



# APPROXIMATIONS OF STOCHASTIC INTEGRALS OF THE POISSON PROCESS

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## ABSTRACT

In this article the asymptotic behavior of the finite sums of the elements depending on approximations, being linear combinations of approximations by I to and Stratonovich, is investigated. The outcomes of the present work are generalizations of the corresponding theorems from

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## Academic Discipline And Sub-Disciplines

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## TYPE (METHOD/APPROACH)

the convergence of the integral sums of Poisson

## INTRODUCTION

For giving a mathematical meaning to the stochastic differential equations, in which the controlling process is a Wiener process, the stochastic differentials (integrals) were defined. Initially stochastic integrals of Ito and Stratonovich were used as a base integrals, the first integral more often is operated by mathematics, and the second one is operated by natural scientists (in particular case, by physicists). Numerous generalizations of these integrals now are known: the  $\theta$  –stochastic integral, Ogawa stochastic integral etc. [2],[3],[4].

On the basis of generalized random processes algebra the space of generalized stochastic differentials was constructed in the article [1]. And the "symmetrical" subsets, which correspond to differentials by I to and Stratonovich at an associated level, were allocated in this space. In the work [5] the problem of approximation of a stochastic  $\theta$  –integral of the Brownian motion random process by the elements of generalized random processes algebra was investigated.

The second important class of exterior fluctuations is an appearance of the specific discrete events at the random moments of time in the mentioned space. The Poisson process is an appropriate model in this case. It is interesting to know that will turn out if refuse similarly the independence of the increments of Poisson process. In the work [6] the convergence of the integral sums of Poisson for the cases, where approximation is I to and Stratonovich approximation, was investigated. In this article the asymptotic behavior of the finite sums of the elements depending on approximations, being linear combinations of approximations by I to and Stratonovich, is investigated. The outcomes of the present work are generalizations of the corresponding theorems from [6].

## PAGE SIZE

### Basic outcomes

Let  $(\Omega, A, P)$  be full probability space;  $\Pi(t, \omega), t \in T, \omega \in \Omega$  be random Poisson process, which trajectories are continuous on the right;  $\rho(t), t \in R$  be non-negative indefinitely differentiable function, which carrier from  $[0,1]$  and  $\int_R \rho(s) ds = 1$ , and  $\rho_n^1(t) = n\rho(nt), \rho_n^2(t) = \varphi(n)\rho(\varphi(n)t), n \in N$ , where  $\varphi(n)$  is a monotonically nondecreasing sequence. Let  $0 < h_n < 2h_n < \dots < mh_n = a$  be partition of the segment  $T = [0, a]$ . Then, for any  $t \in T$  there are  $\tau_t \in [0, h_n]$  and  $m_t \in N$ , that  $t = \tau_t + m_t h_n$ .

As generalized Poisson process let us consider

$$\Pi_n^\theta(t, \omega) = \theta \Pi_n^1(t, \omega) + (1 - \theta) \Pi_n^2(t, \omega), \quad \theta \in [0,1], \quad \text{where}$$

$$\Pi_n^1(t, \omega) = \int_0^{1/n} \Pi(t+s, \omega) \rho_n^1(s) ds, \quad \Pi_n^2(t, \omega) = \int_0^{1/n} \Pi(t+s, \omega) \rho_n^2(s) ds.$$

**Theorem 1. Let**  $f \in C^1(R)$ ,  $f_n = f^* \rho_n^1$ ,  $n \rightarrow \infty$ ,  $h_n \rightarrow 0$ ,  $\frac{1}{\varphi(n)} = o(h_n)$ ,  $h_n = o\left(\frac{1}{n}\right)$ .

Then

$$\sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + (k-1)h_n, \omega)) (\Pi_n^\theta(\tau_t + kh_n, \omega) - \Pi_n^\theta(\tau_t + (k-1)h_n, \omega)) \rightarrow$$

$$\rightarrow \int_0^t [g(\Pi(s-0, \omega) + \theta) - g(\Pi(s-0, \omega))] d\Pi(s, \omega) + (1 - \theta) \int_0^t f(\Pi(s-0, \omega) + \theta) d\Pi(s, \omega)$$

In the  $L_p(T)$ ,  $p \geq 1$ , for almost all  $\omega \in \Omega$ , where  $g(x) = \int_0^x f(s) ds$ .

The integral everywhere is understood in a sense of the Lebesgue - Stieltjes for almost all  $\omega \in \Omega$ .

**Theorem 2. Let**  $f \in C^1(R)$ ,  $f_n = f^* \rho_n^1$ ,  $n \rightarrow \infty$ ,  $h_n \rightarrow 0$ ,  $\frac{1}{\varphi(n)} = o(h_n)$ ,  $h_n = o\left(\frac{1}{n}\right)$ .

Then

$$\sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + kh_n, \omega)) (\Pi_n^\theta(\tau_t + kh_n, \omega) - \Pi_n^\theta(\tau_t + (k-1)h_n, \omega)) \rightarrow$$

$$\rightarrow \int_0^t [g(\Pi(s-0, \omega) + \theta) - g(\Pi(s-0, \omega))] d\Pi(s, \omega) + (1 - \theta) \int_0^t f(\Pi(s, \omega)) d\Pi(s, \omega)$$

In the  $L_p(T)$ ,  $p \geq 1$ , for almost all  $\omega \in \Omega$ .

As generalized Poisson process now let us consider

$$\Pi_n^\theta(t, \omega) = \theta \Pi_n^1(t, \omega) + (1 - \theta) \Pi_n^2(t, \omega), \quad \theta \in [0,1], \quad \text{where}$$

$$\Pi_n^1(t, \omega) = \int_0^{1/n} \Pi(t-s, \omega) \rho_n^1(s) ds, \quad \Pi_n^2(t, \omega) = \int_0^{1/n} \Pi(t-s, \omega) \rho_n^2(s) ds.$$

**Theorem 3. Let**  $f \in C^1(R)$ ,  $f_n = f^* \rho_n^1$ ,  $n \rightarrow \infty$ ,  $h_n \rightarrow 0$ ,  $\frac{1}{\varphi(n)} = o(h_n)$ ,  $h_n = o\left(\frac{1}{n}\right)$ .

Then

$$\sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + (k-1)h_n, \omega)) (\Pi_n^\theta(\tau_t + kh_n, \omega) - \Pi_n^\theta(\tau_t + (k-1)h_n, \omega)) \rightarrow$$

$$\rightarrow \int_0^t [g(\Pi(s, \omega)) - g(\Pi(s, \omega) - \theta)] d\Pi(s, \omega) + (1 - \theta) \int_0^t f(\Pi(s-0, \omega)) d\Pi(s, \omega)$$

In the  $L_p(T)$ ,  $p \geq 1$ , for almost all  $\omega \in \Omega$ .

**Theorem 4.** Let  $f \in C^1(R)$ ,  $f_n = f^* \rho_n^1$ ,  $n \rightarrow \infty$ ,  $h_n \rightarrow 0$ ,  $\frac{1}{\varphi(n)} = o(h_n)$ ,  $h_n = o\left(\frac{1}{n}\right)$ .

Then

$$\sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + kh_n, \omega))(\Pi_n^\theta(\tau_t + kh_n, \omega) - \Pi_n^\theta(\tau_t + (k-1)h_n, \omega)) \rightarrow \int_0^t [g(\Pi(s, \omega)) - g(\Pi(s, \omega) - \theta)] d\Pi(s, \omega) + (1 - \theta) \int_0^t f(\Pi(s, \omega) - \theta) d\Pi(s, \omega)$$

In the  $L_p(T)$ ,  $p \geq 1$ , for almost all  $\omega \in \Omega$ .

**Proof of the theorem 4.**

$$\begin{aligned} & \sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + kh_n, \omega))(\Pi_n^\theta(\tau_t + kh_n, \omega) - \Pi_n^\theta(\tau_t + (k-1)h_n, \omega)) = \\ & = \theta \sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + kh_n, \omega))(\Pi_n^1(\tau_t + kh_n, \omega) - \Pi_n^1(\tau_t + (k-1)h_n, \omega)) + \\ & + (1 - \theta) \sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + kh_n, \omega))(\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k-1)h_n, \omega)) = \\ & = \theta I_n^1(t, \omega) + (1 - \theta) I_n^2(t, \omega). \end{aligned}$$

Let us show that  $\theta I_n^1 \rightarrow \int_0^t [g(\Pi(s, \omega)) - g(\Pi(s, \omega) - \theta)] d\Pi(s, \omega)$

Let  $g_n(x) = \int_0^x f_n(s) ds$ ,  $x \in R$ , then  $g_n \rightarrow g$  uniformly at any compact set from  $R$ .

$$\begin{aligned} & g_n(\Pi_n^\theta(t, \omega)) - g_n(\Pi_n^\theta(0, \omega)) = \\ & = \sum_{k=1}^{m_t} (g_n(\Pi_n^\theta(\tau_t + kh_n, \omega)) - g_n(\Pi_n^\theta(\tau_t + (k-1)h_n, \omega))) = \\ & = \sum_{k=1}^{m_t} (g_n(\Pi_n^\theta(\tau_t + kh_n, \omega)) - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1 - \theta) \Pi_n^2(\tau_t + kh_n, \omega))) + \\ & + \sum_{k=1}^{m_t} (g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1 - \theta) \Pi_n^2(\tau_t + kh_n, \omega)) - g_n(\Pi_n^\theta(\tau_t + (k-1)h_n, \omega))) = \\ & = J_n^1(t, \omega) + J_n^2(t, \omega). \end{aligned}$$

Let us consider the first addend:  $J_n^1(t, \omega) =$

$$= \sum_{k=1}^{m_t} (g_n(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta) \Pi_n^2(\tau_t + kh_n, \omega)) - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1 - \theta) \Pi_n^2(\tau_t + kh_n, \omega)))$$



$$= \theta \sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + kh_n, \omega)) \left( \Pi_n^1(\tau_t + kh_n, \omega) - \Pi_n^1(\tau_t + (k-1)h_n, \omega) \right) + \\ + \frac{1}{2} \theta^2 \sum_{k=1}^{m_t} f_n'(\bar{\theta}_n) \left( \Pi_n^1(\tau_t + kh_n, \omega) - \Pi_n^1(\tau_t + (k-1)h_n, \omega) \right)^2,$$

Where  $\bar{\theta}_n$  is a point laying on a segment, connecting  $\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta) \Pi_n^2(\tau_t + kh_n, \omega)$  and  $\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1-\theta) \Pi_n^2(\tau_t + kh_n, \omega)$

From the lemma 1 [7, page 35] follows that

$$\frac{1}{2} \theta^2 \sum_{k=1}^{m_t} f_n'(\bar{\theta}_n) \left( \Pi_n^1(\tau_t + kh_n, \omega) - \Pi_n^1(\tau_t + (k-1)h_n, \omega) \right)^2 \rightarrow 0 \text{ uniformly on } t \text{ for almost all } \omega \in \Omega$$

Let us designate the norm as  $\|\cdot\|$  in the  $L_p(T)$ ,  $p \geq 1$ . We shall obtain that

$$\|g_n(\theta \Pi_n^1(u, \omega) + (1-\theta) \Pi_n^2(u-\Delta, \omega)) - g(\theta \Pi(u, \omega) + (1-\theta) \Pi(u-\Delta, \omega))\| \leq \\ \leq \|g_n(\theta \Pi_n^1(u, \omega) + (1-\theta) \Pi_n^2(u-\Delta, \omega)) - g(\theta \Pi_n^1(u, \omega) + (1-\theta) \Pi_n^2(u-\Delta, \omega))\| + \\ + \|g_n(\theta \Pi_n^1(u, \omega) + (1-\theta) \Pi_n^1(u-\Delta, \omega)) - g(\theta \Pi(u, \omega) + (1-\theta) \Pi(u-\Delta, \omega))\| \leq \\ \leq \left\| \int_0^{1/n} g(\theta \Pi_n^1(u, \omega) + (1-\theta) \Pi_n^2(u-\Delta, \omega) + s) - g(\theta \Pi_n^1(u, \omega) + (1-\theta) \Pi_n^2(u-\Delta, \omega)) \rho_n(s) ds \right\| + \\ + \|g'(\bar{\theta}_n) \{ \theta [\Pi_n^1(u, \omega) - \Pi(u, \omega)] + (1-\theta) [\Pi_n^2(u-\Delta, \omega) - \Pi(u-\Delta, \omega)] \}\| \leq \\ \leq \frac{C(\omega)}{n} + C \|\Pi_n^1(u, \omega) - \Pi(u, \omega)\| + \|\Pi_n^2(u-\Delta, \omega) - \Pi(u-\Delta, \omega)\|, \text{ where } \Delta \geq 0, u - \Delta \geq 0.$$

The right part of the inequality aspires to zero at the  $n \rightarrow \infty$  for almost all  $\omega \in \Omega$ .

Thus, 
$$g_n(\Pi_n^\theta(t, \omega)) - g_n(\Pi_n^\theta(0, \omega)) \longrightarrow g(\Pi(t, \omega)) - g(\Pi(0, \omega)) = \\ = \int_0^t [g(\Pi(s, \omega)) - g(\Pi(s-0, \omega))] d\Pi(s, \omega) \text{ in the } L_p(T), p \geq 1, \text{ for almost all } \omega \in \Omega.$$

Now we shall prove that

$$A_n(\omega) = \left\| J_n^\theta(t, \omega) - \sum_{i=1}^{\Pi(t, \omega)} [g(\theta \Pi(\mu_i - 0, \omega) + (1-\theta) \Pi(\mu_i, \omega)) - g(\Pi(\mu_i - 0, \omega))] \right\| \longrightarrow 0$$

for almost all  $\omega \in \Omega$ .



$$\begin{aligned}
 A_n(\omega) &\leq \left\| \sum_{k=1}^{m_t} [g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi_n^2(\tau_t + kh_n, \omega)) \right. \\
 &\quad - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi_n^2(\tau_t + (k-1)h_n, \omega))] \\
 &\quad - \sum_{k=1}^{m_t} [g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi_n^2(\tau_t + kh_n, \omega)) \\
 &\quad \left. - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega))] \right\| + \\
 &+ \left\| \sum_{k=1}^{m_t} [g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + kh_n, \omega)) - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) \right. \\
 &\quad \left. + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega))] \right. \\
 &\quad \left. - \sum_{i=1}^{\Pi(t, \omega)} [g(\theta \Pi(\mu_i - 0, \omega) + (1-\theta)\Pi(\mu_i - 0, \omega)) - g(\Pi(\mu_i - 0, \omega))] \right\| = \\
 &= A_n^1(\omega) + A_n^2(\omega).
 \end{aligned}$$

Let us note, that  $\Pi_n^2(\cdot, \omega)$  differs from  $\Pi(\cdot, \omega)$  only at points  $\tau_t + jh_n, 1 \leq j \leq m_t$ , for which performs  $\tau_t + jh_n \in (\mu_i, \mu_i + \frac{1}{\varphi(n)})$ ,  $1 \leq j \leq m_t, 1 \leq i \leq \Pi(t, \omega)$ .

the set  $B_n$  of  $t \in T$ , for which  $\tau_t + jh_n$  hit in the specified intervals, has a measure can be estimated above by  $C(\omega) \frac{1}{\varphi(n)h_n}$ .

Under the investigation of  $A_n^1(\omega)$  we shall use the previous relations:

$$\begin{aligned}
 A_n^1(\omega) &= \int_{E_n} \left\{ \sum_{k=1}^{m_t} [g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi_n^2(\tau_t + kh_n, \omega)) \right. \\
 &\quad - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi_n^2(\tau_t + (k-1)h_n, \omega))] \\
 &\quad - \sum_{k=1}^{m_t} [g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + kh_n, \omega)) \\
 &\quad \left. - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega))] \right\} dt \leq C(\omega) \frac{1}{\varphi(n)h_n}.
 \end{aligned}$$

Let us consider now  $A_n^2(\omega)$ :



$$\begin{aligned}
 A_n^2(\omega) &\leq \left\| \sum_{k=1}^{m_t} \{ [g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + kh_n, \omega)) \right. \\
 &\quad - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega))] \\
 &\quad - [g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega)) \\
 &\quad \left. - g_n(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega))] \right\| + \\
 &+ \left\| \sum_{k=1}^{m_t} \{ [g(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + kh_n, \omega)) - g(\theta \Pi_n^1(\tau_t + (k-1)h_n, \omega) \right. \\
 &\quad \left. + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega))] \right. \\
 &\quad - [g(\theta \Pi(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k-1)h_n, \omega)) \\
 &\quad \left. - g(\Pi(\tau_t + (k-1)h_n, \omega))] \right\| + \\
 &+ \left\| \sum_{k=1}^{m_t} \{ [g(\theta \Pi(\tau_t + (k-1)h_n, \omega) + (1-\theta)\Pi(\tau_t + kh_n, \omega)) - g(\Pi(\tau_t + (k-1)h_n, \omega))] \right. \\
 &\quad \left. - \sum_{i=1}^{\Pi(\tau, \omega)} [g(\theta \Pi(\mu_i - 0, \omega) + (1-\theta)\Pi(\mu_i, \omega)) - g(\Pi(\mu_i - 0, \omega))] \right\| = \\
 &= A_n^{21}(\omega) + A_n^{22}(\omega) + A_n^{23}(\omega) \\
 A_n^{21}(\omega) &= \left\| \sum_{i=1}^{\Pi(\tau, \omega)} \{ [g_n(\theta \Pi_n^1(\tau_t + (k_i - 1)h_n, \omega) + (1-\theta)\Pi(\tau_t + k_i h_n, \omega)) - g_n(\theta \Pi_n^1(\tau_t \right. \\
 &\quad \left. + (k_i - 1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k_i - 1)h_n, \omega))] \right. \\
 &\quad - [g(\theta \Pi_n^1(\tau_t + (k_i - 1)h_n, \omega) + (1-\theta)\Pi(\tau_t + k_i h_n, \omega)) \\
 &\quad \left. - g(\theta \Pi_n^1(\tau_t + (k_i - 1)h_n, \omega) + (1-\theta)\Pi(\tau_t + (k_i - 1)h_n, \omega))] \right\} \rightarrow 0
 \end{aligned}$$

For almost all  $\omega \in \Omega$ , where  $\tau_i + (k_i - 1)h_n < \mu_i \leq \tau_i + k_i h_n$ .

After implementation of the theorem by Lagrange:



$$A_n^{22}(\omega) = \left\| \sum_{i=1}^{\Pi(\tau, \omega)} \{ [g(\theta \Pi_n^1(\tau_t + (k_i - 1)h_n, \omega) + (1 - \theta)\Pi(\tau_t + k_i h_n, \omega)) - g(\theta \Pi_n^1(\tau_t + (k_i - 1)h_n, \omega) + (1 - \theta)\Pi(\tau_t + (k_i - 1)h_n, \omega))) - [g(\theta \Pi(\tau_t + (k_i - 1)h_n, \omega) + (1 - \theta)\Pi(\tau_t + k_i h_n, \omega)) - g(\Pi(\tau_t + (k_i - 1)h_n, \omega)))] \} \right\| \leq$$

$$\leq \left\| \sum_{i=1}^{\Pi(\tau, \omega)} \theta C[\Pi_n^1(\tau_t + (k_i - 1)h_n, \omega) + \Pi(\tau_t + (k_i - 1)h_n, \omega)] \right\| \longrightarrow 0$$

for almost all  $\omega \in \Omega$ .

$$A_n^{23}(\omega) = \left\| \sum_{i=1}^{\Pi(\tau, \omega)} \{ [g(\theta \Pi(\tau_t + (k_i - 1)h_n, \omega) + (1 - \theta)\Pi(\tau_t + k_i h_n, \omega)) - g(\Pi(\tau_t + (k_i - 1)h_n, \omega))] - [g(\theta \Pi(\mu_i - 0, \omega) + (1 - \theta)\Pi(\mu_i, \omega)) - g(\Pi(\mu_i - 0, \omega))] \} \right\| \rightarrow 0$$

As a result for almost all  $\omega \in \Omega$ .

$$\theta I_n^1 \xrightarrow{L_p(T)} \int_0^t (g(\Pi(s, \omega)) - g(\Pi(s - 0, \omega))) d\Pi(s, \omega) - \sum_{i=1}^{\Pi(s, \omega)} [g(\theta \Pi(\mu_i - 0, \omega) + (1 - \theta)\Pi(\mu_i, \omega)) - g(\Pi(\mu_i - 0, \omega))] = \int_0^t [g(\Pi(s, \omega)) - g(\Pi(s, \omega) - \theta)] d\Pi(s, \omega)$$

at realization of the theorem conditions.

Now let us consider  $I_n^2(t, \omega)$ .

Let us show that

$$I_n^2(t, \omega) \longrightarrow \int_0^t f(\Pi(s, \omega) - \theta) d\Pi(s, \omega) \text{ in the } L_p(T), p \geq 1, \text{ for almost all } \omega \in \Omega.$$

$$\left\| \sum_{k=1}^{m_t} f_n(\Pi_n^\theta(\tau_t + kh_n, \omega)) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k - 1)h_n, \omega)) - \int_0^t f(\Pi(s, \omega) - \theta) d\Pi(s, \omega) \right\| \leq \left\| \sum_{k=1}^{m_t} (f_n(\Pi_n^\theta(\tau_t + kh_n, \omega)) - f(\Pi_n^\theta(\tau_t + kh_n, \omega))) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k - 1)h_n, \omega)) \right\| +$$



$$\begin{aligned}
 & + \left\| \sum_{k=1}^{m_t} [f(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta)(\Pi_n^2(\tau_t + kh_n, \omega)) \right. \\
 & \quad \left. - f(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta)\Pi(\tau_t + kh_n, \omega))] (\Pi_n^2(\tau_t + kh_n, \omega)) \right. \\
 & \quad \left. - \Pi_n^2(\tau_t + (k - 1)h_n, \omega) \right\| + \\
 & + \left\| \sum_{k=1}^{m_t} [f(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta)(\Pi(\tau_t + kh_n, \omega)) - f(\Pi(\tau_t \right. \\
 & \quad \left. + kh_n, \omega))] (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k - 1)h_n, \omega)) - \sum_{i=1}^{\Pi(t, \omega)} [f(i - \theta) - f(i)] \right\| \\
 & + \\
 & + \left\| \sum_{k=1}^{m_t} f(\Pi(\tau_t + kh_n, \omega)) \Pi_n^2(\tau_t + kh_n, \omega) - \Pi(\tau_t + kh_n, \omega) \right\| + \\
 & + \left\| \sum_{k=1}^{m_t} f(\Pi(\tau_t + kh_n, \omega)) \Pi_n^2(\tau_t + (k - 1)h_n, \omega) - \Pi(\tau_t + (k - 1)h_n, \omega) \right\| + \\
 & + \left\| \sum_{k=1}^{m_t} f(\Pi(\tau_t + kh_n, \omega)) (\Pi(\tau_t + kh_n, \omega) - \Pi(\tau_t + (k - 1)h_n, \omega)) - \int_0^t f(\Pi(s, \omega)) d\Pi(s, \omega) \right\| = \\
 & = I_n^{21}(t, \omega) + I_n^{22}(t, \omega) + I_n^{24}(t, \omega) + I_n^{25}(t, \omega) + I_n^{26}(t, \omega).
 \end{aligned}$$

$I_n^{26}(t, \omega) \longrightarrow 0$  by the definition.

Using the representation  $f_n$ , we shall obtain that  $I_n^{21}(t, \omega) \longrightarrow 0$ .

$$\begin{aligned}
 I_n^{22}(t, \omega) & = \left\| \sum_{k=1}^{m_t} [f(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta)\Pi_n^2(\tau_t + kh_n, \omega)) - f(\theta \Pi_n^1(\tau_t + kh_n, \omega) \right. \\
 & \quad \left. + (1 - \theta)\Pi(\tau_t + kh_n, \omega))] (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k - 1)h_n, \omega)) \right\| \leq \\
 & \leq (1 - \theta) C \sum_{k=1}^{m_t} \left( \int_{\mathcal{B}_n} |(\Pi_n^2(\tau_t + kh_n, \omega) - \Pi(\tau_t + kh_n, \omega)) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k - 1)h_n, \omega))|^p dt \right)^{1/p} \leq \\
 & \leq \frac{C}{\varphi(n)h_n} \longrightarrow 0
 \end{aligned}$$

at the realization of the theorem conditions.





$$\begin{aligned}
 I_n^{23}(t, \omega) &= \left\| \sum_{k=1}^{m_t} [f(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta)(\Pi(\tau_t + kh_n, \omega)) - f(\Pi(\tau_t + kh_n, \omega))) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k-1)h_n, \omega)) - \sum_{i=1}^{\Pi(t, \omega)} [f(i - \theta) - f(i)] \right\| = \\
 &= \left( \int_{B_n} \left| \sum_{k=1}^{m_t} [f(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta)(\Pi(\tau_t + kh_n, \omega)) - f(\Pi(\tau_t + kh_n, \omega))) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k-1)h_n, \omega)) - \sum_{i=1}^{\Pi(t, \omega)} [f(i - \theta) - f(i)] \right|^p dt \right. \\
 &\quad \left. + \int_{T/B_n} \left| \sum_{k=1}^{m_t} [f(\theta \Pi_n^1(\tau_t + kh_n, \omega) + (1 - \theta)(\Pi(\tau_t + kh_n, \omega)) - f(\Pi(\tau_t + kh_n, \omega))) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi_n^2(\tau_t + (k-1)h_n, \omega)) - \sum_{i=1}^{\Pi(t, \omega)} [f(i - \theta) - f(i)] \right|^p dt \right)^{1/p} \leq \\
 &\leq \left( \frac{C}{\varphi(n)h_n} + \int_{T/B_n} \left| \sum_{i=1}^{\Pi(t, \omega)} [f(\theta \Pi_n^1(\tau_t + k_i h_n, \omega) + (1 - \theta)\Pi(\tau_t + k_i h_n, \omega)) - f(i - \theta)] \right|^p dt \right)^{1/p} \leq \\
 &\leq \left( \frac{C}{\varphi(n)h_n} + C \int_{T/B_n} \left| \sum_{i=1}^{\Pi(t, \omega)} \theta \int_0^{1/n} (\Pi(\tau_t + k_i h_n - s, \omega) - \Pi(\tau_t + (k_i - 1)h_n - s, \omega)) P_n(s) ds \right|^p dt \right)^{1/p} \leq \\
 &\leq \left( \frac{C}{\varphi(n)h_n} + C(nh_n)^p \right)^{1/p} \longrightarrow 0.
 \end{aligned}$$

$$\begin{aligned}
 I_n^{24}(t, \omega) &= \left\| \sum_{k=1}^{m_t} f(\Pi(\tau_t + kh_n, \omega)) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi(\tau_t + kh_n, \omega)) \right\| = \\
 &= \left( \int_{B_n} \left| \sum_{k=1}^{m_t} f(\Pi(\tau_t + kh_n, \omega)) (\Pi_n^2(\tau_t + kh_n, \omega) - \Pi(\tau_t + kh_n, \omega)) \right|^p dt \right)^{1/p} \leq \frac{C}{\varphi(n)h_n} \longrightarrow 0.
 \end{aligned}$$

Similarly,  $I_n^{25}(t, \omega) \longrightarrow 0$ .

Thus validity of the theorem is established.

Note: the theorems 1,2 and 3 are proved similarly.



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