



HPM for Solving the Time-fractional Coupled Burger's Equations

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ABSTRACT

This paper is devoted to derive the approximate solutions for the time-fractional (Caputo sense) coupled Burger's equations with implementation of the homotopy perturbation method. The numerical results are compared with the exact solution. The results reveal that the presented technique is very effective and simple.

Keywords: Fractional differential equations; Caputo's derivative; Homotopy perturbation method.

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INTRODUCTION

In this paper, we will implement one of these methods, namely, homotopy perturbation method (HPM) ([7]-[10]), to solve the proposed problem ([1], [3], [6], [12]). Unlike the traditional numerical methods, such as, the finite element method [5] and the finite difference method [19], HPM does not need, discretization or linearization. Recently, the applications of homotopy theory among scientists appeared, and the homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with perturbation theory. Many authors (see [4], [11], [15]-[17]) are pointed out that HPM can overcome the difficulties arising in calculation of Adomian's polynomials in Adomian decomposition method.

Definition 1.

The Caputo fractional derivative of the function $g(t)$ is defined as follows:

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha \leq n, \quad t > 0. \quad (1)$$

In the following proposition, we will state a basic property.

Proposition 1. [13]

If $m-1 < \alpha < m$, $m \in \mathbb{N}$, and $f(x) \in C_\mu^m$, $\mu \geq -1$, then:

$$D^\alpha J^\alpha f(x) = f(x), \quad \text{and} \quad J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}, \quad x > 0. \quad (2)$$

The Caputo fractional derivative operator is satisfied the linear relationship and the so-called Leibnitz rule, that's:

$$\begin{aligned} D^\alpha [c_1 f(x) + c_2 g(x)] &= c_1 D^\alpha f(x) + c_2 D^\alpha g(x); \\ D^\alpha [f(x)g(x)] &= \sum_{r=0}^{\infty} \binom{\alpha}{r} f^{(r)}(x) D^{\alpha-r} g(x). \end{aligned} \quad (3)$$

where c_1 and c_2 are constants, $f(x)$ is continuous in $[a, x]$ and $g(x)$ has $n+1$ continuous derivatives in $[a, x]$. For the Caputo's derivative we have $D^\alpha C = 0$, C is a constant, and [18]



$$D^{\alpha} x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}.$$

(4)

For more details on fractional derivatives definitions and their properties see ([13], [18]).

Approximate Solution using HPM

In this section, we will illustrate how implement HPM to obtain the approximate solutions of the fraction non-linear coupled system of Burger's equations of the following form:

$$D^{\alpha} u - u_{xx} - 2uu_x + (uv)_x = 0, \quad 0 < \alpha \leq 1, t > 0, \quad (5)$$

$$D^{\beta} v - v_{xx} - 2vv_x + (uv)_x = 0, \quad 0 < \beta \leq 1, \quad (6)$$

subject to the following initial conditions:

$$u(x, 0) = v(x, 0) = f(x) = \sin(x). \quad (7)$$

The exact solution of (5)-(6) for the special case $\alpha = \beta = 1$ is given by:

$$u(x, t) = v(x, t) = e^{-t} \sin(x). \quad (8)$$

In fact, different response systems can be obtained when at less one of the parameters α and β varies. when $\alpha = \beta = 1$ the fractional equations reduce to the classical coupled Burger's equations.

Now, we will apply the HPM to obtain the approximate solutions of the system (5)-(6).

Using the HPM, we can construct the following simple homotopies:

$$H_1(u, q) = D^{\alpha} u - q[u_{xx} + 2uu_x - (uv)_x] = 0, \quad (9)$$

$$H_2(v, q) = D^{\beta} v - q[v_{xx} + 2vv_x - (uv)_x] = 0, \quad (10)$$

where $q \in [0, 1]$ is an embedding parameter. In case $q = 0$, Eqs. (9)-(10) become linear equations, which are easy to solve, and when $q = 1$, Eqs.(9)-(10) turn out to be the original equations (5)-(6).

Now, we will use the homotopy parameter q to expand the solutions:

$$u(x, t) = u_0 + qu_1 + q^2 u_2 + q^3 u_3 + \dots, \quad (11)$$

$$v(x, t) = v_0 + qv_1 + q^2 v_2 + q^3 v_3 + \dots, \quad (12)$$

the approximate solution can be obtained by setting $q = 1$ in equations (11)-(12):

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots, \quad v(x, t) = v_0 + v_1 + v_2 + v_3 + \dots. \quad (13)$$

Substituting from (11)-(12) into (5) and (6), respectively, and equating the terms with the identical powers of q , we can obtain a system of linear equations. Here we only write the first few linear equations:



$$D^\alpha u_0(x, t) = 0, \quad u_0(x, 0) = \sin(x), \quad (14)$$

$$D^\beta v_0(x, t) = 0, \quad v_0(x, 0) = \sin(x), \quad (15)$$

$$D^\alpha u_1(x, t) = u_{0xx} + 2u_0 u_{0x} - (u_0 v_{0x} + v_0 u_{0x}), \quad u_1(x, 0) = 0, \quad (16)$$

$$D^\beta v_1(x, t) = v_{0xx} + 2v_0 v_{0x} - (u_0 v_{0x} + v_0 u_{0x}), \quad v_1(x, 0) = 0, \quad (17)$$

$$D^\alpha u_2(x, t) = u_{1xx} + 2(u_0 u_{1x} + u_1 u_{0x}) - (u_0 v_{1x} + u_1 v_{0x} + v_0 u_{1x} + v_1 u_{0x}), \quad u_2(x, 0) = 0, \quad (18)$$

$$D^\beta v_2(x, t) = v_{1xx} + 2(v_0 v_{1x} + v_1 v_{0x}) - (u_0 v_{1x} + u_1 v_{0x} + v_0 u_{1x} + v_1 u_{0x}), \quad v_2(x, 0) = 0. \quad (19)$$

Take the inverse operator J^α and J^β on both sides of the Eqs.(14)-(19), we will use the initial conditions in the following form:

$$u_0(x, t) = \sin(x), \quad v_0(x, t) = \sin(x). \quad (20)$$

So, we can obtain the approximate solutions of the system (14)-(19) in the following form:

$$u_1(x, t) = J^\alpha [u_{0xx} + 2u_0 u_{0x} - (u_0 v_{0x} + v_0 u_{0x})] = -\sin(x) \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad (21)$$

$$v_1(x, t) = J^\beta [v_{0xx} + 2v_0 v_{0x} - (u_0 v_{0x} + v_0 u_{0x})] = -\sin(x) \frac{t^\beta}{\Gamma(1+\beta)}, \quad (22)$$

$$\begin{aligned} u_2(x, t) &= J^\alpha [u_{1xx} + 2(u_0 u_{1x} + u_1 u_{0x}) - (u_0 v_{1x} + u_1 v_{0x} + v_0 u_{1x} + v_1 u_{0x})] \\ &= \sin(x)(1 - 3 \cos(x)) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \sin(x) \cos(x) \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}, \end{aligned} \quad (23)$$

$$\begin{aligned} v_2(x, t) &= J^\beta [v_{1xx} + 2(v_0 v_{1x} + v_1 v_{0x}) - (u_0 v_{1x} + u_1 v_{0x} + v_0 u_{1x} + v_1 u_{0x})] \\ &= \sin(x)(1 - 3 \cos(x)) \frac{t^{2\beta}}{\Gamma(1+2\beta)} + \sin(x) \cos(x) \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}, \dots \end{aligned} \quad (24)$$

Therefore, the approximate solution can be presented by the same iterative process, in the following series form:

$$u(x, t) = f - f \frac{t^\alpha}{\Gamma(1+\alpha)} + f(1 - 3f_x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f f_x \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \dots, \quad (25)$$

$$v(x, t) = f - f \frac{t^\beta}{\Gamma(1+\beta)} + f(1 - 3f_x) \frac{t^{2\beta}}{\Gamma(1+2\beta)} + f f_x \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \dots \quad (26)$$

Where $f = f(x) = \sin(x)$. In order to illustrate the advantages and the accuracy of HPM for solving the presented problem, we calculate the second order approximation, i.e.:

$$u_{HPM}(x, t) = u_0 + u_1 + u_2 + \dots, \quad v_{HPM}(x, t) = v_0 + v_1 + v_2 + \dots \quad (27)$$

The behavior of the approximate solution is given the figure 1 and compared with the exact solution in figure 2.

Conclusion and General Remarks

In this article, we implemented HPM for obtaining the approximate solution of the time-fractional coupled Burger's equations. From the introduced numerical results, we can conclude the following properties of the presented method:

- (1) HPM gives a simple and a powerful mathematical tool for solving the FDEs;
- (2) HPM is an efficient and powerful method for solving a wide class of fractional differential equations;
- (3) HPM is straightforward without any restrictive assumptions and special techniques;
- (4) HPM gives the approximate solution with higher accuracy and fast convergence rate.

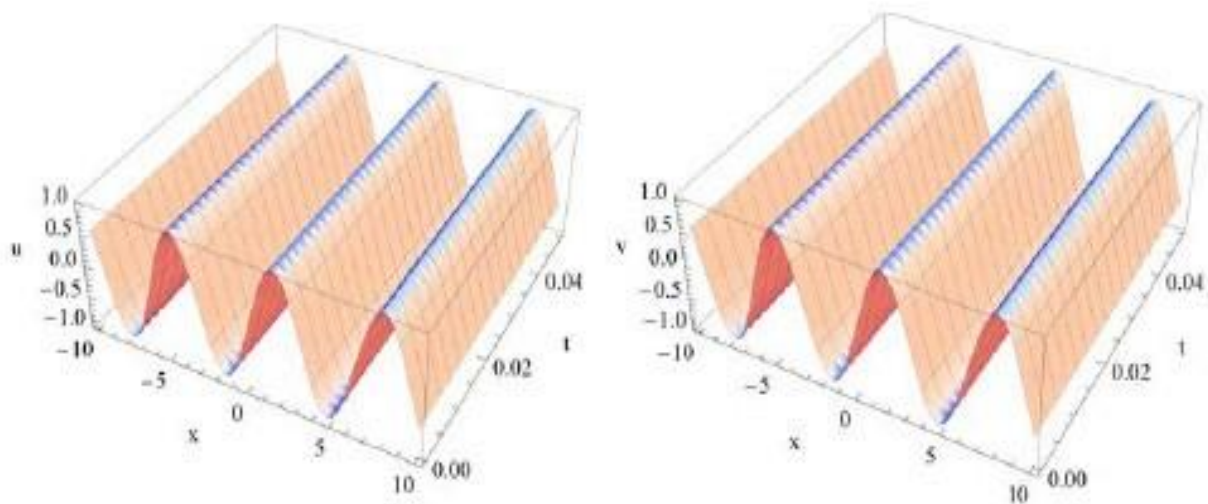


Figure 1: The behavior of the approximate solutions $u_{HPM}(x,t)$ (Left), $v_{HPM}(x,t)$ (Right).

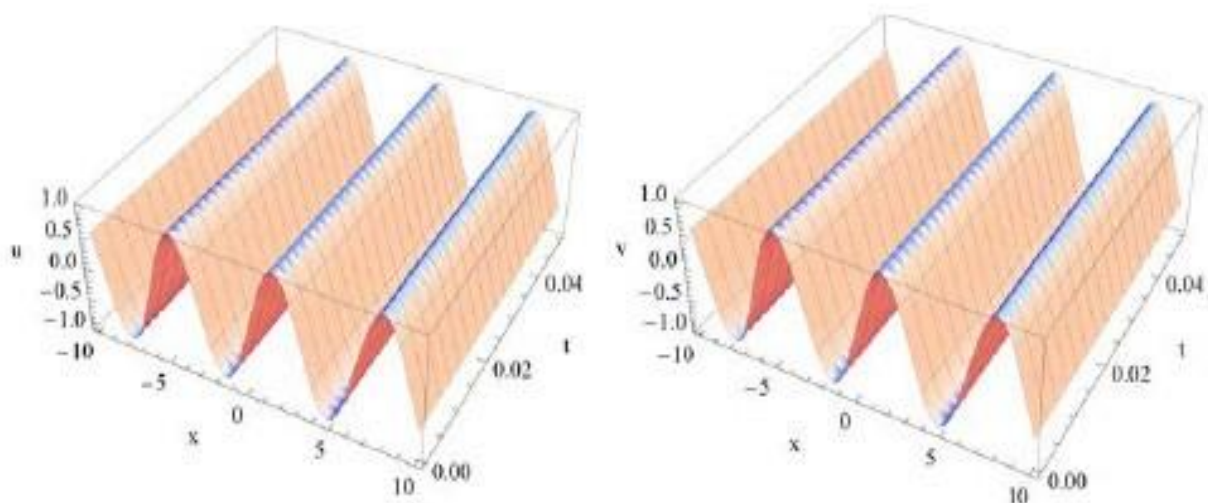


Figure 2: The behavior of the exact solution $u(x,t)$ (Left), $v(x,t)$ (Right).



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Author' biography

Khadijah Mohammed Abualnaja was born in 1974 in KSA. She received her B.S. degree from Taif University, M.S. and Ph.D degrees from King Abdulaziz University, in 1998, 2006 and 2010, respectively, and is currently Associate Professor in the Mathematics and Statistic Department of Taif University, KSA. Her main research interests are the Numerical Analysis for all kinds of equations.

