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Oscillation Results for First Order Nonlinear Neutral Difference Equation with "Maxima"

S.Rajan[1] and K.Malathi[2]

[1] Department of Mathematics , Erode Arts and Science College , Erode – 638009 , Tamil Nadu , India .

[2] Department of Mathematics ,Kongu Arts and Science College , Erode – 638107 , Tamil Nadu , India .

malathiilan@gmail.com

ABSTRACT

In this paper we consider the first order nonlinear neutral difference equation with maxima of the form

$$\Delta (x_n + px_{n-k}) + q_n \max_{[[n-m,n]} x_s^{\alpha} = 0 , \ n \in N_0$$

and established some sufficient conditions for the oscillation of all solutions of the above equation . Examples are provided to illustrate the main results .

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1 INTRODUCTION

Consider the first order nonlinear neutral difference equation of the form

$$\Delta (x_n + px_{n-k}) + q_n \max_{[n-m,n]} x_s^{\alpha} = 0 , \quad n \in N_0 , \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} – x_n$ and $N_0 = \{ n_0 , n_0 + 1 , n_0 + 2 , \ldots \}$, subject to the following conditions :

(C_1) $\{ q_n \}$ is a positive real sequence ;

(C_2) k and ℓ are positive integers and $0 \le p < \infty$;

(C_3) α is a ratio of odd positive integers .

Let $\theta = \max \{ k , \ell \}$. By a solution of equation (1.1) we mean a real sequence $\{ x_n \}$ defined for all $n \ge n_0 – \theta$ and satisfying equation(1.1) for all $n \ge n_0$. A solution $\{ x_n \}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years there is a great interest in studying the oscillatory behaviour of first order nonlinear neutral type difference equations without " maxima " , see for example [1 , 2 , 3 , 5 , 7] and the references cited therein. In [5 , 7] , the authors studied the oscillatory behaviour of solutions of equation (1.1) when α = 1 and without " maxima ". Motivated by these observation , in this paper we obtain some sufficient conditions for the oscillation of all solutions of equation (1.1) when α < 1 , α > 1and α = 1.

In Section 2 , we establish some sufficient conditions for the oscillation of all solutions of equation (1.1) and in Section 3 , we present some examples to illustrate the main results.

2 Main Results

To prove our main results we need the following lemmas.

Lemma 2.1 . If $A \ge 0 , B \ge 0$ and $0 < \alpha \le 1$, then

$$A^{\alpha} + B^{\alpha} \ge (A + B)^{\alpha} . \tag{2.1}$$

Lemma 2.2 . If $A \ge 0 , B \ge 0$ and $\alpha > 1$, then

$$A^{\alpha} + B^{\alpha} \ge [1 / (2^{\alpha-1})] (A + B)^{\alpha} . \tag{2.2}$$

For the proof of Lemmas 2.1 and 2.2 , see [4].

Lemma 2.3 . If $0 < \alpha < 1$, ℓ is a positive integer and $\{ q_n \}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} q_n = \infty$, then every solution of equation

$$\Delta x_n + q_n x_{n-\ell}^{\alpha} = 0 , \qquad (2.3)$$

is oscillatory.

Lemma 2.4 . If $\alpha = 1$ and

$$\lim_{n\to\infty} \inf \sum_{s=n-\ell}^{n-1} q_s > [\ell / (\ell + 1)]^{\ell+1} , \qquad (2.4)$$

then every solution of equation (2.3) is oscillatory.

Lemma 2.5 . Let $\alpha > 1$. If there exists a $\lambda > (1 / \ell) \log \alpha$ such that

$$\lim_{n\to\infty} \inf [q_n \exp (- e^{\lambda n})] > 0 , \qquad (2.5)$$

then every solution of equation (2.3) is oscillatory.

For the proof of Lemmas 2.3 and 2.5 , see [6] , and Lemma 2.4 , see [3] .

Lemma 2.6 . The sequence $\{ x_n \}$ is an eventually negative solution of equation (1.1) if and only if $\{ - x_n \}$ is an eventually positive solution of equation

$$\Delta (x_n + p x_{n-k}) + q_n \max_{[n-m,n]} x_s^{\alpha} = 0 , \quad n \in N_0 .$$

The assertion of Lemma 2.6 can be verified easily.

Before stating the next theorem , let us define

$$Q_n = \min \{ q_n , q_{n-k} \} \text{ for } n \in N_0 . \qquad (2.6)$$

Theorem 2.1 . Let $0 < \alpha \le 1$. If the first order neutral difference inequality

$$\Delta w_n + [1 / (1 + p^{\alpha})^{\alpha}] Q_n \max_{[n-m,n]} w_{s+k}^{\alpha} \le 0 , \qquad (2.7)$$

has no positive solution , then every solution of equation (1.1) is oscillatory .

Proof . Let $\{ x_n \}$ be a nonoscillatory solution of equation (1.1) . Without loss of generality we may assume that $x_n > 0$ and $x_{n-k} > 0$ for all $n \ge n_1 \ge n_0 + \theta$. Then $z_n = x_n + p x_{n-k} > 0$ for all $n \ge n_1$.

From the equation (1.1) , we have

$$\Delta z_n + q_n \max_{[n-m,n]} x_s^{\alpha} = 0 , \qquad (2.8)$$

and

$$p^{\alpha} \Delta z_{n-k} + p^{\alpha} q_{n-k} \max_{[n-k-m,n-k]} x_s^{\alpha} = 0 . \qquad (2.9)$$

Combining (2.8) and (2.9) , and then using (2.6) we get

$$\Delta (z_n + p^{\alpha} z_{n-k}) + Q_n (\max_{[n-m,n]} x_s^{\alpha} + p^{\alpha} \max_{[n-k-m,n-k]} x_s^{\alpha}) \le 0 . \qquad (2.10)$$

Applying Lemma 2.1 in inequality (2.10) , we obtain

$$\Delta (z_n + p^\alpha z_{n-k}) + Q_n \max_{[n-m,n]} (x_s + p x_{s-k})^\alpha \le 0$$

Or

$$\Delta (z_n + p^\alpha z_{n-k}) + Q_n \max_{[n-m,n]} z_s^\alpha \le 0 . \qquad (2.11)$$

Let $w_n = z_n + p^\alpha z_{n-k}$. Then $w_n > 0$ and using the decreasing nature of z_n , we obtain

$$W_n \le (1 + p^\alpha) z_{n-k}$$

Or

$$(w_{n+k}) / (1 + p^\alpha) \le z_n . \qquad (2.12)$$

Substituting (2.12) in (2.11) , we get that $\{ w_n \}$ is a positive solution of the inequality

$$\Delta w_n + [1 / (1 + p^\alpha)^\alpha] Q_n \max_{[n-m,n]} w_{s+k}^\alpha \le 0 ,$$

which is a contradiction . The proof is now complete.

Theorem 2.2 . Let $\alpha > 1$. If the first order neutral difference inequality

$$\Delta w_n + [1 / (1 + p^\alpha)^\alpha] 2^{1-\alpha} Q_n \max_{[n-m,n]} w_{s+k}^\alpha \le 0 , \qquad (2.13)$$

has no positive solution , then every solution of equation (1.1) is oscillatory .

Proof . Let $\{ x_n \}$ be a nonoscillatory solution of equation (1.1) . From the proof of Theorem 2.1 , we have (2.10) . Now applying Lemma 2.2 to (2.10) , we obtain

$$\Delta (z_n + p^\alpha z_{n-k}) + 2^{1-\alpha} Q_n \max_{[n-m,n]} z_s^\alpha \le 0 . \qquad (2.14)$$

Let $w_n = z_n + p^\alpha z_{n-k}$. Then $w_n > 0$ and using the decreasing nature of z_n , we obtain

$$W_n \le (1 + p^\alpha) z_{n-k}$$

Or

$$(w_{n+k}) / (1 + p^\alpha) \le z_n . \qquad (2.15)$$

Substituting (2.15) in (2.14) , we get that $\{ w_n \}$ is a positive solution of the inequality

$$\Delta w_n + [1 / (1 + p^\alpha)^\alpha] 2^{1-\alpha} Q_n \max_{[n-m,n]} w_{s+k}^\alpha \le 0 ,$$

which is a contradiction . The proof is now complete .

Corollary 2.1 . Let $m > k$ and $0 < \alpha < 1$ in equation (1.1) . If

$$\sum_{n=n_0}^{\infty} Q_n = \infty , \qquad (2.16)$$

then every solution of equation (1.1) is oscillatory .

Proof . From Lemma 2.3 we see that the condition (2.16) implies that the inequality (2.7) has no positive solution and hence the proof follows from Theorem 2.1 .

Corollary 2.2 . Let $m > k$ and $\alpha = 1$ in equation (1.1) . If

$$\lim_{n\to\infty} \inf \sum_{s=n-m+k}^{n-1} Q_s > (1 + p) [(m - k) / (m - k - + 1)]^{\ell - k + 1} \qquad (2.17)$$

then every solution of equation (1.1) is oscillatory .

Proof . From Lemma 2.4 we see that the condition (2.17) implies that the inequality (2.7) has no positive solution and hence the proof follows from Theorem 2.1 .

Corollary 2.3 . Let $m > k$ and $\alpha > 1$ in equation (1.1) . If there exists a $\lambda > 0$ such that $\lambda > [1/(m-k)] \log \alpha$ and

$$\lim_{n\to\infty} \inf [Q_n \exp (- e^{\lambda n})] > 0 , \qquad (2.18)$$

then every solution of equation (1.1) is oscillatory.

Proof . From Lemma 2.5 we see that the condition (2.18) implies that the inequality (2.13) has no positive solution and hence the proof follows from Theorem 2.2 .

3 Examples

In this section , we present some examples to illustrate the main results .

Example 3.1 . Consider the neutral difference equation

$$\Delta (x_n + 2x_{n-2}) + 6 \max_{[n-4,n]} x_s^{1/3} = 0 , \; n \geq 1 . \qquad (3.1)$$

Here $p = 2$, $q_n = 6$, $k = 2$, $m = 4$, $\alpha = 1/3$. It is easy to see that all conditions of Corollary 2.1 are satisfied . Hence every solution of equation (3.1) is oscillatory . In fact $\{ x_n \} = [(-1)^{3n}]$ is one such solution of equation (3.1) .

Example 3.2 . Consider the neutral difference equation

$$\Delta (x_n + 2x_{n-2}) + [(6n - 5) / (n - 4)] \max_{[n-4,n]} x_s = 0 , \; n \geq 5 . \qquad (3.2)$$

Here $p = 2$, $q_n = (6n - 5) / (n - 4)$, $k = 2$, $m = 4$, $\alpha = 1$. It is easy to see that all conditions of Corollary 2.2 are satisfied . Hence every solution of equation (3.2) is oscillatory . In fact $\{ x_n \} = [n (-1)^n]$ is one such solution of equation (3.2) .

Example 3.3 . Consider the neutral difference equation

$$\Delta (x_n + 3x_{n-2}) + [1 + (1 / n)] e^{e^{2n}} \max_{[n-4,n]} x_s^3 = 0 , \; n \geq 1 . \qquad (3.3)$$

Here $p = 3$, $q_n = [1 + (1 / n)] e^{e^{2n}}$, $k = 2$, $m = 4$, $\alpha = 3$. Choose $\lambda = 2$, then it is easy to see that all conditions of Corollary 2.3 are satisfied . Hence every solution of equation (3.3) is oscillatory .

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Author's Biography

K.Malathi was born in 1978 in Tamil nadu , India. She received her B.Sc. degree from Avinashilingam Deemed University, Coimbatore , India and her M.Sc. degree

in pure Mathematics from Bharathiar University , Coimbatore ,India . She is now a PhD scholar in Difference Equations . Her research interest include Difference Equations and Graph Theo