A Note on the Perron Value of Brualdi-Li Matrices

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ABSTRACT

Let $\mathbf{B_{2n}}$ denote the Brualdi-Li matrix, and let $\boldsymbol{\rho_{2n}} = \boldsymbol{\rho}(\mathbf{B_{2n}})$ denote the Perron value of the Brualdi-Li matrix of order 2n. We prove that $\mathbf{n^3}\left(\mathbf{n} - \frac{\mathbf{1}}{\mathbf{2}} - \frac{\boldsymbol{\gamma}}{\mathbf{4n}} - \boldsymbol{\rho_{2n}}\right)$ is monotonically decreasing for all $\mathbf{n} > 1$, where $\mathbf{e}$=2.71828⋯, $\boldsymbol{\gamma} = \frac{\mathbf{e^2-1}}{\mathbf{e^2+1}}$=0.76159⋯ .

Keywords

Tournament matrix; Brualdi-Li matrix;Perron value.

SUBJECT CLASSIFICATION

AMS Classification : 05C20, 05C50,15A18, 15B34, 15B48.

TYPE

Research article

1.INTRODUCTION

We denote by J the matrix of all ones and by I the identity matrix.Tournament matrices are square matrices T with entries in$\{0,1\}$and suchthat$T + T' = J - I$. An $n \times n$ tournament matrix T is called regular if eachof the row sums ofTis$\frac{n-1}{2}$. Since the row sums are integers, a tournamentmatrix can only be regular if n is odd and in this case it is known that regular tournament matrices maximize the Perron value over the class of all$n \times n$tournament matrices. For even n Brualdi and Li [1] conjectured thatthematrix$B_{2n} = \begin{pmatrix} U_n & U_n' \\ I + U_n' & U_n \end{pmatrix}$, where U_n is strictly upper triangular tournament matrix (all of whose entriesabove the main diagonal are equal to one), i.e.

$$U_n = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is among the tournament matrices that maximize the Perron value. Thisconjecture has recently been confirmed in [2]. The matrixB_{2n} is called theBrualdi-Li matrix.

Let $\rho_{2n} = \rho(B_{2n})$denote the Perron value of the Brualdi-Li matrix oforder$2n$. Friedland and Katz [3] paid particular attention to the behavior ofthe Perron value of B_{2n} and posed the problem of determining the behaviorof the sequence $2n\left(n - \frac{1}{2} - \rho_{2n}\right)$. Steve Kirkland [4] showed that

$$\lim_{n\to\infty} n^3\left(n - \frac{1}{2} - \frac{\gamma}{4n} - \rho_{2n}\right) = \frac{\gamma^2}{16} + \frac{e^2(3\gamma+1)}{12(e^2+1)^2},$$

where$e = 2.71828\cdots, \gamma = \frac{e^2-1}{e^2+1} = 0.76159\cdots$. There are several interestingproperties of the Perron value of the Brualdi-Li matrix, see e.g. [5]~[11].We investigate the sequence $n^3\left(n - \frac{1}{2} - \frac{\gamma}{4n} - \rho_{2n}\right)$and obtain the followingresults.

Theorem 1.1 Let$n \geq 2$ be an integer, and let $\rho_{2n} = \rho(B_{2n})$ be thePerron value of the Brualdi-Li matrix B_{2n}: Then $n^3\left(n - \frac{1}{2} - \frac{\gamma}{4n} - \rho_{2n}\right)$is monotonically decreasing for all n, where $e = 2.71828\cdots, \gamma = \frac{e^2-1}{e^2+1} = 0.76159\cdots$.

By theorem 1.1 and lemma 2.5, one can easily prove the following corollaries.

Corollary 1.2 Let$n \geq 2$ be an integer, and let $\rho_{2n} = \rho(B_{2n})$ be thePerron value of the Brualdi-Li matrix B_{2n}. Then

$$\rho_{2n} < n - \frac{1}{2} - \frac{\gamma}{4n} - \left(\frac{\gamma^2}{16} + \frac{e^2(3\gamma+1)}{12(e^2+1)^2}\right)\frac{1}{n^3}.$$

We give some lemmas in section 2 and prove theorem 1.1 in section 3.

2. SOME LEMMAS

For two variables, the Implicit Function Theorem is as follow.

Implicit Function Theorem If a real-valued function F is defined onan open disk containing point(x_0, y_0), satisfying

(i) $F(x_0, y_0) = 0$,

(ii)$\frac{\partial F}{\partial y}\Big|_{(x_0,y_0)} \neq 0$,

(iii) $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$are continuous on the open disk,

then the equation $F(x, y)$ = 0 defines y as a function of $x$near the point(x_0, y_0) and the derivative of this function is given by$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$, where$\frac{\partial F}{\partial x}and\frac{\partial F}{\partial y}$indicate the partial derivatives of F with respect to x and$y$.

Lemma 2.1([4]) Let$n \geq 2$ be an integer, and let $\rho_{2n} = \rho(B_{2n})$be thePerron value of the Brualdi-Li matrix B_{2n}. Then

$$(2\rho_{2n}^2 + 2(1-n)\rho_{2n} + 1 - n)\left(\left(1 + \frac{1}{\rho_{2n}}\right)^{2n} + 1\right) - 1 = 0.$$

It is wellknownthat $n - 1 < \rho_{2n} < n$.

Let$F = F(m,\rho) = (2\rho^2 + 2(1-m)\rho + 1 - m)\left((1+\frac{1}{\rho})^{2m} + 1\right) - 1andD = \{(m,\rho)|m > 2, m-1 < \rho < m, m$ and ρ are real numbers$\}$, Then the partial derivatives of$F$ are

$$\frac{\partial F}{\partial m} = -(2\rho+1)\left(\left(1+\frac{1}{\rho}\right)^{2m} + 1\right) + 2(2\rho^2 + 2(1-m)\rho + 1 - m)\left(1+\frac{1}{\rho}\right)^{2m} ln(1+\frac{1}{\rho})\text{and}$$

$$\frac{\partial F}{\partial \rho} = 2(2\rho - m + 1)\left(\left(1+\frac{1}{\rho}\right)^{2m} + 1\right) - \frac{2m}{\rho^2}(2\rho^2 + 2(1-m)\rho + 1 - m)\left(1+\frac{1}{\rho}\right)^{2m-1}.$$

Evidently,$\frac{\partial F}{\partial x}$ and$\frac{\partial F}{\partial x}$are derivable on the open disk $D$and

$$\frac{\partial F}{\partial \rho} = 2(2\rho - m + 1) + 2\left((2\rho - m + 1)\left(1+\frac{1}{\rho}\right) - \frac{m}{\rho^2}(2\rho^2 + 2(1-m)\rho + 1 - m)\right)\left(1+\frac{1}{\rho}\right)^{2m-1} > 0,\ (m,\rho) \in D.$$

By the Implicit Function Theorem, the equation$F(m,\rho) = 0$defines ρ asa function in terms of m and the derivative ofthis function is given by

$$\frac{d\rho}{dm} = -\frac{\frac{\partial F}{\partial m}}{\frac{\partial F}{\partial \rho}}.$$

Lemma 2.2[6] Let m >1 be a real number, and letρ be a functioninterms of m satisfying the equation

$$F(m,\rho) = (2\rho^2 + 2(1-m)\rho + 1 - m)\left((1+\frac{1}{\rho})^{2m} + 1\right) - 1, (m,\rho) \in D.$$

Then

$$m - \frac{1}{2} - \frac{1}{5m} < \rho < m - \frac{1}{2} - \frac{\gamma}{4m},$$

where $\gamma = \frac{e^2-1}{e^2+1} = 0.76159\cdots$.

Lemma 2.3 Let$m \geq 3$ be a real number, and let ρbe a function interms of m satisfying the equation $F(m,\rho) = 0$, and let

$$G = G(m,\rho) = 2\rho^2 + 2(1-m)\rho + 1 - m,\ (m,\rho) \in D.$$

Then

$$0 < G(1-G) < \frac{9}{20}\left(1 - \gamma + \frac{\gamma^2}{4m^2}\right).$$

ProofForfixed real number $m \geq 3$, $G(m,\rho)$is continuous real functionin terms of ρ.

Let$\rho_1 = m - \frac{1}{2} - \frac{1}{5m}and\rho_2 = m - \frac{1}{2} - \frac{\gamma}{4m}$, then

$$G(m,\rho_1) = \frac{1}{10} + \frac{2}{25m^2}, \qquad G(m,\rho_2) = \frac{1-\gamma}{2} + \frac{\gamma^2}{8m^2}.$$

By lemma 2.2, we have

$$G(m,\rho_1) < G(m,\rho) = 2\rho^2 + 2(1-m)\rho + 1 - m < G(m,\rho_2).$$

Hence

$$\frac{1}{10} + \frac{2}{25m^2} < G(m,\rho) < \frac{1-\gamma}{2} + \frac{\gamma^2}{8m^2},$$

that is

$$0 < \frac{1-\gamma}{2} - \frac{\gamma^2}{8m^2} < 1 - G(m,\rho) < \frac{9}{10} - \frac{2}{25m^2}.$$

We have the desired result.

Lemma 2.4([6])Let$m \geq 3$ be a real number, and let ρbe a function interms of m satisfying the equation $F(m,\rho) = 0$, then

$$\frac{d\rho}{dm} = \frac{2\rho(1+\rho)(G-1)G\,ln\left(1+\frac{1}{\rho}\right) + \rho(1+\rho)(2\rho+1)}{2m(G-1)G + \rho(1+\rho)(4\rho+2-2m)}.$$

Lemma 2.5([4])

$$\lim_{n\to\infty} n^3\left(n - \frac{1}{2} - \frac{\gamma}{4n} - \rho_{2n}\right) = \frac{\gamma^2}{16} + \frac{e^2(3\gamma+1)}{12(e^2+1)^2},$$

Where$e = 2.71828\cdots, \gamma = \frac{e^2-1}{e^2+1} = 0.76159\cdots$.

3. PROOF OF THEOREM 1.1

In section 2,$F = F(m,\rho) = (2\rho^2 + 2(1-m)\rho + 1 - m)\left((1+\frac{1}{\rho})^{2m} + 1\right) - 1andG = G(m,\rho) = 2\rho^2 + 2(1-m)\rho + 1 - m$, $(m,\rho) \in D$. $\rho$is a function interms of m satisfying the equation $F(m,\rho) = 0$. Clearly, F and $G$are thefunctions in terms of m.

Let $m \geq 3$ be a real number,

$$F_1 = \rho(1+\rho)(4\rho + 2 - 2m) - 2m(1-G)G,$$

$$F_2 = 8m^2 - \gamma - 3(1+2\rho)m - 2m\rho(1+\rho)\ln(1+\frac{1}{\rho}),$$

$$F_3 = m^2(2\rho+1) + (2\rho+1-m)(\gamma + 3(1+2\rho)m - 8m^2).$$

By lemma 2.3, then

$$F_1 = \rho(1+\rho)(4\rho+2-2m) - 2m(1-G)G$$

$$> (m-1)m(4(m-1)+2-2m) - 2m\cdot\frac{9}{20}\left(1-\gamma+\frac{\gamma^2}{4m^2}\right)$$

$$=2m\left((m-1)^2 - \frac{9}{20}(1-\gamma+\frac{\gamma^2}{4m^2})\right)>0.$$

By lemma 2.2 and Taylor's Theorem, we have

$$F_2 = 8m^2 - \gamma - 3(1+2\rho)m - 2m\rho(1+\rho)\ln(1+\frac{1}{\rho})$$

$$> 8m^2 - \gamma - 3(1+2\rho)m - 2m\rho(1+\rho)\left(\frac{1}{\rho} - \frac{1}{2\rho^2} + \frac{1}{3\rho^3}\right)$$

$$= 8m^2 - \gamma - 3(1+2\rho)m - 2m\rho(1+\rho)\left(\frac{1}{2} + \rho - \frac{1}{6\rho} + \frac{1}{3\rho^2}\right)$$

$$= 8m^2 - \gamma - 4m - 8m\rho + \frac{m}{3\rho} - \frac{2m}{3\rho^2}$$

$$> 8m^2 - \gamma - 4m - 8m(m - \frac{1}{2} - \frac{\gamma}{4m}) + \frac{m}{3m} - \frac{2m}{3(m-1)^2}$$

$$= \gamma + \frac{1}{3} - \frac{2m}{3(m-1)^2} > 0.$$

By lemma 2.2, we have

$$F_3 = m^2(2\rho+1) + (2\rho+1-m)(\gamma + 3(1+2\rho)m - 8m^2)$$

$$> m^2\left(2\left(m-\frac{1}{2}-\frac{1}{5m}\right)+1\right)-\left(2\left(\mathrm{m}-\frac{1}{2}-\frac{\gamma}{4m}\right)+1-m\right)\left(8m^2-\gamma+3m\left(1+2(m-\frac{1}{2}-\frac{1}{5m})\right)\right)$$

$$=(2\gamma-1)\mathrm{m}+\frac{3\gamma}{10}-\gamma^2>0.$$

By lemma 2.4, we have

$$\frac{d\rho}{dm}-4+\frac{3}{2m}+\frac{3\rho}{m}+\frac{\gamma}{2m^2}$$

$$=\frac{d\rho}{dm}+\frac{\gamma+3(1+2\rho)m-8m^2}{2m^2}$$

$$=\frac{2\rho(1+\rho)(G-1)G\,ln\left(1+\frac{1}{\rho}\right)+\rho(1+\rho)(2\rho+1)}{2m(G-1)G+\rho(1+\rho)(4\rho+2-2m)}+\frac{\gamma+3(1+2\rho)m-8m^2}{2m^2}$$

$$=\frac{2\mathrm{m}^2\left(2\rho(1+\rho)(\mathrm{G}-1)\mathrm{G}\ln\left(1+\frac{1}{\rho}\right)+\rho(1+\rho)(2\rho+1)\right)}{2\mathrm{m}^2\big(2\mathrm{m}(\mathrm{G}-1)\mathrm{G}+\rho(1+\rho)(4\rho+2-2\mathrm{m})\big)}$$

$$+\frac{\big(2\mathrm{m}(\mathrm{G}-1)\mathrm{G}+\rho(1+\rho)(4\rho+2-2\mathrm{m})\big)(\gamma+3(1+2\rho)m-8m^2)}{2m^2\big(2\mathrm{m}(\mathrm{G}-1)\mathrm{G}+\rho(1+\rho)(4\rho+2-2\mathrm{m})\big)}$$

$$=\frac{m(1-\mathrm{G})\mathrm{G}\left(8m^2-\gamma-3(1+2\rho)m-2m\rho(1+\rho)\ln\left(1+\frac{1}{\rho}\right)\right)}{\mathrm{m}^2(\rho(1+\rho)(4\rho+2-2\mathrm{m})-2\mathrm{m}(1-\mathrm{G})\mathrm{G})}$$

$$+\frac{\rho(1+\rho)\big(m^2(2\rho+1)+(2\rho+1-\mathrm{m})(\gamma+3(1+2\rho)m-8m^2)\big)}{m^2(\rho(1+\rho)(4\rho+2-2\mathrm{m})-2\mathrm{m}(1-\mathrm{G})\mathrm{G})}$$

$$=\frac{\mathrm{m}(1-\mathrm{G})\mathrm{G}\mathrm{F}_2+\rho(1+\rho)\mathrm{F}_3}{\mathrm{m}^2\mathrm{F}_1}.$$

Due to $(1-G)G\ >\ 0, F_i>0, i=\ 1,2,3$, then

$$\frac{d\rho}{dm}-4+\frac{3}{2m}+\frac{3\rho}{m}+\frac{\gamma}{2m^2}>0.$$

Hence, as $\mathrm{m}\geq 3$,

$$\frac{d}{dm}\left(m^3\left(m-\frac{1}{2}-\frac{\gamma}{4m}-\rho\right)\right)=-m^3\left(\frac{d\rho}{dm}-4+\frac{3}{2m}+\frac{3\rho}{m}+\frac{\gamma}{2m^2}\right)<0.$$

To sum up, we obtain the following result.

Let$\mathrm{m}\geq\ 3$ be a real number, and let ρbe a function of m satisfying theequation $F(m,\rho)=0,(m,\rho)\in D$,then $m^3\left(m-\frac{1}{2}-\frac{\gamma}{4m}-\rho\right)$is a monotonicallydecreasing function for all m.

Note that $\rho_{2n}=\rho(B_{2n})$ is the Perron value of the Brualdi-Li matrix B_{2n},bylemma 2.1, ρ_{2n} satisfy the equation

$$F(n,\rho_{2n})=(2\rho_{2n}^2+2(1-n)\rho_{2n}+1-m)\left((1+\frac{1}{\rho_{2n}})^{2n}+1\right)-1.$$

Where $\mathrm{n}\geq 3$ is an integer. It is easy to see that$n^3\left(n-\frac{1}{2}-\frac{\gamma}{4n}-\rho_{2n}\right)$ is a monotonicallydecreasing function for all $n\geq 3$.

Since$2^3\left(2-\frac{1}{2}-\frac{0.76159}{8}-\rho_{2\times2}\right)=8\left(1.5-\frac{0.76159}{8}-1.3953\right)=0.07601$

$$>3^3\left(3-\frac{1}{2}-\frac{0.76159}{12}-\rho_{2\times3}\right)=27\left(2.5-\frac{0.76159}{12}-2.4340\right)=0.06842.$$

Evidently, it follows that $n^3\left(n-\frac{1}{2}-\frac{\gamma}{4n}-\rho_{2n}\right)$ismonotonically decreasingfor all integers $\mathrm{n}\geq 2$. We complete the proof of Theorem 1.1.

We close the paper by giving some numerical data. For $n=3,4,\cdots,20$. Wecomputed the quantities using MATLAB. Here are the results:

n	$\rho(B_{2n})$	$n-\frac{1}{2}-\frac{\gamma}{4n}-\left(\frac{\gamma^2}{16}+\frac{e^2(3\gamma+1)}{12(e^2+1)^2}\right)\frac{1}{n^3}$

3	2.4339681553	2.434126720545780
4	3.4513485104	3.451384870028336
5	4.4613886265	4.461400358647304
6	5.4679613465	5.467966022694605
7	6.4726085757	6.472610728557371
8	7.4760721450	7.476073245723328
9	8.4787548455	8.478755454934886
10	9.4808947951	9.480895154406742
11	10.4826419899	10.482642212688186
12	11.4840957000	11.484095844150017
13	12.4853242800	12.485324376520939
14	13.4863763527	13.486376419338121
15	14.4872874601	14.487287507269073
16	15.4880841901	15.488084224200309
17	16.4887868322	16.488786857426792
18	17.4894111405	17.489411159408988
19	18.4899695344	18.489969548874562
20	19.4904719379	19.490471949088757

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