Weak injective dimension and almost perfect rings

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Abstract

In this paper*, we study the weak-injective dimension and we characterize the global weak-injective dimension of rings. After we study the transfer of the global weak-injective dimension in some known ring construction. Finally we study the transfer of almost perfect property in pullback and D+M constructions.

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1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital. For an R-module M, we use $pd_R(M)$ to denote the usual projective dimension of M. $gldim(R)$ and $wdim(R)$ are, respectively, the classical global and weak global dimensions of R.

In 2006, Lee in [9] introduced the class of weak injective modules which are generalization of cotorsion modules.

Definition 1.1 An R-module M is said to be weak-injective if $Ext_R^1(F,M)=0$ for all R-modules F of flat dimension $\leq$ 1.

After in 2009, Fuchs and Lee in [8] introduced the weak injective dimension of a module M to be the smallest integer n such that $Ext_R^{n+1}(F,M)=0$ for all R-modules F of $fd(F)\leq 1$. And they introduced also the global weak-injective dimension of ring R to be the supremum of weak-injective dimension of all R-modules.

On the other hand in 2003, the notion of almost perfect ring has been introduced by Bazzoni and Salce in [1].

Definition 1.2 A ring R is almost perfect if R/I is perfect for any proper ideal I of R.

The main aim of this paper is to study the transfer of weak-injective global dimension of ring and almost perfect property to polynomial rings direct product of rings and pullbacks constructions. Also we study some properties of weak-injective modules and dimension.

In Section 2 we study the class of weak injective modules, in Proposition 2.2 we show the behavior theorem of weak injective modules and we see that they are stable with direct product. After we see the definition and the characterization of weak injective dimension Theorem 2.1.

In Section 3 we study weak-injective global dimension, in Proposition 3.2 we give its characterization. Also we see the relation between weak-injective global dimension and the global dimension and the cotorsion global dimension. In the end of this section we see the characterization of perfect and almost perfect rings using weak-injective global dimension.

In Section 4 we give the main results of this paper in studying the transfer of weak-injective global dimensions in polynomial rings Theorem 4.1, direct product of rings 4.3 and $D+M$ constructions Theorem 4.5.

2 Weak-injective dimension of modules

In this section we study the properties of weak injective modules. After we characterize the weak-injective dimension of module.

We start by giving a characterization of weak-injective modules.

Proposition 2.1 Let M be an R-module. Then M is weak-injective if and only if $Ext_R^i(F,M)=0$ for any $i>0$ and for R-module F of flat dimension ≤ 1

Proof: We prove by induction on i. If $i=1$ it follows from the definition, suppose that it for $i-1$ and we prove it for i. Let F be a R-module of flat dimension ≤ 1. Applying the long exact sequence of the functor $Hom_R(.,M)$ to the short exact sequence of R-modules $0\to K\to L\to F\to 0$ where L is free and K is flat,

we have for any $i>0$:

$$0=Ext^{i-1}(L,M)\to Ext^{i-1}(K,M)\to Ext^{i}(F,M)\to Ext^{i}(L,M)=0.$$

Since K is flat and by induction $Ext^{i-1}(K,M)=0$, then $Ext^{i}(F,M)=0$.

In the following proposition we show that weak-injective modules behave in short exact sequence and it is stable over direct product.

Proposition 2.2

1. Let $0\rightarrow A\rightarrow W\rightarrow B\rightarrow 0$ be a short exact sequence of R-modules, such that W is a weak-injective module. If A is weak-injective, then so is B.

2. Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then $\prod_{i\in I}M_i$ is a weak-injective module if and only if every M_i is weak-injective.

Proof:

1. Suppose that A is weak-injective and let F be an R-module of $fd(F)\leq 1$, applying the functor $Hom_R(F,.)$ to the short exact sequence $0\rightarrow A\rightarrow W\rightarrow B\rightarrow 0$, we get:

$$\cdots\rightarrow Ext_R(F,W)\rightarrow Ext_R(F,B)\rightarrow Ext_R^2(F,A)\rightarrow Ext_R^2(F,W),$$

since W and A are weak-injective we have $Ext_R(F,W)=Ext_R^2(F,A)=0$. Then also $Ext_R(F,B)=0$ and B is weak-injective as desired.

2. Follows from the isomorphism $Ext^n(F,\prod_i A_i)\cong\prod_i Ext^n(F,A_i)$ (see [10, Theorem 7.14]).

New we give the definition of weak-injective dimension introduced by Fuchs and Lee in [8].

Definition 2.3 The weak-injective dimension of an R-module M is the smallest integer n such that $Ext_R^{n+1}(F,M)=0$ for all R-modules F of $fd(F)\leq 1$, denoted $wid(M)=n$

This result is a characterization of weak-injective dimension

Theorem 2.4 Let M be an R-module, the following conditions are equivalent for a positive integer n:

1. $wid(M)\leq n$;
2. $Ext_R^{n+1}(F,M)=0$ for any R-module F of $fd(F)\leq 1$.
3. $Ext_R^{n+i}(F,M)=0$ for any $i>0$ and for R-module F of $fd(F)\leq 1$.
4. For any exact sequence $0\rightarrow M\rightarrow W_0\rightarrow W_1\rightarrow\cdots\rightarrow W_{n-1}\rightarrow W_n\rightarrow 0$, if $W_0,\cdots,W_{n-1}$ are all weak-injective modules, then the R-module W_n is also weak-injective.

Proof: $1\Leftrightarrow 2$ Follows immediately from the definition of weak-injective dimension.

$2\Rightarrow 3$. Let F be a R-module of flat dimension ≤ 1. Let $0\rightarrow F_0\rightarrow L\rightarrow F\rightarrow 0$ be an exact sequence, where L is a free R-module, then F_0 is a flat R-module. Applying the functor $Hom_R(-,M)$, we get the exact sequence, for $i>0$:

$$0=Ext^i(L,M)\rightarrow Ext^i(F_0,M)\rightarrow Ext^{i+1}(F,M)\rightarrow Ext^{i+1}(L,M)=0$$

and by induction we get the desired result.

$3\Rightarrow 4$. First, consider an exact sequence $0\rightarrow M\rightarrow I_0\rightarrow\cdots\rightarrow I_{n-1}\rightarrow I_n\rightarrow 0$, where $I_0,\cdots,I_{n-1}$ are injective R-modules. We have $Ext_R^{n+1}(F,M)\cong Ext_R(F,I_n)$ for all R-modules F. If $fd_R(F)\leq 1$, then $Ext_R^{n+1}(F,M)=0$, and so $Ext_R(F,I_n)=0$. Then, I_n is a weak-injective R-module. Now, since each I_i is injective R-module, with $0\leq i\leq n-1$, we get the following commutative diagram:

$$0\rightarrow M\rightarrow W_0\rightarrow\cdots\rightarrow W_n\rightarrow 0$$

$$\begin{array}{ccc} \| & & \downarrow \\ 0\to M\to I_0 \to & \cdots \to & I_n \to 0 \end{array}$$

This diagram gives a chain map between complexes:

$$\begin{array}{ccc} 0\to W_0 \to & \cdots \to & W_n \to 0 \\ \downarrow & & \downarrow \\ 0\to I_0 \to & \cdots \to & I_n \to 0 \end{array}$$

which induces an isomorphism in homology. Then, from [10, Exercises 6.13-6.15] its mapping cone is exact. That is, the following exact sequence:

$$W_0 \to I_0 \oplus W_1 \to \cdots \to I_{n-1} \oplus W_n \to I_n \to 0$$

Finally, decomposing this sequence on short exact sequences and using Proposition 2.2 we deduce that W_n is a weak-injective R-module.

$4 \Rightarrow 2.$ Consider an exact sequence:

$$0 \to I_0 \to \cdots \to I_{n-1} \to I_n \to 0,$$

where $I_0, \cdots, I_{n-1}$ are injective R-modules. Then, by hypothesis, I_n is weak-injective. Then, $Ext_R^{n+1}(F,M) = Ext_R(F,I_n) = 0$, as desired.

Proposition 2.5 Let $\{A_i\}_{i\in I}$ a family of modules. Then:

$$wid(\prod_i A_i) = \sup\{wid(A_i), \quad i \in I\}$$

Proof: Follows from the isomorphism $Ext^n(F, \prod_i A_i) \cong \prod_i Ext^n(F, A_i)$ (see [10, Theorem 7.14]).

3 Global weak-injective dimension of rings

In this section we give definition the global weak-injective dimension of rings and we give its characterization.

Definition 3.1 The global weak-injective dimension of R is the supremum of weak-injective dimensions of all R -modules, denoted:

$$Wi-gldim(R) = \sup\{wid(M)/M \quad R-module\}$$

The following proposition gives a characterization of global weak-injective dimension

Proposition 3.2 Let R be a ring and let n be a positive integer. The following are equivalent:

1. $Wi-gldim(R) \le n$;
2. $Ext_R^{n+1}(F,M) = 0$ for all R -module F of $fd(F) \le 1$ and M an R -module.
3. $Ext_R^{i}(F,M) = 0$ for any $i > n+1$ and for R -module F of $fd(F) \le 1$ and M an R -module.
4. $pd(F) \le n$ for all R -module F of $fd(F) \le 1$.
5. $wid(M) \le n$ for R -module M .

Proof: The proof is obvious it follows from the definition and Theorem 2.4.

The global cotorsion dimension of a ring R is denoted $C-gldim(R)$ is the supremum of cotorsion dimensions of all R-modules, denoted, $C-gldim(R)=\sup\{cod(M)/M \quad R-module\}$ (see [4]). In the following proposition we see the relation between $Wi-gldim(R)$ and $C-gldim(R)$ and global dimension of R $gldim(R)$.

Proposition 3.3 Let R be a ring and let n be a positive integer. Then:

$$C-gldim(R)\leq Wi-gldim(R)\leq gldim(R)$$

Proof: Suppose that $Wi-gldim(R)$=n and let F be a flat module, then since $fd_R(F)\leq 1$ and from Proposition 3.2 we have $pd_R(F)\leq n$ and then $C-gldim(R)=n\leq Wi-gldim(R)$. The second inequality is easy since $gldim(R)=sup\{pd_R(M)/M \quad R-module\}$.

In [1] bazzoni and salce introduced the almost perfect rings which are the rings since R/I is perfect for any proper ideal I of R. in the following proposition we see a characterization of perfect and almost perfect rings using weak-injective global dimensions.

Proposition 3.4 Let R be a ring. Then:

1. $Wi-gldim(R)=0$, then R is perfect.
2. If R is an integral domain, then $Wi-gldim(R)\leq 1$ if and only if R is almost perfect.

Proof:

1. Suppose that $Wi-gldim(R)=0$, and let F be a flat module from Proposition 3.2 $pd_R(F)=0$, then F is projective and R is perfect.
2. See [8, Theorem 6.3].

4 Weak-injective dimension under change of rings.

In this section we are interesting in finding some change of rings results for weak-injective global dimension in some known ring extension and ring constructions.

We begin by the weak-injective global dimension of polynomial rings.

Theorem 4.1 Let $R[X_1,X_2,\cdots,X_n]$ be the polynomial ring in n indeterminates over a ring R. Then:

$$Wi-gldim(R[X_1,X_2,\cdots,X_n])=Wi-gldim(R)+n$$

Proof: By induction we can prove it only for $n=1$, we prove that $Wi-gldim(R[X])=Wi-gldim(R)+1$.

The first inequality $C-gldim(R[X])\leq C-gldim(R)+1$ is same [7, Example (iv)].

Conversely, Assume that $C-gldim(R[X])=n+1<\infty$. Let F be an R-module such that $fd_R(F)\leq 1$, then it is easy to see that $fd_{R[X]}(F[X])\leq 1$. Then, $pd_R(F)=pd_R(F[X])=pd_{R[X]}(F[X])=n+1$. This means that $Wi-gldim(R)\leq n+1$. Assume that $Wi-gldim(R)$=n+1. Then, there exists, from Theorem 3.2, an R-module F of $fd_R(F)\leq 1$ such that $pd_R(F)=n+1$. Thus, there exists an R-module E such that $fd_R(E)\leq 1$, such that $Ext^{n+1}(E,F)=0$. From [2, Example (7), page 9], the endomorphism $\mu:F\to F$, defined by $\mu(f)=Xf$, is injective. Then, we may apply the Rees's theorem [10, Theorem 9.37], which gives:

$$Ext^{n+2}_{R[X]}(E,F[X])\cong Ext^{n+1}_R(E,F)\neq 0$$

Then, From [10, Exercice 9.20, page 258]

$$Ext^{n+2}_{R[X]}(E[X],F[X])\cong Hom_R(R[X],Ext^{n+2}_{R[X]}(E,F[X]))\cong(Ext^{n+2}_{R[X]}(E,F[X]))^{\mathbb{N}}\neq 0.$$

Then, $wid_{R[X]}(F[X])=n+2$, which contradicts with $Wi-gldim(R[X])=n+1$, so $Wi-gldim(R)=n$

Example 4.2 Let R be an integral domain which is not a field. Then from Proposition 3.4 $R[X_1,X_2,\cdots,X_n]$ is never almost perfect for any $n \geq 1$.

In this theorem we study the transfer of global weak-injective dimension in finite product of rings.

Theorem 4.3 Let $\{R_i\}_{i=1,\ldots,m}$ be a family of rings. Then:

$$Wi-gldim(\prod R_i) = sup\{Wi-gldim(R_i), 1 \leq i \leq m\}$$

Proof: The equality follows by induction on m and using Proportion 2.5 and the following lemma.

Lemma 4.4 Let $R_1 \times R_2$ be a direct product of rings R_1 and R_2 , and let F_i be an R_i-module for $i = 1,2$. Then, $fd_{R_1 \times R_2}(F_1 \times F_2) = sup\{fd_{R_1}(F_1), fd_{R_2}(F_2)\}$.

Proof: Since R_1 is a projective $R_1 \times R_2$-module, [3, Exercise 10, page 123] gives:

$$fd_{R_1}(F_1 \times 0) \leq fd_{R_1}((F_1 \times F_2) \times (R_1 \times 0)) \leq fd_{R_1 \times R_2}(F_1 \times F_2).$$

Similarly, we obtain: $fd_{R_1}(0 \times F_2) \leq fd_{R_1 \times R_2}(F_1 \times F_2)$.

Thus $sup\{fd_{R_1}(F_1), fd_{R_2}(F_2)\} \leq fd_{R_1 \times R_2}(F_1 \times F_2)$.

Conversely, from [3, Exercise 10, page 123], we have:

$fd_{R_1 \times R_2}(F_1 \times 0) \leq fd_{R_1 \times 0}(F_1 \times 0) = fd_{R_1}(F_1)$ and

$fd_{R_1 \times R_2}(0 \times F_2) \leq fd_{R_2}(F_2)$.

Therefore, $fd_{R_1 \times R_2}(F_1 \times F_2) = sup\{fd_{R_1 \times R_2}(F_1 \times 0), fd_{R_1 \times R_2}(0 \times F_2)\} \leq sup\{fd_{R_1}(F_1), fd_{R_2}(F_2)\}$, as desired.

Let $T = K + M$ be an integral domain where K is a field and M is a maximal ideal of T. Let D a subring of K, and consider the ring $R = D + M$. Now we study the transfer transfer of global weak-injective dimension in $D + M$ constructions. This construction have proven to be useful in solving many open problems and conjectures for various contexts in ring theory (see for example [5, 6]).

Theorem 4.5 Let T be a ring of the form $T = K + M$ where K is a field an M a maximal ideal of T. Let D be a subring of K where $frac(D) = K$. Consider $R = D + M$, then:

$$Wi-gldim(R) = sup\{Wi-gldim(T), Wi-gldim(D)\}$$

The proof of the theorem concludes from the following lemma.

Lemma 4.6 Let T be a ring of the form $T = K + M$ where K is a field an M a maximal ideal of T. Let D be a subring of K where $frac(D) = K$. Consider $R = D + M$, then for any R-module F such that $fd_R(F) \leq 1$ we have:

$$pd_R(F) = sup\{pd_T(F \otimes_R T), pd_D(F/MF)\}$$

Proof: Suppose that $pd_R(F) = n$ and consider the following exact sequence of R-modules:

$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to F \to 0$, where P_i are projective modules. Since T is flat, we obtain the following exact sequence of S-modules :

$$0 \to P_n \otimes_R T \to P_{n-1} \otimes_R T \to \cdots \to P_0 \otimes_R T \to F \otimes_R T \to 0$$

Then $pd_T(F \otimes_R T) \leq n$. On the other hand from [11, Proof of Theorem1.1], $Tor_1^R(D, F) = 0$, and since $fd_R(F) \leq 1$, then $Tor_p^R(D, F) = 0$ for any $p > 0$ and by [3, Prpopsition 4.1.3], for any D-module C and for any intege $n > 1$ we have $Ext_R^{n+1}(F, C) \cong Ext_D^{n+1}(F \otimes_R D, C)$, so $pd_D(F/MF) \leq n$ and then

$sup\{pd_T(F \otimes_R T), pd_D(F/MF)\} \leq pd_R(F)$

Conversely, suppose that $sup\{fd_T(F \otimes_R T), fd_D(F/MF)\} = n$, for some positive integer n. And let the exact sequence of R-modules $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to p_0 \to F \to 0$ such that $P_0 \cdots P_{n-1}$ are projective. then, $P_n \otimes_R T$ and P_n/MP_n are projective T-module and D-module, respectively. Thus, from [11, Theorem 1.1] P_n is a projective R-module. Then $pd_R(F) \leq n$, so $pd_R(F) = sup\{pd_T(F \otimes_R T), pd_D(F/MF)\}$ as desired.

In this theorem we study the transfer of almost n-perfect ring in $D+M$ constructions.

Theorem 4.7 Let T be a ring of the form $T = K + M$ where K is a field an M a maximal ideal of T. Let D a subring of K where $frac(D) = K$. Consider $R = D + M$, then:

R is almost perfect $\Leftrightarrow$ D and T are almost perfect.

Proof: Follows from Proposition 3.4 and Theorem 4.5 above.

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