2
Journal of Advance in Mathematics

Generalized (α,β)-derivations and Left Ideals in Prime and Semiprime Rings

Asma Ali and Md. Hamidur Rahaman
Department of Mathematics
Aligarh Muslim University, Aligarh-202002 (India)

Abstract

Let R be an associative ring, α, β be the automorphisms of R, λ be a nonzero left ideal of R, $F: R \to R$ be a generalized (α,β)-derivation and $d: R \to R$ be an (α,β)-derivation. In the present paper we discuss the following situations: (i) F(xoy)= $a\,\alpha$(xy$\pm$ yx), (ii) $F([x,y]) = a\,\alpha(xy \pm yx)$, (iii) $d(x)od(y) = a\alpha(xy \pm yx)$ for all $x,y \in \lambda$ and $a \in \{0,1,-1\}$. Also some related results have been obtained.

Keywords : Semiprime rings, Generalized (α,β) −derivations, (α,β)-derivations.

Mathematics Subject Classification 2010. 16N60, 16U80, 16W25 .

1 Intrduction

Throughout the paper let R be an associative ring with centre $Z(R)$, α,β be the automorphisms of R, λ be a left ideal of R, F be a generalized (α,β) −derivation and d be an (α,β) −derivation of R. For any pair of element x,y in R, $[x,y]$ denotes the commutator $xy-yx$ and xoy denotes the anti-commutator $xy+yx$. If $S \subseteq R$, then we can define the left (resp. right) annihilator of S as $l(S) = \{x \in R \mid xs = 0 \ for\ all\ s \in S\}$ (resp. $r(S) = \{x \in R \mid sx = 0 \ for\ all\ s \in S\}$). Recall that a ring R is called prime if for any $x,y \in R$, $xRy = 0$ implies that either $x = 0$ or $y = 0$ and is called semiprime if $xRx = 0$ implies that $x = 0$.

An additive mapping $d: R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x,y \in R$ is called a derivation. In [1] Bresar introduced the concept of a generalized derivation. An additive mapping $F: R \to R$ associated with a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$ is called a generalized derivation. So, every derivation is a generalized derivation but the converse is not true in general. Let $a,b \in R$, an additive mapping $F: R \to R$ defined as $F(x) = ax + xb$ for all $x \in R$ is an example of a generalized derivation. An additive mapping $d: R \to R$ is called an (α,β) −derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x,y \in R$. An additive mapping $F: R \to R$ is called a generalized (α,β) −derivation associated with (α,β) −derivation d if $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x,y \in R$. If $d = 0$, then we have $F(xy) = F(x)\alpha(y)$ $for\ all\ x,y \in R$ which is called left α-multiplier mapping of R. Thus generalized (α,β)-derivation generalizes both the concepts, (α,β)-derivation as well as left α-multiplier mapping.

In [2], Daif and Bell proved that if R is a semiprime ring, I be a nonzero ideal of R and $d: R \to R$ is a derivation such that $d([x,y]) = \pm[x,y]$ for all $x,y \in I$, then I is a central ideal. In particular , if $I = R$, then R is commutative. Recently in [4] Dhara studied the above results in semiprime rings. In the present paper, our goal is to study the following identities:
(i) $F(xoy) = a\alpha(xy \pm yx), (ii)\ F([x,y]) = a\alpha(xy \pm yx), (iii)\ d(x)od(y) = a\alpha(xy \pm yx)$ $for\ all\ x,y \in \lambda$, a one sided ideal of semiprime (prime) ring R and $a \in \{1,-1,0\}$.

2 Main Results

We shall do a great deal of calculation with commutators and anti-commutators, routinely using the following basic identities:

$$x\ o\ (yz) = (x\ oy)z - y[x,z] = y(x\ o\ z) + [x,y]z$$
$$(xy)oz = x(y\ o\ z) - [x,z]y = (x\ o\ z)y + x[y,z]$$
$$[x,yz] = (x\ oy)z - y(xoz) = y[x,z] + [x,y]z$$

Lemma 2.1 [5, lemma 3] If the prime ring R contains a commutative nonzero right ideal I, then R is commutative.

Theorem 2.2 Let R be a semiprime ring and λ be a nonzero left ideal of R. If F is a generalized (α,β)-derivation of R associated with an (α,β)-derivation d of R such that $F(x\ o\ y) = a\alpha(xy \pm yx)$ for all $x,y \in \lambda$, where $a \in \{0,1,-1\}$, then $[\beta(\lambda),\beta(\lambda)]d(\lambda) = 0$.

Proof. If $F(\lambda) = 0$, then $F(\lambda^2) = 0 = F(\lambda)\alpha(\lambda) + \beta(\lambda)d(\lambda)$ and hence, $\beta(\lambda)d(\lambda) = 0$, which gives our conclusion. Now assume that $F(\lambda) \neq 0$. Then by our assumption, we have

$$F(x\,o\,y) = a\alpha(xy \pm yx) \quad for\ all\ \ x,y \in \lambda$$
(2.1)

Replacing y by yx in the above equation (2.1), we have

$$F((x\,o\,y)x) = a\alpha((xy \pm yx)x) \quad for\ all\ x,y \in \lambda. \tag{2.2}$$

Since F is a generalized (α,β)-derivation, we have

$$F(x\,o\,y)\alpha(x) + \beta(xoy)d(x) = a\alpha(xy \pm yx)\alpha(x) \quad for\ all\ \ x,y \in \lambda.$$

Therefore, we have by using equation (2.1)

$$\beta(xoy)d(x) = 0 \quad for\ all\ x,y \in \lambda. \tag{2.3}$$

Again we replace y by $zy, z \in \lambda$ in equation (2.3), to have

$$\beta[x,z]\beta(y)d(x) = 0 \quad for\ all\ \ x,y \in \lambda. \tag{2.4}$$

Replacing y by $ry,\ r \in R$, we get

$$\beta[x,z]R\beta(y)d(x) = 0 \quad for\ all\ \ x,y \in \lambda.$$
(2.5)

Since R is semiprime, it must contain a family $\Im = \{I_\alpha : \alpha \in \Lambda\}$ of prime ideals such that $\bigcap I_\alpha = 0$. If I is a typical element of $\Im$ and $y \in \lambda$, we have either [β(x),β(z)]⊆ I or β(y)d(x)⊆ I . For a fixed I, the following two sets are additive subgroup of λ such that $T_1 \cup T_2 = \lambda$:

$$T_1 = \{x \in \lambda : [\beta(x),\beta(\lambda)] \subseteq I\}$$
$$T_2 = \{x \in \lambda : \beta(\lambda)d(x) \subseteq I\}.$$

By Brauer's trick, we have either $T_1 = \lambda$ or $T_2 = \lambda$ i.e either [β(x),β(λ)]⊆ I or β(λ)d(λ)⊆ I. Above two conditions together implies that $[\beta(x),\beta(\lambda)]d(\lambda) \subseteq I$ for any $I \in \Im$. Therefore, we have $[\beta(\lambda),\beta(\lambda)]d(\lambda) \subseteq \cap I_\alpha = 0$.

Corollary 2.3 Let R be a prime ring and λ be a nonzero left ideal of R. If R admits a generalized (α,β) −derivation F associated with an (α,β)-derivation d such that $F(x\,o\,y) = a\alpha(x\,o\,y)$ for all $x,y \in \lambda$, where $a \in \{0,1,-1\}$, then one of the following holds:

(i) $\beta(\lambda)d(\lambda) = 0$

(ii) R is commutative ring with $char(R) = 2$

(iii) R is commutative ring with $char(R) \neq 2$ and $F(x) = a\alpha(x)$ for all $x \in R$.

Proof. By Theorem 2.2 we have $[\beta(\lambda),\beta(\lambda)]d(\lambda) = 0$. This gives that $0 = [\beta(\lambda),\beta(\lambda^2)]d(\lambda) = [\beta(\lambda),\ \beta(\lambda)]\beta(\lambda)d(\lambda) = [\beta(\lambda),\beta(\lambda)]\beta(R\lambda)d(\lambda) = [\beta(\lambda),\beta(\lambda)]R\beta(\lambda)d(\lambda)$. Using primeness of R, we have either $[\beta(\lambda),\beta(\lambda)] = 0$ or $\beta(\lambda)d(\lambda) = 0$. Now $\beta(\lambda)d(\lambda) = 0$ gives our conclusion (i). Let $[\beta(\lambda),\beta(\lambda)] = 0$. Then $\beta[\lambda,\lambda] = 0$. left multiplying by β^{−1}, we have $[\lambda,\lambda] = 0$ which implies that $[\lambda,R\lambda] = 0 = [\lambda,R]\lambda = [\lambda,R]$. Again this gives that $[R\lambda,R] = 0 = [R,R]\lambda$. In a prime ring left annihilator of a left ideal is zero, we have $0 = [R,R]$ and hence R is commutative. If $char(R) = 2$, we obtain our conclusion (ii). On the other hand if $char(R) \neq 2$, then from hypothesis we have

$$F(xy) = a\alpha(xy) \ for\ all\ x,y \in \lambda.$$

This gives that

$$(F(x) - a\alpha(x))\alpha(y) + \beta(x)d(y) = 0. \tag{2.6}$$

Now replacing x by xz in the equation (2.6), we have

$$((F(x) - a\alpha(x))\alpha(z) + \beta(x)d(z))\alpha(y) + \beta(xz)d(y) = 0 \quad for\ all\ x,y,z \in \lambda.$$

Using equation (2.6) in the above expression, we have

$$\beta(xz)d(y) = 0 \quad for\ all\ x,y \in \lambda.$$

Replacing y by $yr, r \in R$ in the above expression, we have

$$\beta(xz)(d(y)\alpha(r) + \beta(y)d(r)) = 0 \quad for\ all\ z,y,z \in \lambda\ and\ \ r \in R.$$

That is

$$\beta(xzy)d(r) = 0 \quad for\ all\ x,y,z \in \lambda\ \ and\ \ r \in R.$$

Again replacing y by ys, we have

$$\beta(xzy)\beta(s)d(r) = 0 \quad for\ all\ x,z,y \in \lambda\ and\ \ r,s \in R.$$

That is

$$\beta(xzy)Rd(r) = 0 \quad for\ all\ x,y,z \in \lambda\ \ and\ r \in R.$$

Since R is a prime ring, we have either $d(R) = 0$ or $\beta(xzy) = 0$. Let $\beta(xzy) = 0$ i.e $\lambda^3 = 0$. Since R is a prime ring, we have λ=0 which is a contradiction.

Using $d(R) = 0$ *in equation (2.6), we have*

$$(F(x) - a\alpha(x))\alpha(y) = 0 \quad for\ all\ x, y \in \lambda.$$

Therefor, we have $F(x) = a\alpha(x)$ for all $x \in \lambda$. Replacing x by rx, $r \in R$ and using $d(R) = 0$, we get $F(r) = a\alpha(r)$ for all $r \in R$.

Theorem 2.4 Let R be a semiprime ring and λ be a nonzero left ideal of R. If F is a generalized (α, β)-derivation of R associated with an (α, β)-derivation d of R such that $F([x,y]) = a\alpha(xy \pm yx)$ for all $x, y \in \lambda$, where $a \in \{0, 1, -1\}$, then $[\beta(\lambda), \beta(\lambda)]d(\lambda) = 0$.

Proof. $If\ F(\lambda) = 0$, then for any $x, y \in \lambda$ we have $0 = F(xy) = F(x)\alpha(y) + \beta(x)d(y) = \beta(x)d(y)$ i.e $\beta(\lambda)d(\lambda) = 0$. This gives our conclusion. So we assume that $F(\lambda) \neq 0$. Then by our assumption, we have

$$F([x,y]) = a\alpha(xy \pm yx)\ for\ all\ x, y \in \lambda.$$

(2.7)

Substituting yx for y in equation (2.7), we have $F([x,y]x) = a\alpha(xy \pm yx)\alpha(x)$. That is

$$F([x,y])\alpha(x) + \beta([x,y])d(y) = a\alpha(xy \pm yx)\alpha(x) \quad \text{for all } x, y \in \lambda.$$

(2.8)

Now using equation (2.7) in equation (2.8), we get

$$\beta([x,y])d(y) = 0 \ \ for\ all\ x, y \in \lambda.$$

(2.9)

Substituting zy for $y, z \in \lambda$, we have $\beta([x,y])\beta(z)d(y) = 0$ for $all\ x, y \in \lambda$ which is same as equation (2.4). Arguing as in Theorem 2.2 we can conclude the result.

Corollary 2.5 Let R be a prime ring and λ be a nonzero left ideal of R. If F is a generalized (α, β)-derivation of R associated with an (α, β)-derivation d of R such that $F([x,y]) = a\alpha(xy \pm yx)$ for all $x, y \in \lambda$, where $a \in \{0, 1, -1\}$, then either R is commutative or $\beta(\lambda)d(\lambda) = 0$ and one of the following holds:

(i) $\alpha(\lambda)[\alpha(\lambda), \alpha(\lambda)] = 0$

(ii) $F(x) = a\alpha(x) for\ all\ x \in \lambda. Also\ in\ this\ case, if\ a \neq 0\ then\ char(R)\ is\ 2.$

Proof. By Theorem 2.4, we have

$$[\beta(\lambda), \beta(\lambda)]d(\lambda) = 0. \quad (2.10)$$

Again by corollary (2.3), we have either R is commutative or $\beta(\lambda)d(\lambda) = 0$. We assume that R is not commutative. Then for any $x, y \in \lambda$, $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ implying that $F(xy) = F(x)\alpha(y)$ i.e F acts as a left α-multiplier on λ. Replacing $y\ by\ yz, z \in \lambda$ in our hypothesis, we get

$$F([x,y]z + y[x,z]) = a\alpha((xy \pm yx)z \mp y[x,z]).$$

(2.11)

Since F acts as left α $-$multiplier on λ, we have

$$F([x,y])\alpha(z) + F(y)\alpha([x,z]) = a\alpha\big((xy \pm yx)z \mp y[x,z]\big)\ for\ all\ x, y \in \lambda.$$

(2.12)

By hypothesis, we have

$$F(y)\alpha([x,z]) = a\alpha(y[x,z])\ for\ all\ x, y \in \lambda.$$

That is

$$\big(F(y) - a\alpha(y)\big)\alpha([x,z]) = 0\ \ for\ all\ x, y \in \lambda.$$

(2.13)

Replacing $x\ by\ xv$ in above equation (2.13), we have

$$\big(F(y) - a\alpha(y)\big)\alpha(x)\alpha([v,z]) = 0\ \ for\ all\ x, y, v \in \lambda.$$

(2.14)

That is

$$\big(F(y) - a\alpha(y)\big)R\alpha(x)\alpha([v,z]) = 0 \ for\ all\ x,y,v \in \lambda.$$
(2.15)

Using primeness of R in the above expression, we get either $F(y) - a\alpha(y) = 0$ or $\alpha(x)\alpha([v,z]) = 0$ i.e either $F(y) = a\alpha(y)$ or $\alpha(\lambda)[\alpha(\lambda),\alpha(\lambda)] = 0$. When $F(y) = a\alpha(y)$, then by hypothesis we get $a\alpha([x,y]) = a\alpha(xy \pm yx)$ for all $x,y \in \lambda$. If $a\alpha([x,y]) = a\alpha(xy - yx)$, then it is trivial and nothing is to prove. So we consider the following $a\alpha([x,y]) = a\alpha(xy + yx)$ i.e $2a\alpha(yx) = 0$ for all $x,y \in \lambda$. This can be written as $2a\alpha(R\lambda^2) = 0$ i.e $2aR\alpha(\lambda^2) = 0$. Since λ is a nonzero left ideal of R, we have either $a = 0\ or\ char(R) = 2.$

Theorem 2.6 Let R be a semiprime ring and λ be a nonzero left ideal of R and d is an (α,α)-derivation of R such that $d(x)od(y) = a\alpha(xy \pm yx)\ \ for\ all\ x,y \in \lambda$, where $a \in \{0,1,-1\}$, then $\alpha(\lambda)[[\alpha(\lambda),d(\lambda)],d(\lambda)] = 0$.

Proof. By our hypothesis, we have

$$d(x)d(y) + d(y)d(x) = a\alpha(xy \pm yx)\ for\ all\ x,y \in \lambda. \qquad (2.16)$$

Replacing $y\ by\ yx$ in the above expression, we have for all $x,y \in \lambda$

$$d(x)(d(y)\alpha((x) + \alpha(y)d(x)) + (d(y)\alpha((x) + \alpha(y)d(x))d(x)) = a\alpha((xy \pm yx)x). \qquad (2.17)$$

Now using equation (2.16) in equation (2.17), we get

$$d(x)\alpha(y)d(x) + d(y)[\alpha(x),d(x)] + \alpha(y)d(x)^2 = 0. \qquad (2.18)$$

Again replacing $y\ by\ xy$ in the above equation (2.18), we have

$$d(x)\alpha(xy)d(x) + (d(y)\alpha(y) + \alpha(x)d(y))[\alpha(x),d(x)] + \alpha(y)d(x)^2 = 0 \qquad (2.19)$$

Now left multiplying equation (2.18) by $\alpha(x)$ and subtracting from equation (2.19), we get

$$[d(x),\alpha(x)]\alpha(y)d(x) + d(x)\alpha(y)[\alpha(x),d(x)] = 0 \qquad (2.20)$$

Replacing y by $\alpha^{-1}(d(x))y$ in the above expression, we have

$$[d(x),\alpha(x)]d(x)\alpha(y)d(x) + d(x)^2\alpha(y)[\alpha(x),d(x)] = 0 \qquad (2.21)$$

left multiplying equation (2.20) by $d(x)$ and subtracting from equation (2.21), we have

$$\big[[\alpha(x),d(x)],d(x)\big]\alpha(y)d(x) = 0\ for\ all\ x,y \in \lambda. \qquad (2.22)$$

This implies that $\big[[\alpha(x),d(x)],d(x)\big]\alpha(y)[[\alpha(x),d(x)],d(x)] = 0$ and hence $\alpha(y)\big[[\alpha(x),d(x)],d(x)\big]R\alpha(y)[[\alpha(x),d(x)],d(x)] = 0$. Since R is a semiprime ring we have $\alpha(\lambda)[[\alpha(\lambda),d(\lambda)],d(\lambda)] = 0$.

Theorem 2.7 Let R be a prime ring and λ be a nonzero left ideal of R such that $r(\lambda) = 0$. If R admits a generalized (α,β)-derivation F associated with a nonzero (α,β)-derivation d such that $F\big(\alpha([x,y])\big) = 0\ for\ all\ x,y \in \lambda$, then R is commutative.

Proof. By assumption, we have

$$F\big(\alpha([x,y])\big) = 0\ for\ all\ x,y \in \lambda.$$
(2.23)

Replacing $y\ by\ yx$ in (2.23) and using (2.23), we get $\beta\alpha([x,y])d(\alpha(x)) = 0$ which implies that

$$[x,y]\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0\ \ for\ all\ x,y \in \lambda.$$
(2.24)

Now substituting ry for y in (2.24) and using (2.24), we obtain $[x,r]y\alpha^{-1}\beta^{-1}(d(\alpha(x)) = 0$ for $all\ x,y \in \lambda\ and\ r \in$ R. In particular, $[x,R]R\lambda\alpha^{-1}(d(\alpha(x))) = 0\ for\ all\ x \in \lambda$. The primeness of R yields that for $each\ x \in \lambda$, either $[x,R] = 0\ or$ $\lambda\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0, in\ this\ case\ d(\alpha(x)) = 0$. Set $\lambda_1 = \{x \in \lambda \mid [x,R] = 0\}$ and $\lambda_2 = \{x \in \lambda \mid d(\alpha(x)) = 0\}$. Then, λ_1 and λ_2 are both additive subgroups of λ such that $\lambda = \lambda_1 \cup \lambda_2$. Thus, by Brauer's trick, we have either $\lambda = \lambda_1\ or\ \lambda = \lambda_2$. If $\lambda = \lambda_1$, then $[\lambda,R] = 0$ i.e $\lambda \subseteq Z(R)$ and hence R is commutative by Lemma 2.1. If $\lambda = \lambda_2$, then $d(\alpha(\lambda)) = 0$ and $0 = d(\alpha(R\lambda))$=$d\big(\alpha(R)\big)\alpha^2(\lambda) + \beta(\alpha(R))d(\alpha(\lambda))$, which reduces to $d\big(\alpha(R)\big)\alpha^2(\lambda) = 0$ and hence

$d(\alpha(R))\alpha^2(R\lambda) = 0 = d(\alpha(R))\alpha^2(R)\alpha^2(\lambda) = d(\alpha(R))R\alpha^2(\lambda)$. Since λ is nonzero left ideal and the last relation forces that $d(\alpha(R)) = 0$ $i.e$ $d = 0$, contradiction.

Theorem 2.8 Let R be a prime ring and λ be a nonzero left ideal of R such that $r(\lambda) = 0$. If R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $F(\alpha(x \circ y)) = 0$ $for\ all$ $x, y \in \lambda$, then R is commutative.

Proof. By assumption, we have

$$F(\alpha(xoy)) = 0 \ for\ all\ \ x,y \in \lambda. \quad (2.25)$$

Replacing y by yx in (2.25) and using (2.25), we get $\beta\alpha(x \circ y)d(\alpha(x)) = 0$, which implies that

$$(x \circ y)\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0 \quad for \quad all \quad x,y \in \lambda. \quad (2.26)$$

Now substituting ry for y in (2.26}) and using (2.26), We obtain $[x,r]y\alpha^{-1}\beta^{-1}(d(\alpha(x)) = 0$ for all $x, y \in \lambda$ and $r \in R$. In particular, $[x,R]R\lambda\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0$ for all $x \in \lambda$. The primeness of R yields that for each $x \in \lambda$, either $[x,R] = 0$ or $\lambda\,\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0$, in this case $d(\alpha(x)) = 0$. Arguing in the similar manner as in Theorem 2.7 we get the result.

Acknowledgements

The author would like to expres his sincere thanks to the referee for valuable suggestions.

References

1. Bresar M. On the distance of the composition of two derivations to the generalized derivations, Glasgow Mathematical Journal, 33(1991),89-93.
2. Daif M. N. and Bell H.E. Remarks on derivations on semiprime rings, International Journal of Mathematics and Mathematical Sciences, 5(1992), 205-206.
4. Dhara B. Remarks on generalized derivations in prime and semiprime rings, International Journal of Mathematics and Mathematical Sciences, 2010, Article ID 646587, 6 pages.
5. Mayne J. H. Centralizing mappings of prime rings, Canad. Math. Bull. 27(1984), 122-126.
6. Anderson F.W. Lectures on noncommutative rings. University of Oregon, Oregon (2002).

Authors' biography with photos

Asma Ali is working as Professor in the Department of Mathematics, Faculty of Science, Aligarh Muslim University (India), since June 23, 2009. She earned her Ph.D. in Mathematics from Aligarh Muslim University, India. Her field of specialization is Ring theory/Near Ring theory/Module theory. She has teaching and research experience of 23 years. She has been awarded N. Nahar OSU Distiguished Scientist Award for excellent research and teaching in 2017. She has published a number of research papers in reputed referred International Journals. She edited a Research Volume entitled "Algebra and its Applications" by Springer Verlag coedited S.Tariq Rizvi, Ohio State University, Columbus, USA and Vincenzo De Filippis of Messina University, Italy. She is a Life member of various academic bobies like American Mathematical Society, Indian Mathematical Society, Indian Science Congress, Jammu Mathematical society and Calcutta Mathematical Society.

Md. Hamidur Rahaman is doing Ph.D under the supervision of Professor Asma Ali in the Department of Mathematics, Faculty of Science, Aligarh Muslim University, India. He has a bright academic career. He was awarded Junior Research Felloship (JRF) in 2016 from Council of Scientific and Industrial Research (CSIR), New Delhi, India.